Online Appendix
to "Efficient Ex-Ante Stabilization of Firms"

David M. Frankel[39]
Melbourne Business School and Iowa State University

March 2, 2017

6 Introduction

This is the online appendix to Frankel (2017).[40] It consists of the omitted formal results and proofs that underlie the extensions discussed in section 3 of that paper. Section 7 studies the case of large noise; section 8 studies a two-period learning model; and section 9 studies a model of duopoly competition.

In order to avoid ambiguity, numbering of sections, footnotes, and results in this paper continues where Frankel (2017) ends. Bibliographic items that do not appear in this document may be found in Frankel (2017, pp. 143-144).

7 Large Noise

This section contains the results from the large-noise extension of Frankel (2017, section 3.5). The efficient schemes we will study here do not satisfy all of the assumptions that underlie the iterative dominance argument of the base model.[41] We will therefore rely on the following stronger solution concept.

Threshold PBE. A Threshold PBE is a perfect Bayesian equilibrium[42] of the whole game in which, for any price $p \in [0, \overline{p}]$ (including prices that the firm does not choose in equilibrium), each agent i invests if and only if her signal x_i exceeds a common threshold k that may depend on the price.[43]

[39] Contact address: Melbourne Business School, 200 Leicester Street, Carlton, VIC 3053, Australia.

[40] "Efficient Ex-Ante Stabilization of Firms." Journal of Economic Theory 170:112-144 (2017).

[41] In particular, they do not satisfy the technical assumption of One-Sided Lipschitz Continuity (Frankel 2017, section 2.3).

[42] See Fudenberg, Drew, and Jean Tirole. "Perfect Bayesian Equilibrium and Sequential Equilibrium." Journal of Economic Theory 53:236-260 (1991).

[43] Other papers on global games that restrict to threshold equilibria include Angeletos, Hellwig, and Pavan (2007), Mathevet and Steiner (2013), and Morris and Shin (2004a).

With large noise and a nonuniform prior there can exist multiple threshold equilibria (Morris and Shin 2000a). In order to avoid this and to render the model tractable, we also assume the state θ is uniformly distributed on the unit interval.[44]

We begin with an intuition for why the agents' demand at a state θ equals the mean relative payoff R_θ in the small-noise limit as discussed in Frankel (2017, section 2.1). For any given price p, the subgame played by the agents is a global game. In such a game, there is a unique equilibrium, in which each agent invests if and only if her signal falls below a common cutoff k. We refer to this strategy as the "switching strategy around k". For this strategy to be an equilibrium, it must be a best response to itself. Hence, if an agent's signal equals k, she must be indifferent between investing and not investing under the belief that all other agents will play a switching strategy around k. What does this indifference imply about k? To answer, we need to know the agent's posterior beliefs about the joint distribution of the state θ and the investment rate ℓ in the given situation.

The agent's beliefs about the state are simple: as the noise vanishes, she believes that it equals her signal k. As for the investment rate ℓ, it is just the proportion of agents j whose signals x_j are less than k which, in turn, equals i's signal x_i. But agent j's signal $x_j = \theta + \sigma\varepsilon_j$ is less than agent i's signal $x_i = \theta + \sigma\varepsilon_i$ if and only if agent j's signal error ε_j is less than agent i's error ε_i. Thus, the proportion ℓ who invest is simply the realized rank of i's signal error among all agents' signal errors. Since signal errors are i.i.d., each such rank ℓ is equally likely *ex ante*. And because the prior has a continuous density, agent i's posterior over the state is approximately uniform in the set $S = [x_i - \sigma/2, x_i + \sigma/2]$ of states that are possible given her signal x_i. Thus, when i learns her signal she learns nothing about her signal error ε_i.[45] And she learns nothing about the signal error ε_j of any other agent j, as ε_j is independent of the state and of i's signal error - and thus of i's signal $x_i = \theta + \varepsilon_i$. Hence, learning her signal does not change i's beliefs about the difference $\varepsilon_i - \varepsilon_j$ between her signal and that of any other agent j. So after learning her signal, agent i's beliefs over her rank ℓ are unchanged: she still thinks that all values of ℓ in $[0, 1]$ are equally likely.

[44] Other papers that assume a uniform prior include Goldstein and Pauzner (2005). The uniqueness argument in the base model also relies on a type of uniformity: the prior of the state is smooth and thus, when σ is small, approximately uniform on the set $[x - \sigma/2, x + \sigma/2]$ of states that can occur for a given signal x.

[45] The proof is simple. By Bayes's Rule, the probability given x_i that the signal error ε_i equals a given ε' in $[-1/2, 1/2]$ is $\frac{\phi(x_i - \sigma\varepsilon')f(\varepsilon')}{\int_{\varepsilon=-1/2}^{1/2} \phi(x_i - \sigma\varepsilon) f(\varepsilon) d\varepsilon}$. This is approximately equal to $f(\varepsilon') \left[\int_{\varepsilon=-1/2}^{1/2} f(\varepsilon) d\varepsilon\right]^{-1} = f(\varepsilon')$ since, for σ close to zero, $\phi(x_i - \sigma\varepsilon)$ is approximately constant for all ε in $[-1/2, 1/2]$.

Summarizing, if the agent's signal equals the common threshold k, she believes that the state equals k and that the investment rate ℓ is uniform on $[0,1]$. So she is indifferent between investing and not if her expected net relative payoff under these beliefs, $\int_{\ell=0}^{1}\left(r_k^\ell - p\right)d\ell$, is zero. It then follows from (1) that the common threshold k is defined by the condition $p = R_k$. And since k is the agents' investment threshold, they invest whenever their signals fall below k. But in the small-noise limit, their signals are arbitrarily close to the true state θ. Hence, the agents invest whenever the state θ is less than k or, equivalently, whenever R_θ exceeds R_k which, in turn, equals p. Summarizing, they invest at θ if and only if $p < R_\theta$. It follows that their willingness to pay at the state θ is their mean relative payoff, R_θ, as claimed. This completes the intuition.

Now consider how things change with large noise: when the scale factor σ is fixed and positive. In this setting, for any given price p, the agents still play a threshold equilibrium: an agent invests if and only if her signal falls below a common threshold k. However, this threshold no longer equates the mean relative payoff R_k to p. Rather, it is given by equality between the price p and the *large-noise mean relative payoff*[46]

$$R_k^\sigma \stackrel{d}{=} \int_{\ell=0}^{1} r_{k-\sigma F^{-1}(\ell)}^\ell d\ell. \tag{34}$$

Why? Consider an agent whose signal equals a common investment threshold k. Assume k is far enough from the endpoints of the prior distribution of the state that all states in $[k-\sigma/2, k+\sigma/2]$ have positive prior probability.[47] Thus, no investment rate $\ell = F\left(\frac{k-\theta}{\sigma}\right)$ in $[0,1]$ can be ruled out on the basis of the signal k. As in the small-noise case, the agent still believes the investment rate ℓ is uniform on the unit interval.[48] But the state is not simply equal to the signal k for each investment rate ℓ. Rather, it is determined by the condition that, by the law of large numbers, the proportion ℓ who invest at a given state θ equals the probability $F\left(\frac{k-\theta}{\sigma}\right)$ that an agent j's signal is less than the threshold k.[49] Solving this equality for the state θ yields $\theta = k - \sigma F^{-1}(\ell)$: a higher investment rate ℓ arises from

[46] One can easily see that there is continuity in the limit: R_k^σ converges to the mean relative payoff R_k as the noise scale factor σ shrinks to zero.

[47] Since the prior support is $[0,1]$, this means that k lies in the interval $I = [\sigma/2, 1-\sigma/2]$. Below we will make assumptions on payoffs that ensure that any threshold k chosen by the agents must lie in I.

[48] The proof is nearly identical to that for the small-noise case above. Since $k \in I$, all states θ in $[k-\sigma/2, k+\sigma/2]$ have the same unit prior density as this interval is a subset of the prior support of θ. This fact is then used in the argument in n. 45 to show that learning that her signal equals k tells the agent nothing about her signal error ε_i. The rest of the argument is unchanged.

[49] This is because $\Pr\left(x_j = \theta + \sigma\varepsilon_j < k|\theta\right) = F\left(\frac{k-\theta}{\sigma}\right)$.

a higher probability that an agent's signal lies below the threshold k, which can only result from a lower state. Hence, at the common threshold k, an agent is just willing to invest under the belief that ℓ is uniform on $[0,1]$ and, given each such ℓ, the state is given by $k-\sigma F^{-1}(\ell)$. It follows that the threshold k must satisfy $p=R_k^\sigma$ as claimed.

Armed with this result, we can now visualize the large-noise economy by reinterpreting Figure 1 in Frankel (2017) as follows. The horizontal axis now represents the unconditional probability

$$G(k)=\int_{\theta=0}^{1} F\left(\frac{k-\theta}{\sigma}\right) d\theta \in (0,1) \tag{35}$$

that an agent's signal is less than k as the threshold k ranges from $-\infty$ to ∞.[50] The curve R is now interpreted as the large-noise demand curve R_k^σ and r^1 now represents the maximum value $r^1_{k-\sigma/2}$ of the integrand in (34). The marginal social benefit curve s and the marginal revenue curve m are reinterpreted as their large-noise analogues, defined below. As before, the laissez-faire and efficient thresholds are given by points H and N, respectively. Since H lies to the right of N, the agents invest too infrequently as in the small-noise case: there is again a rationale for intervention.

We now fill in the gaps in this analogy; some technical details are deferred to section 7.1, below. We begin with our payoff assumptions. We weaken State Monotonicity to let r jump downwards in the state, requiring only that its mean R_θ^σ be Lipschitz-continuous in θ.[51] More precisely, we assume:

Weak State Monotonicity (WSM). There are constants $k_4>k_3>0$ such that for every pair of states $\theta'>\theta$ and each investment rate ℓ, $\frac{r^\ell_{\theta'}-r^\ell_\theta}{\theta'-\theta}<-k_3$ and $\frac{R^\sigma_{\theta'}-R^\sigma_\theta}{\theta'-\theta}\in(-k_4,-k_3)$. Moreover, r_θ^ℓ is discontinuous at at most a finite number of points θ_n for $n=1,...,N$ (where $N=0$ corresponds to no discontinuities) and it is Lipschitz continuous with constant k_4 between any two such points: for any $\theta'>\theta$ such that there is no θ_n in $[\theta,\theta']$, $\frac{r^\ell_{\theta'}-r^\ell_\theta}{\theta'-\theta}>-k_4$.

In our base model, the infinite prior support of the state together with State Monotonicity implies that there are dominance regions. In the present context, since the prior support of the state is finite, dominance regions must be explicitly assumed. Let n be a natural

[50] $G(k)$ is the integral, over all states θ in $[0,1]$, of the prior density $\phi(\theta)=1$ of the state times the probability $F\left(\frac{k-\theta}{\sigma}\right)$ that the signal x_i of an agent is less than the threshold k given the state θ.

[51] We are able to tolerate downwards jumps in r since we are now using the stronger solution concept of Threshold PBE rather than iterative strict dominance. These jumps are useful: they yield a result (Theorem 5) that is general enough to apply to the learning model studied below in section 8.

number to be specified later. Let h denote either the primitive or the augmented relative payoff function (r or $\widetilde{r}$, respectively).

Dominance Regions of Type n (DR$_n$). For any investment rate ℓ, the relative payoff h_θ^ℓ exceeds the highest price $\overline{p}$ (resp., is less than the lowest price 0) for all $\theta \leq n\sigma$ (resp., $\theta \geq 1 - n\sigma$).

We will assume the primitive relative payoff function r satisfies DR$_2$, and produce an augmented relative payoff function $\widetilde{r}$ that satisfies DR$_1$. This latter property, in turn, implies that an agent has a dominant action when her signal lies within $\sigma/2$ of zero or one since the gap between an agent's signal $x_i = \theta + \sigma\varepsilon_i$ and the state θ is at most $\sigma/2$.[52]

We now give expressions for agent and social welfare for a general price p and investment threshold k. As the firm gets its price p per agent who invests, its expected payoff is simply $pG(k)$. Since, as noted above, the investment rate ℓ at a given state θ is $F\left(\frac{k-\theta}{\sigma}\right)$, agent welfare is

$$AW(p,k) = \int_{\theta=0}^{1} \left[\ell\left(v_\theta^\ell - p\right) + (1-\ell)\, o_\theta^\ell\right]_{\ell = F\left(\frac{k-\theta}{\sigma}\right)} d\theta.$$

Omitting the transfer p, we obtain social welfare:

$$SW(k) = \int_{\theta=0}^{1} \left[\ell v_\theta^\ell + (1-\ell)\, o_\theta^\ell\right]_{\ell = F\left(\frac{k-\theta}{\sigma}\right)} d\theta.$$

Differentiating and applying the change of variables $\ell = F\left(\frac{k-\theta}{\sigma}\right)$, we obtain the first order condition for an efficient threshold:

$$0 = SW'(k) = s_k^\sigma \overset{d}{=} \int_{\ell=0}^{1} \frac{\partial}{\partial \ell}\left[w_\theta^\ell\right]_{\theta = k - \sigma F^{-1}(\ell)} d\ell \tag{36}$$

where s_k^σ is the *large-noise marginal social benefit* of raising the investment threshold k and

$$w_\theta^\ell = \ell v_\theta^\ell + (1-\ell)\, o_\theta^\ell. \tag{37}$$

[52] An agent who sees a signal $x < \sigma/2$ knows that the state θ is less than σ and thus, by DR$_1$, that the relative payoff h_θ^ℓ exceeds the highest price $\overline{p}$: she will adopt. And if $x > 1 - \sigma/2$, the agent knows that the state θ exceeds $1 - \sigma$ and thus that the relative payoff h_θ^ℓ is negative: she will not adopt.

is realized social welfare.[53] Intuitively, a small increase in the threshold k induces the investment of agents whose signals equal k. For these agents, the proportion ℓ who invest is uniform on the unit interval and the state is given by $\theta = k - \sigma F^{-1}(\ell)$ as explained above. Thus, their decision to invest changes social welfare by the integral in (36).

We assume the large-noise marginal social benefit s_k^σ is continuous and decreasing in the threshold k, and takes on both positive and negative values:

DLMB. Decreasing Large-Noise Marginal Social Benefit. The large-noise marginal social benefit s_k^σ is continuous and decreasing in the threshold k and satisfies $s_{\sigma/2}^\sigma > 0$ and $s_{1-\sigma/2}^\sigma < 0$.

DLMB ensures that s_k^σ equals zero at a unique threshold k^*, which lies in $I = [\sigma/2, 1-\sigma/2]$. It is the efficient investment threshold.

The laissez-faire equilibrium outcome is as follows. Since the agents' demand at the threshold k is R_k^σ, the firm chooses a threshold k to maximize its expected payoff $\Pi_r^\sigma(k) = R_k^\sigma G(k)$. The firm's marginal revenue from raising its expected quantity $G(k)$ by one infinitesimal unit is thus $m_k^\sigma \overset{d}{=} \frac{d\Pi_r^\sigma(k)}{dG(k)} = R_k^\sigma + \frac{\partial R_k^\sigma}{\partial k}\frac{G(k)}{G'(k)}$. The agents' demand curve $p = R_k^\sigma$ is downwards sloping by WSM, so the agents' willingness to pay exceeds the firm's marginal revenue:

$$R_k^\sigma > m_k^\sigma. \tag{38}$$

This is just the usual monopoly pricing distortion. We also assume:

DLMR. Decreasing Large-Noise Marginal Revenue. Marginal revenue m_k^σ is continuous and decreasing in the threshold k, and satisfies $m_{\sigma/2}^\sigma > 0$ and $m_{1-\sigma/2}^\sigma < 0$.

Hence the firm's marginal revenue equals zero at a unique threshold $\widehat{k}$, which lies in $I = [\sigma/2, 1-\sigma/2]$; it is the laissez-faire equilibrium threshold. In analogy to (8) we also assume that the agents' maximum willingness to pay at the equilibrium threshold $\widehat{k}$ is less than the firm's maximum price:

$$r_{\widehat{k}-\sigma/2}^1 < \overline{p}. \tag{39}$$

[53] As the signal errors vanish, the large-noise marginal social benefit s_k^σ converges to the marginal social benefit s_k defined in equation (2):

$$\lim_{\sigma\to 0} s_k^\sigma = \int_{\ell=0}^1 \frac{\partial}{\partial \ell}\left[w_k^\ell\right] d\ell = \int_{\ell=0}^1 dw_k^\ell = v_k^1 - o_k^0 = s_k.$$

As in (5), we also assume that for any investment threshold k, the large-noise marginal social benefit s_k^σ exceeds the agents' primitive demand R_k^σ.[54] Finally, we follow (3) in assuming that $r^1_{k-\sigma/2} > s_k^\sigma$, which converges to assumption (3) in the small-noise limit.[55] Combining these two inequalities with (38), we obtain the large-noise analogue to (9):

$$r^1_{k-\sigma/2} > s_k^\sigma > R_k^\sigma > m_k^\sigma. \tag{40}$$

By (40), we obtain a depiction of the large-noise model by reinterpreting θ, r^1_θ, s_θ, R_θ, m_θ, and $\Phi(\theta)$ in Figure 1 of Frankel (2017, p. 121) as k, $r^1_{k-\sigma/2}$, s_k^σ, R_k^σ, m_k^σ, and $G(k)$, respectively. The laissez-faire and efficient thresholds $\widehat{k}$ and k^* then appear at points H and N, respectively. Hence, the efficient threshold k^* exceeds the laissez-faire threshold $\widehat{k}$: as in the base model, the agents invest too infrequently. Moreover, since (by DLMR) marginal profits m_k^σ are negative for all k in $\left(\widehat{k}, k^*\right]$, the laissez-faire equilibrium threshold $\widehat{k}$ gives the firm a higher payoff than the efficient threshold k^*. Thus, some sort of public intervention is needed if we wish to induce the firm to choose the efficient threshold k^*.

As in our base model (Frankel 2017, section 2), we now consider schemes in which an agent who invests receives a nonnegative transfer τ_θ^ℓ that depends on the state is θ and the investment rate ℓ. Substituting the augmented relative payoff function $\widetilde{r}$ for r in (34), we obtain the augmented mean relative payoff function $\widetilde{R}_k^\sigma = R_k^\sigma + T_k^\sigma$ where $T_k^\sigma = \int_{\ell=0}^1 \tau^\ell_{k-\sigma F^{-1}(\ell)} d\ell$ is the mean transfer function.

The analogue of an APSS is the following class of schemes.

PSS (Informal). A Predictable Subsidy Scheme (PSS) is a function τ with the following two properties.

1. Predictability. The augmented relative payoff function $\widetilde{r} = r + \tau$ satisfies the conditions for predictability in the agent subgame.

[54] This can be rearranged to yield $\int_{\ell=0}^1 \left[\ell \frac{\partial}{\partial \ell} v_\theta^\ell + (1-\ell) \frac{\partial}{\partial \ell} o_\theta^\ell\right]_{\theta = k - \sigma F^{-1}(\ell)} d\ell > 0$: if the agents invest when their signals fall below some threshold k, then inducing an agent to invest when her signal equals k raises the mean surplus generated by all other agents - both investors and noninvestors. Roughly, this means that the mean externality (over all possible investment rates ℓ and corresponding states θ) is stronger for the firm than for the outside option. A similar intuition holds for (4), which is equivalent to (5).

[55] By definition of r_θ^ℓ and s_k^σ, this inequality holds if and only if the effect $o^0_{k+\sigma/2} - o^1_{k-\sigma/2}$ on the outside option payoff, of an agent's switching from her most pessimistic to her her most optimistic beliefs regarding this payoff, exceeds the integral $\int_{\ell=0}^1 \frac{\partial}{\partial \theta} \left[w_\theta^\ell\right]_{\theta = k-\sigma F^{-1}(\ell)} \sigma dF^{-1}(\ell)$. As the noise scale factor σ vanishes, this integral shrinks to zero and thus the inequality $r^1_{k-\sigma/2} > s_k^\sigma$ converges to $o_k^0 > o_k^1$ which, in turn, is equivalent to (3).

2. No Taxation. Transfers are nonnegative: for all ℓ and θ, $\tau_\theta^\ell \geq 0$.

An efficient PSS is one that induces a unique Threshold PBE, in which the firm chooses the efficient threshold k^*. Under any PSS, the firm's payoff from the threshold k is $\Pi_{\widetilde{r}}^\sigma(k) = \widetilde{R}_k^\sigma G(k)$. As taxes are ruled out, augmented demand $\widetilde{R}_k^\sigma$ cannot fall below primitive demand R_k^σ. Thus, the firm gets at least its laissez-faire payoff $\Pi_r^\sigma(k)$ from any threshold k. So in order to induce the firm to choose the efficient threshold k^* rather than $\widehat{k}$, augmented demand $\widetilde{R}_{k^*}^\sigma$ cannot be lower than the "minimum price"

$$p_m^\sigma = \Pi_r^\sigma\left(\widehat{k}\right)/G(k^*), \tag{41}$$

as in (12).

We now derive an expression for the revenue cost $C^\sigma(\tau)$ of any efficient PSS τ. This cost is the integral, over all states θ in $[0,1]$, of the aggregate transfer $\ell\tau_\theta^\ell$ given to the ℓ investors, evaluated at the investment rate $\ell = F\left(\frac{k^*-\theta}{\sigma}\right)$ that results from the state θ when agents choose the efficient threshold k^*. It is convenient to separate this cost into a coordination cost $C_0^\sigma(\tau)$, which is incurred when all agents invest, and a miscoordination cost $C_1^\sigma(\tau)$, which is incurred when some but not all invest. The coordination cost is incurred when the state θ falls below the cutoff $k^*-\sigma/2$ and thus equals $C_0^\sigma(\tau) = \int_{\theta=0}^{k^*-\sigma/2}\tau_\theta^1 d\theta$. Miscoordination occurs when the state θ lies strictly between $k^*-\sigma/2$ and $k^*+\sigma/2$, whence (since k^* lies in $I = [\sigma/2, 1-\sigma/2]$) the miscoordination cost is $C_1^\sigma(\tau) = \int_{\theta=k^*-\sigma/2}^{k^*+\sigma/2}\left[\ell\tau_\theta^\ell\right]_{\ell=F\left(\frac{k^*-\theta}{\sigma}\right)} d\theta$.[56] The scheme's total cost $C^\sigma(\tau)$ is then the sum $C_0^\sigma(\tau) + C_1^\sigma(\tau)$. A least-cost efficient PSS is an efficient PSS τ such that, for any other efficient PSS τ', $C^\sigma(\tau') \geq C^\sigma(\tau)$.[57]

We now define a floor-based PSS. If the agents use the threshold k and the realized investment rate is ℓ, then the state θ must equal $k - \sigma F^{-1}(\ell)$ as noted above: the agents' augmented relative payoff is $\widetilde{r}_{k-\sigma F^{-1}(\ell)}^\ell$. A floor-based PSS is a PSS τ such that for each k in $I = [\sigma/2, 1-\sigma/2]$ there is a floor κ_k such that, for each investment rate $\ell > 0$, the augmented relative payoff $\widetilde{r}_{k-\sigma F^{-1}(\ell)}^\ell$ equals the primitive relative payoff $r_{k-\sigma F^{-1}(\ell)}^\ell$ or the floor κ_k, whichever is greater. Hence, such a scheme ensures that for any threshold k that the agents use, an agent's relative payoff from investing will not fall below some floor κ_k in

[56] In the definitions of $C_0(\tau)$ and $C_1(\tau)$, we assume the prior support $\phi(\theta)$ equals one in each interval of integration $[0, k^*-\sigma/2]$ and $[k^*-\sigma/2, k^*+\sigma/2]$. This is so since, by DLMB, the efficient cutoff k^* lies in $I = [\sigma/2, 1-\sigma/2]$.

[57] In a more fully developed model with a separate source of distortionary taxation, a planner might prefer to accept some deviation from the efficient threshold in order to lower taxes elsewhere. For brevity, we do not analyze this here.

a partial run.[58]

Theorem 5 below will show that the following floor-based scheme τ^* is an efficient PSS, and is cost-minimizing when the agents' error distribution function F is log-concave. Fix some small constant $k_3' \in (0, k_3)$.[59] The augmented demand function that will result from τ^* is

$$\widetilde{R}_k^\sigma = \begin{cases} R_k^\sigma & \text{if} \quad k \notin [k^1, k^2] \\ p_m^\sigma & \text{if} \quad k \in [k^1, k^*] \\ p_m^\sigma - (k - k^*)/k_3' & \text{if} \quad k \in [k^*, k^2] \end{cases} \tag{42}$$

Let $\zeta_k(\kappa) = \int_{\ell=0}^1 \max\left\{r_{k-\sigma F^{-1}(\ell)}^\ell, \kappa\right\} d\ell$ denote augmented demand at the threshold k if a floor κ on is given at this threshold. The floor we need at k to yield augmented demand $\widetilde{R}_k^\sigma$ is then $\kappa_k^* = \zeta_k^{-1}\left(\widetilde{R}_k^\sigma\right)$. This yields the subsidy function

$$\tau_{k-\sigma F^{-1}(\ell)}^{*\ell} = \begin{cases} \max\left\{0, \kappa_k^* - r_{k-\sigma F^{-1}(\ell)}^\ell\right\} & \text{if} \quad \ell > 0 \\ 0 & \text{if} \quad \ell = 0 \end{cases}. \tag{43}$$

Let $r_{\theta-}^\ell$ denote the left limit $\lim_{\theta' \uparrow \theta} r_{\theta'}^\ell$ of the primitive relative payoff function at the state θ. The "minimum price" p_m^σ is defined in (41). In order for a minimum-cost policy to exist, we assume that the firm chooses the efficient price whenever it is indifferent between this price and some other price(s).[60]

Theorem 5 *Assume the primitive relative payoff function r satisfies AM, WSM, DR_2, DLMB, and DLMR, as well as equations (39) and (40). For sufficiently small k_3' in $(0, k_3)$:*

1. *The Floor-Based PSS τ^* defined in (43) is an efficient PSS. It has the following properties:*

   (a) *If $r_{(k^*-\sigma/2)-}^1 \geq p_m^\sigma$, no subsidy is ever paid when agents coordinate: τ_θ^{*1} is identically zero.*

[58] For technical reasons, the floor is not imposed when the investment rate ℓ is zero.

[59] The precise meaning of "small" is given in the proof of Theorem 5.

[60] This can be obtained as an equilibrium prediction, if we include the planner as a player who wishes to implement the efficient outcome at minimum cost. Intuitively, if the firm puts positive weight on an inefficient price when it is willing to choose the efficient one as well, the planner would deviate to a scheme that makes the efficient price slightly more appealing than any alternative - thus inducing the firm to choose the efficient price with probability one. Hence, the firm must choose the efficient price in any PBE.

(b) *If* $r^1_{(k^*-\sigma/2)-} < p^\sigma_m$*, full insurance plus a subsidy is given when agents miscoordinate. Moreover, a positive subsidy* τ^{*1}_θ *is paid when all agents invest for states* θ *in a nonempty interval* $[\theta_3, k^* - \sigma/2]$.

2. *If* $\ln F$ *is concave, then the scheme* τ^* *is also a least-cost efficient PSS.*

Proof. See p. 30 below. ■

7.1 Large Noise: Technical Details

Additional assumptions for the large-noise extension are as follows. We assume the primitive relative payoff function r satisfies AM, WSM, and DR_2 (which implies DR_1). Claim 5 and Theorem 6 then immediately yield the following characterization of the agent subgame, which relies on the fact that R^σ_k is decreasing in k by WSM.

Claim 4 *Assume the primitive relative payoff* r *satisfies AM, WSM, and* DR_1 *and there is no public intervention. Then for any price* $p \in [0, \overline{p}]$*, there is a unique threshold equilibrium* k*, which satisfies* $R^\sigma_k = p$ *and lies in* $I = [\sigma/2, 1 - \sigma/2]$*. This threshold is decreasing in the price* p*.*

Proof. Follows directly from Claim 5 and Theorem 6. ■

By Claim 4, choosing a price is equivalent to choosing a threshold. Hence, the firm chooses a threshold $k \in I$ to maximize its expected profit function $\Pi^\sigma_r(k) = R^\sigma_k G(k)$.

Let $h \in \{r, \widetilde{r}\}$ denote either the primitive or the augmented relative payoff function and let

$$H^\sigma_\theta \overset{d}{=} \int_{\ell=0}^{1} h^\ell_{\theta-\sigma F^{-1}(\ell)} d\ell \tag{44}$$

denote the corresponding large-noise mean relative payoff.

Theorem 5 assumes that the primitive relative payoff function r satisfies AM, WSM, and DR_2. It will be shown that the transfer scheme in (43) induces an augmented relative payoff function that satisfies DR_1 together with the following two properties. First, for a threshold k in I to be an equilibrium at the price p, the agents' willingness to pay at k must equal the price: $H^\sigma_k = p$. The following property will be used to show that there exists such a threshold k.

Large-Noise Mean State Monotonicity (LMSM). H^σ_k is continuous and nonincreasing in $k \in I$.

However, it is not enough for an agent to be indifferent at a threshold k. She must also be willing (not) to invest for signals that lie below (above) k. For this to hold, it will suffice that the agent's realized relative payoff - her relative payoff h_θ^ℓ evaluated at the investment rate $\ell = F\left(\frac{k-\theta}{\sigma}\right)$ that results from the state θ - is nonincreasing in the state:

Threshold Monotonicity (TM). For any constant $k \in I$, $h_\theta^{F\left(\frac{k-\theta}{\sigma}\right)}$ is nonincreasing in θ.

The following result is immediate:

Claim 5 *AM*(h) *and WSM*(h) *jointly imply LMSM*(h) *and TM*(h).

Proof. Trivial. ■

The following result shows that the three properties DR$_1$, LMSM, and TM suffice for there to exist a threshold equilibrium for each price p that the firm may charge. At any such equilibrium threshold k, the mean relative payoff H_k^σ must equal the price p. Thus, the agents' equilibrium threshold is unique if H_k^σ is decreasing in k. Moreover, a higher price leads the agents to choose a lower threshold; intuitively, a higher price leads agents to invest at a smaller range of signals.

Theorem 6 *Let h satisfy DR$_1$, LMSM, and TM. For any price $p \in [0, \overline{p}]$, the set of threshold equilibria k of the investment subgame is nonempty and equals the set $\left[\underline{k}_p^h, \overline{k}_p^h\right] \subset I$ of solutions k to $H_k^\sigma = p$. If $p' > p$, then $\underline{k}_p^h > \overline{k}_{p'}^h$: a higher price induces the agents to choose a lower threshold. If, for some price $p \in [0, \overline{p}]$, the mean relative payoff H_k^σ is decreasing at any k in $\left[\underline{k}_p^h, \overline{k}_p^h\right]$, then $\underline{k}_p^h = \overline{k}_p^h$: there is only one threshold the agents can choose in response to p.*

Proof. See p. 29 below. ■

We now turn to the firm's problem when the agents' relative payoff function h satisfies DR$_1$, LMSM, and TM. For any price p in $[0, \overline{p}]$, the resulting threshold k must satisfy $p = H_k^\sigma$ by Theorem 6. Hence, the firm's payoff must be $\Pi_h^\sigma(k) = H_k^\sigma G(k)$. We now show that if Π_h^σ is uniquely maximized at a given threshold k, then the agents must choose the threshold k and the firm must choose the corresponding price H_k^σ.

Claim 6 *Let the agents' relative payoff function h satisfy DR$_1$, LMSM, and TM. Suppose Π_h^σ is uniquely maximized at some threshold k. Then in any Threshold PBE, the firm must choose the price $p = H_k^\sigma$ and the agents must respond with the threshold k.*

Proof. Let $k = k_0$ uniquely maximize $\Pi_h^\sigma(k)$. By DR$_1$, $H_{\sigma/2}^\sigma > 0$, so $H_{k_0}^\sigma > 0$. Now consider a Threshold PBE in which the firm's payoff is some $\pi < \Pi_h^\sigma(k_0)$. Let the firm deviate to the price $p_0 - \varepsilon$ for ε in $(0, p_0)$. By Theorem 6, the agents must respond by choosing a threshold $k' > k_0$. The firm's payoff is thus at least $(p_0 - \varepsilon) G(k_0)$ which, in turn, is at least $\Pi_h^\sigma(k_0) - \varepsilon$ which, in turn, exceeds π as long as ε is less than the positive amount $\Pi_h^\sigma(k_0) - \pi$. Hence the firm will deviate. It follows that the firm must get $\Pi_h^\sigma(k_0)$. But by Theorem 6, k_0 is the only threshold that yields this payoff. Accordingly, the agents must select the threshold k_0 and thus, by Theorem 6, the firm's price must be $p_0 = H_{k_0}^\sigma$. ■

The exact definition of a PSS is as follows. The conditions for predictability are now those on which Claim 6 relies to ensure a unique Threshold PBE.

PSS. A Predictable Subsidy Scheme (PSS) is a function τ with the following two properties.

1. Predictability. The augmented relative payoff function $\widetilde{r} = r + \tau$ satisfies DR$_1$, LMSM, and TM.
2. No Taxation. Transfers are nonnegative: for all ℓ and θ, $\tau_\theta^\ell \geq 0$.

By Theorem 6, when a PSS τ is imposed and the firm then picks a price p in $[0, \overline{p}]$, the agents choose the investment threshold k that satisfies $\widetilde{R}_k^\sigma = p$ and the firm's expected quantity sold is $G(k)$.

8 Learning about the State

We now formally present the two-period model with learning that is discussed in Frankel (2017, section 3.6). A dynamic setting is considerably more complex than a one-shot game. Thus, we correct the model's inefficiencies sequentially. That is, we first show how to correct period 2's inefficiencies, following any period-1 outcome. With these corrections in place, we then show how to correct period 1's inefficiencies. This approach does not reveal every inefficiency of the laissez-faire outcome: it omits inefficiencies in period 1 that result from anticipated inefficient behavior in period 2. However, it is simple and intuitive, and is all we need to identify an efficient scheme. We show, in particular, that if taxes cannot be levied, the least-cost way to attain the efficient outcome in the two-period model is through a floor-based subsidy scheme in period 1, followed by a floor-based subsidy scheme in period 2 that depends on the period-1 outcome.

There are two periods, $t = 1, 2$, a single firm with zero costs, and a unit measure of agents. An agent can invest in either period, or not at all. Investment is irreversible: if an agent invests in period 1, she does business with the firm in period 2 as well as period 1. Let ℓ_t denote the measure of agents who invest in period $t = 1, 2$.[61] Let the *cumulative investment rate* $\ell^t = \sum_{s \leq t} \ell_s$ denote the proportion of agents who invest in or before period t. The period-t utility of an agent who invests in or before period t is then $v_{t,\theta}^{\ell^t}$; an agent who does not invest in or before period t gets $o_{t,\theta}^{\ell^t}$. When ambiguity is unlikely, we will write ℓ instead of ℓ^t. For instance, the realized period-t relative payoff of an agent who invests in some period $t' \leq t$ may be written as $r_{t,\theta}^{\ell} = v_{t,\theta}^{\ell} - o_{t,\theta}^{\ell}$.

An agent who invests in period t pays the price p_t that the firm charges in that period. That is, an agent who invests in period 1 pays p_1 but not p_2. The firm's realized payoff in the game is simply its revenue, $\ell_1 p_1 + \ell_2 p_2$. We assume the primitive payoffs $v_{t,\theta}^{\ell}$ and $o_{t,\theta}^{\ell}$ are differentiable with respect to the cumulative investment rate ℓ. We also assume the primitive relative payoff functions r_1 and r_2 satisfy action and state monotonicity as well as dominance regions, where ℓ in those properties is now interpreted as the cumulative investment rate ℓ^t:

Dynamic Action Monotonicity (DAM). For $t = 1, 2$, AM(r_t) holds with $\ell = \ell^t$.

Dynamic State Monotonicity (DSM). For $t = 1, 2$, SM(r_t) holds with $\ell = \ell^t$.

Dynamic Dominance Regions (DDR). For $t = 1, 2$, DR$_2\,(r_t)$ holds with $\ell = \ell^t$.

The timing is as follows. Period 1 is identical to the large-noise model of section 7 above. The firm first chooses a price p_1 in $[0, \overline{p}]$.[62] Each agent $i \in [0, 1]$ then sees a private signal $x_i = \theta + \sigma\varepsilon_i$ where the state θ is uniform on $[0, 1]$ and $\sigma > 0$ is a fixed scale factor. The noise terms ε_i (which are independent of each other and of θ) are identically distributed with continuous density f, distribution function F, and connected support contained in $[-1/2, 1/2]$. The agents then simultaneously decide whether to invest in period 1 or to wait.

In period 2, the firm and agents perfectly observe the state θ and the proportion ℓ_1 who invested in period 1. If $\ell_1 < 1$, the firm then chooses a second price p_2 and the remaining

[61] Thus, $\ell_1 \in [0, 1]$ and $\ell_2 \in [0, 1 - \ell_1]$.

[62] In principle, the firm might prefer to charge a negative price in period 1 if by doing so it could induce more agents to adopt early and thus raise the remaining agents' willingness to pay in period 2. (A negative price, if not too low, can be interpreted as a price that lies below the firm's marginal cost as we normalize the latter to zero.) This can be accommodated by assuming a stronger dominance regions property, with no essential change in the results.

$1-\ell_1$ agents then simultaneously decide whether or not to invest. We assume that in period 2, agents use the global games criterion: they invest whenever doing so is optimal if the proportion ℓ_2 of period-2 investors is uniform on $[0, 1-\ell_1]$. This assumption is supported by the finding of Heinemann, Nagel, and Ockenfels (2009) that experimental subjects play according to the global games solution even when the state is common knowledge.[63] We also assume, as in section 7 above, that the agents play a threshold equilibrium in period 1 whenever such an equilibrium exists.

Let $w_{t,\theta}^{\ell} = \ell v_{t,\theta}^{\ell} + (1-\ell)\, o_{t,\theta}^{\ell}$ denote realized welfare in period $t = 1, 2$ for a given state θ and cumulative investment rate ℓ. Let $m_{2,\theta}^{\ell}$ denote the marginal social benefit $\partial w_{2,\theta}^{\ell}/\partial \ell$ of raising the cumulative investment rate in period 2. We assume $m_{2,\theta}^{\ell}$ is decreasing in the state and increasing in the cumulative investment rate, and is positive (negative) at sufficiently low (high) states:[64]

Monotone Marginal Social Benefit (MMSB). For $t = 1, 2$, the marginal social benefit $m_{t,\theta}^{\ell} = \partial w_{t,\theta}^{\ell}/\partial \ell$ exists. It is continuously differentiable in both arguments, increasing in $\ell \in [0, 1]$, decreasing in $\theta \in [0, 1]$, and positive (negative) for all $\theta \leq 2\sigma$ ($\theta \geq 1-2\sigma$).

We begin with the efficient outcome. In period 2, for a given state θ and period-1 investment rate ℓ_1, realized period-2 social welfare is $w_{t,\theta}^{\ell_1+\ell_2}$ which, by MMSB, is strictly convex in the proportion $\ell_2 \in [0, 1-\ell_1]$ who invest in period 2. Hence, a corner solution is

[63] It can also be derived formally as follows. Assume an agent's relative period-2 payoff r_2 depends not on the state θ but rather on $\theta + \sigma'\nu$ where ν is a common taste shock with a continuous density and support on $\Re$ and $\sigma' > 0$ is a known parameter that captures the importance of the shock. The firm has no information about the shock ν, so its price p_2 does not signal any information about ν. At the beginning of period 2, each agent i sees a signal $\nu + \sigma''\eta_i$ of the shock, where η_i is i.i.d. noise with a continuous density and support equal to $[-1/2, 1/2]$. In the limit as σ'' goes to zero, the agents can predict the shock ν perfectly; thus, by the usual global games argument using DAM and DSM, they invest if and only if doing so is optimal under the belief that ℓ_2 is uniform on $[0, 1-\ell_1]$. We then take the limit as the taste shocks vanish: as $\sigma' \to 0$. In this double limit, all participants know θ as well.

[64] The assumption that $m_{2,\theta}^{\ell}$ is increasing in ℓ can be justified as follows. If the payoffs v and o of investors and noninvestors are differentiable, then the marginal social benefit $dw_{\theta}^{\ell}/d\ell = \lim_{\Delta \downarrow 0}\left[\frac{1}{\Delta}\left(w_{\theta}^{\ell+\Delta} - w_{\theta}^{\ell}\right)\right]$ of an increase in the cumulative investment rate is the sum $r_{\theta}^{\ell} + \ell\frac{dv_{\theta}^{\ell}}{d\ell} + (1-\ell)\frac{do_{\theta}^{\ell}}{d\ell}$ of three terms: the change r_{θ}^{ℓ} in the new investors' payoffs, the marginal spillover $dv_{\theta}^{\ell}/d\ell$ accruing to each of the ℓ initial investors, and the marginal spillover $do_{\theta}^{\ell}/d\ell$ (which will typically be negative) accruing to each of the $1-\ell$ initial noninvestors. By DSM, the relative payoff r_{θ}^{ℓ} is decreasing in θ, so $dw_{\theta}^{\ell}/d\ell$ is decreasing in θ as long as the marginal spillovers $dv_{\theta}^{\ell}/d\ell$ and $do_{\theta}^{\ell}/d\ell$ are not too sensitive to θ. This explains the first property. Now by DAM, the relative payoff r_{θ}^{ℓ} is nondecreasing in the cumulative investment rate ℓ. Moreover, positive externalities among both investors and noninvestors suggest that $dv_{\theta}^{\ell}/d\ell$ plausibly exceeds $do_{\theta}^{\ell}/d\ell$. Thus, if neither of these two marginal spillovers is very sensitive to ℓ, their weighted average $\ell\frac{dv_{\theta}^{\ell}}{d\ell} + (1-\ell)\frac{do_{\theta}^{\ell}}{d\ell}$ should also be increasing in ℓ. If so, the marginal social benefit $dw_{\theta}^{\ell}/d\ell$ of a higher cumulative investment rate is increasing in ℓ.

optimal: the efficient period-2 investment rate ℓ_2 is either zero or $1-\ell_1$. Realized welfare in period 2 is therefore $w_{2,\theta}^{\ell_1}$ or $w_{2,\theta}^{1}$, whichever is greater.

We now turn to period 1. We assume threshold behavior: each agent invests in period 1 if and only if her signal x lies below some common threshold k_1. Realized whole-game welfare at any given state θ thus equals realized welfare $w_{1,\theta}^{\ell_1}$ in period 1 plus realized welfare $\max\left\{w_{2,\theta}^{\ell_1}, w_{2,\theta}^{1}\right\}$ in period 2. Ex-ante social welfare $SW\left(k_1\right)$ given the threshold k_1 is simply the expectation of realized welfare over all states θ where, by the law of large numbers, the period-1 investment rate ℓ_1 at θ equals the probability $F\left(\frac{k_1-\theta}{\sigma}\right)$ that an agent's signal falls below k_1 conditional on θ:

$$SW\left(k_1\right)=\int_{\theta=0}^{1}\left[w_{1,\theta}^{\ell_1}+\max\left\{w_{2,\theta}^{\ell_1}, w_{2,\theta}^{1}\right\}\right]_{\ell_1=F\left(\frac{k_1-\theta}{\sigma}\right)} d\theta. \tag{45}$$

Suppose the state is θ and the period-1 investment rate is ℓ_1. Then in period 2, at state θ, the *residual social benefit* of the remaining agents investing, per remaining agent who invests, is denoted

$$s_{2,\theta}^{\ell_1}=\begin{cases}\frac{w_{t,\theta}^{1}-w_{t,\theta}^{\ell_1}}{1-\ell_1}=\frac{1}{1-\ell_1}\int_{\ell=\ell_1}^{1} m_{2,\theta}^{\ell} d\ell & \text{if} \quad \ell_1<1 \\ \lim_{\ell_1'\uparrow 1} s_{2,\theta}^{\ell_1'}=m_{2,\theta}^{1} & \text{if} \quad \ell_1=1\end{cases}. \tag{46}$$

MMSB implies several useful properties of the residual social benefit $s_{2,\theta}^{\ell_1}$.

Claim 7 *Assume MMSB. Then:*

1. *The residual social benefit $s_{2,\theta}^{\ell_1}$ is increasing in ℓ_1, decreasing in θ, and differentiable in both arguments.*

2. *It is always (never) efficient for all remaining agents to invest in sufficiently low (high) states: for all ℓ_1, $s_{2,\theta}^{\ell_1}$ is positive for all $\theta\leq 2\sigma$ and negative for all $\theta\geq 1-2\sigma$.*

3. *For any period-1 threshold k_1 in $[0,1]$, $s_{2,\theta}^{F\left(\frac{k_1-\theta}{\sigma}\right)}$ has a unique root $\theta=k_2^*\left(k_1\right)$, which lies in $(2\sigma, 1-2\sigma)$. For any given k_1, it is (not) efficient for the remaining agents to invest if $\theta<(>)\, k_2^*\left(k_1\right)$. The function $k_2^*\left(k_1\right)$ (resp., $k_1-k_2^*\left(k_1\right)$) is a differentiable, nondecreasing (resp., increasing) function of the period-1 threshold k_1.*

Proof. See p. 35 below. ■

We can now characterize the efficient period-1 threshold k_1^*. Let $\ell_1^*(k_1) = F\left(\frac{k_1 - k_2^*(k_1)}{\sigma}\right)$ denote the period-1 investment rate that occurs when the state θ exactly equals the threshold $k_2^*(k_1)$ at which it is equally efficient for all or none of the remaining agents to invest in period 2.

Claim 8 *The marginal social benefit $SW'(k_1)$ of raising the first-period threshold k_1 is positive (negative) for $k_1 \leq 3\sigma/2$ (resp., for $k_1 \geq 1 - 3\sigma/2$). For k_1 in $[3\sigma/2, 1 - 3\sigma/2]$, the marginal social benefit $SW'(k_1)$ equals*

$$s_{k_1}^{\sigma} \stackrel{d}{=} \int_{\ell_1=0}^{1} m_{1,k_1-\sigma F^{-1}(\ell_1)}^{\ell_1} d\ell_1 + \int_{\ell_1=0}^{\ell_1^*(k_1)} m_{2,k_1-\sigma F^{-1}(\ell_1)}^{\ell_1} d\ell_1 \tag{47}$$

which is decreasing and continuous in k_1.

Proof. See p. 36 below. ■

Claim 8 yields the following immediate characterization of the social optimum.

Corollary 1 *The efficient period-1 threshold k_1^* is uniquely given by $s_{k_1^*}^{\sigma} = 0$ and lies in $[3\sigma/2, 1 - 3\sigma/2]$. The efficient period-2 threshold is $k_2^*(k_1^*)$.*

An intuition for (47) is as follows. A small increase in k_1 only affects an agent's investment choice if her period-1 signal equals k_1. And conditional on an agent's signal equalling k_1, the proportion ℓ_1 who invest in period 1 is uniform on $[0, 1]$ and the state is given by $\theta = k - \sigma F^{-1}(\ell_1)$ as explained on p. 3 of this document. Thus, the effect on social welfare of a unit increase in k_1 equals the expected effect on realized welfare $w_{1,\theta}^{\ell_1} + \max\left\{w_{2,\theta}^{\ell_1}, w_{2,\theta}^{1}\right\}$ of a unit increase in the period-1 investment rate ℓ_1 under the assumption that ℓ_1 is uniform on $[0, 1]$ and, given ℓ_1, the state θ is $k - \sigma F^{-1}(\ell_1)$. Clearly, the first integral in (47) is the effect that occurs via the change in the first term $w_{1,\theta}^{\ell_1}$. The second integral is the effect that occurs via the change in the second term $\max\left\{w_{2,\theta}^{\ell_1}, w_{2,\theta}^{1}\right\}$. Its region of integration is $\ell_1 \in [0, \ell_1^*(k_1)]$ since an increase in the investment rate ℓ_1 affects the period-2 realized surplus only when this rate is less than $\ell_1^*(k_1)$, so that it is efficient for the remaining agents not to invest.

We now produce a subsidy scheme that yields the efficient outcome given in Corollary 1. Such a scheme consists of two nonnegative functions, τ_1 and τ_2, one for each period. Given the state θ and the proportion ℓ_t who invest in each period $t = 1, 2$, an agent receives the transfer $\tau_{1,\theta}^{\ell_1,\ell_2}$ if she invests in period 1 and $\tau_{2,\theta}^{\ell_1,\ell_2}$ if she invests in period 2. Thus, her realized payoff as a function of the timing of her investment is as follows.

$$\text{An agent gets} \begin{cases} v_{1,\theta}^{\ell_1} + v_{2,\theta}^{\ell_1+\ell_2} + \tau_{1,\theta}^{\ell_1,\ell_2} - p_1 & \text{if she invests in period 1} \\ o_{1,\theta}^{\ell_1} + v_{2,\theta}^{\ell_1+\ell_2} + \tau_{2,\theta}^{\ell_1,\ell_2} - p_2 & \text{if she invests in period 2} \\ o_{1,\theta}^{\ell_1} + o_{2,\theta}^{\ell_1+\ell_2} & \text{if she never invests} \end{cases} \tag{48}$$

We solve the model backwards. Consider the period-2 choice of an agent who did not invest in period 1, when ℓ_1 agents did so invest and the state is θ. By (48), her augmented relative payoff from investing in period 2 vs. never investing is $\widetilde{r}_{2,\theta}^{\ell_1+\ell_2} \stackrel{d}{=} r_{2,\theta}^{\ell_1+\ell_2} + \tau_{2,\theta}^{\ell_1,\ell_2} - p_2$. Let $T_{2,\theta}^{\ell_1} = \frac{1}{1-\ell_1}\int_{\ell_2=0}^{1-\ell_1} \tau_{2,\theta}^{\ell_1,\ell_2} d\ell_2$ denote the mean period-2 transfer over all possible measures $\ell_2 \in [0, 1-\ell_1]$ of period-2 investors. Let $R_{2,\theta}^{\ell_1}$ denote the analogous mean $\frac{1}{1-\ell_1}\int_{\ell_2=0}^{1-\ell_1} r_{2,\theta}^{\ell_1+\ell_2} d\ell_2$ of primitive relative payoffs $r_{2,\theta}^{\ell_1+\ell_2} = v_{2,\theta}^{\ell_1+\ell_2} - o_{2,\theta}^{\ell_1+\ell_2}$. Since we assume agents use the global games criterion, they invest whenever the price p_2 does not exceed the mean

$$\widetilde{R}_{2,\theta}^{\ell_1} = \frac{1}{1-\ell_1}\int_{\ell_2=0}^{1-\ell_1} \widetilde{r}_{2,\theta}^{\ell_1+\ell_2} d\ell_2 = R_{2,\theta}^{\ell_1} + T_{2,\theta}^{\ell_1} \tag{49}$$

of the augmented relative payoff $\widetilde{r}_{2,\theta}^{\ell_1+\ell_2}$ over all possible measures $\ell_2 \in [0, 1-\ell_1]$ of period-2 investors. As it knows the state θ, the firm can perfectly predict the agents' augmented willingness to pay, $\widetilde{R}_{2,\theta}^{\ell_1}$. Thus, as its costs are zero, it will charge the price $p_2 = \max\left\{0, \widetilde{R}_{2,\theta}^{\ell_1}\right\}$ and the agents will invest if and only if their augmented willingness to pay, $\widetilde{R}_{2,\theta}^{\ell_1}$, is positive.

On the other hand, it is efficient for the remaining agents to invest if and only if the residual social benefit $s_{2,\theta}^{\ell_1}$ is positive. This suggests a simple way to implement the efficient outcome in period 2. Let

$$e_{2,\theta}^{\ell} = \ell \frac{\partial v_{2,\theta}^{\ell}}{\partial \ell} + (1-\ell)\frac{\partial o_{2,\theta}^{\ell}}{\partial \ell} \tag{50}$$

denote the marginal period-2 investment externality when a total proportion ℓ invest (in any period), and let the mean remaining externality $E_{2,\theta}^{\ell_1} = \frac{1}{1-\ell_1}\int_{\ell_2=0}^{1-\ell_1} e_{2,\theta}^{\ell_1+\ell_2} d\ell_2$ denote the mean of the marginal externality over all proportions of remaining agents who invest. In order for the scheme in period 2 to rely only on subsidies, we assume the following property.

Nondecreasing Marginal Externality in Period 2 (NME2). The period-2 marginal externality $e_{2,\theta}^{\ell}$ is nondecreasing in the investment rate ℓ.

As in equation (4), we also assume that investing in the firm displays stronger spillovers in period 2 than the outside option:

$$v_{2,\theta}^{1} - \int_{\ell=0}^{1} v_{2,\theta}^{\ell} d\ell > o_{2,\theta}^{0} - \int_{\ell=0}^{1} o_{2,\theta}^{\ell} d\ell. \tag{51}$$

Claim 9 *Assume NME2 and (51). Then the mean remaining externality $E_{2,\theta}^{\ell_1}$ is nonnegative for any period-1 investment rate and state:*

$$E_{2,\theta}^{\ell_1} \geq 0 \text{ for all } \ell_1, \theta \in [0,1]. \tag{52}$$

Proof. Equation (51) immediately implies that the mean relative payoff $R_{2,\theta}^{0}$ in period 2 is less than the social benefit $v_{2,\theta}^{1} - o_{2,\theta}^{0} = s_{2,\theta}^{0}$ of all agents investing in period 2. Hence, by (53), the mean externality $E_{2,\theta}^{0} = s_{2,\theta}^{0} - R_{2,\theta}^{0}$ is positive when no agents invest in period 1. It follows from NME2 that the mean externality $E_{2,\theta}^{\ell_1}$ is nondecreasing in the period-1 investment rate ℓ_1, which implies (52). ■

Since the firm knows the state, in order to eliminate the period-2 inefficiency it suffices to equate the agents' willingness to pay $\widetilde{R}_{2,\theta}^{\ell_1}$ at any state θ to the *residual social benefit*[65]

$$s_{2,\theta}^{\ell_1} = \frac{1}{1-\ell_1} \int_{\ell=\ell_1}^{1} \frac{\partial w_{2,\theta}^{\ell}}{\partial \ell} d\ell = R_{2,\theta}^{\ell_1} + E_{2,\theta}^{\ell_1}. \tag{53}$$

The least-cost way to accomplish this is with a floor-based PSS. In particular, the transfer $\tau_{2,\theta}^{\ell_1,\ell_2}$ to period-2 investors is such that $\widetilde{r}_{2,\theta}^{\ell_1+\ell_2} = \max\left\{r_{2,\theta}^{\ell_1+\ell_2}, \kappa\right\}$ where κ is set so that agent demand equals the residual social benefit:

$$\widetilde{R}_{2,\theta}^{\ell_1} = \frac{1}{1-\ell_1} \int_{\ell_2=0}^{1-\ell_1} \max\left\{r_{2,\theta}^{\ell_1+\ell_2}, \kappa\right\} d\ell_2 = s_{2,\theta}^{\ell_1}. \tag{54}$$

By (54), the period-2 outcome is efficient for any state θ and period-1 investment rate ℓ_1. Also, $\widetilde{r}_{2,\theta}^{1} = \max\left\{r_{2,\theta}^{1}, s_{2,\theta}^{\ell_1}\right\}$ and hence[66]

$$\tau_{2,\theta}^{\ell_1,1-\ell_1} = \max\left\{0, s_{2,\theta}^{\ell_1} - r_{2,\theta}^{1}\right\} \tag{55}$$

[65] If the firm did not know the state, such a scheme would be inefficient as the firm would have an incentive to forego sales in some states in order to obtain a higher price in others.

[66] There are two cases. If $\kappa \leq r_{2,\theta}^{1}$ (whence $\widetilde{r}_{2,\theta}^{1} = \max\left\{r_{2,\theta}^{1}, \kappa\right\} = r_{2,\theta}^{1}$), then $s_{2,\theta}^{\ell_1} = \widetilde{R}_{2,\theta}^{\ell_1} \leq r_{2,\theta}^{1}$. And if $\kappa > r_{2,\theta}^{1}$ (whence $\widetilde{r}_{2,\theta}^{1} = \max\left\{r_{2,\theta}^{1}, \kappa\right\} = \kappa$) then $s_{2,\theta}^{\ell_1} = \widetilde{R}_{2,\theta}^{\ell_1} = \kappa > r_{2,\theta}^{1}$.

which is nonnegative as required.

Having corrected the externalities in period 2, we now turn to period 1. We first compute an agent's realized payoff from investing and not in period 1, given the state θ and the proportion ℓ_1 who invest. As noted above, the period-2 insurance scheme equates the agents' willingness to pay $\widetilde{R}^{\ell_1}_{2,\theta}$ to the residual social benefit $s^{\ell_1}_{2,\theta}$ of the remaining agents' investing. Hence there are two cases. If $s^{\ell_1}_{2,\theta} < 0$, the agent will not invest in period 2 if she does not in period 1 and neither will the other agents: $\ell_2 = 0$. The agent thus gets $v^{\ell_1}_{1,\theta} + v^{\ell_1}_{2,\theta} + \tau^{\ell_1,0}_{1,\theta} - p_1$ if she invests in period 1 and $o^{\ell_1}_{1,\theta} + o^{\ell_1}_{2,\theta}$ if she does not. On the other hand, if $s^{\ell_1}_{2,\theta} > 0$ then the agent will invest in period 2 if she does not invest in period 1 and all others will do so as well: $\ell_2 = 1 - \ell_1$. Hence, she gets $v^{\ell_1}_{1,\theta} + v^{1}_{2,\theta} + \tau^{\ell_1,1-\ell_1}_{1,\theta} - p_1$ if she invests in period 1 and

$$o^{\ell_1}_{1,\theta} + v^{1}_{2,\theta} + \tau^{\ell_1,1-\ell_1}_{2,\theta} - \widetilde{R}^{\ell_1}_{2,\theta} = o^{\ell_1}_{1,\theta} + v^{1}_{2,\theta} - \min\left\{s^{\ell_1}_{2,\theta}, r^{1}_{2,\theta}\right\}$$

if she does not (where the equality holds as the period-2 price $p_2 = \widetilde{R}^{\ell_1}_{2,\theta}$ equals $s^{\ell_1}_{2,\theta}$ and, by (55), the subsidy $\tau^{\ell_1,1-\ell_1}_{2,\theta}$ equals $\max\left\{0, s^{\ell_1}_{2,\theta} - r^{1}_{2,\theta}\right\}$). Combining the two cases, the agent's whole-game augmented relative payoff from investing in period 1, gross of the price p_1, is

$$\widetilde{r}^{\ell_1}_{\theta} = r^{\ell_1}_{\theta} + \tau^{\ell_1}_{\theta} \tag{56}$$

which is the sum of the whole-game primitive relative payoff

$$r^{\ell_1}_{\theta} = \begin{cases} r^{\ell_1}_{1,\theta} + r^{\ell_1}_{2,\theta} & \text{if} \quad s^{\ell_1}_{2,\theta} < 0 \\ r^{\ell_1}_{1,\theta} + \min\left\{s^{\ell_1}_{2,\theta}, r^{1}_{2,\theta}\right\} & \text{if} \quad s^{\ell_1}_{2,\theta} > 0 \end{cases} \tag{57}$$

plus the period-1 transfer,

$$\tau^{\ell_1}_{\theta} = \begin{cases} \tau^{\ell_1,0}_{1,\theta} & \text{if} \quad s^{\ell_1}_{2,\theta} < 0 \\ \tau^{\ell_1,1-\ell_1}_{1,\theta} & \text{if} \quad s^{\ell_1}_{2,\theta} > 0 \end{cases}$$

For period 1, as in section 7, we restrict to equilibria in which each agent invests if and only if her signal is less than a common threshold k_1. Under this belief, the measure ℓ_1 who invest in period 1 at the state θ is $F\left(\frac{k_1-\theta}{\sigma}\right)$ and hence, at the state θ, the remaining agents' willingness to pay will be $s^{F\left(\frac{k_1-\theta}{\sigma}\right)}_{2,\theta}$. The remaining agents thus invest whenever the state is less than the unique threshold $k^*_2(k_1)$ defined in part 3 of Claim 7.

Let $R^{\sigma}_{\theta} = \int_{\ell_1=0}^{1} r^{\ell_1}_{\theta} d\ell_1$ denote the mean whole-game primitive relative payoff. The firm's whole-game marginal revenue at the period-1 threshold k_1 is then defined as $m^{\sigma}_{k_1} \stackrel{d}{=}$

$\frac{d}{dG(k_1)}\left[R^{\sigma}_{k_1}G(k_1)\right]$, exactly as in section 7 above. We will assume the following property, which is analogous to the property DLMR that is defined in that section.

DWMR. Decreasing Whole-Game Marginal Revenue. Whole-game marginal revenue $m^{\sigma}_{k_1}$ is continuous and decreasing in the period-1 threshold k_1, and satisfies $m^{\sigma}_{\sigma/2} > 0$ and $m^{\sigma}_{1-\sigma/2} < 0$.

Moreover, by analogy to (40) and (39), we assume that

$$r^1_{k_1-\sigma/2} > s^{\sigma}_{k_1} > R^{\sigma}_{k_1} > m^{\sigma}_{k_1} \tag{58}$$

where $s^{\sigma}_{k_1}$, defined in (47), is the marginal social benefit of raising the period-1 threshold. We also assume

$$r^1_{\widehat{k}_1-\sigma/2} < \overline{p} \tag{59}$$

where $\widehat{k}_1$ is the firm's optimal period-1 threshold in the absence of intervention.

We can now rely on the results of section 7 to find the minimum-cost efficient period-1 transfer scheme $\tau^{\ell_1}_{\theta}$. First we verify that the assumptions of Theorem 5 are satisfied:

Claim 10 *The function $r^{\ell_1}_{\theta}$ defined in (57) satisfies AM, WSM, DR$_2$, and DLMR, as well as equations (40) and (39) with k and $\widehat{k}$ in those equations replaced by k_1 and $\widehat{k}_1$, respectively. The whole-game marginal social benefit $s^{\sigma}_{k_1}$ satisfies DLMB with k in that property replaced by k_1.*

Proof. See p. 37 below. ■

The definition of PSS from section 7 then extends to the present model *mutatis mutandum*, where τ in that definition is now interpreted as the transfer $\tau^{\ell_1}_{\theta}$ that period-1 investors receive when a proportion ℓ_1 invest in period 1 and the state is θ. In particular, we define an efficient period-1 PSS to be one that induces the agents to choose the efficient period-1 threshold k^*_1, which is characterized above in Corollary 1. (By assumption, an efficient policy is then used in period 2 so the full outcome is efficient.) The definition of a *floor-based* PSS also extends to the present model.[67] With a floor-based PSS with floor κ, the augmented willingness to pay $\widetilde{R}^{\sigma}_{k^*_1}$ of an agent who sees the signal k^*_1 equals $\int_{\ell=0}^{1} \max\left\{r^{\ell}_{k^*_1-\sigma F^{-1}(\ell)}, \kappa\right\} d\ell$,

[67] In a floor-based PSS in the present model, an agent whose signal equals the efficient period-1 threshold threshold k^*_1 receives a subsidy equal to $\kappa^* - r^{\ell}_{k^*_1-\sigma F^{-1}(\ell_1)}$ if this is positive and zero otherwise, where κ is the floor. See p. 8 of this document.

which we denote as $\zeta_{k_1^*}(\kappa)$. By analogy to (41), let p_m^1 denote the minimum price the firm must receive in period 1 so as to be willing to choose the efficient threshold k_1^* when it anticipates the period-2 demand in (54). Finally, let $r_{\theta-}^1$ denote the left-limit $\lim_{\theta' \downarrow \theta} r_{\theta'}^1$.

Claim 10 shows that whole-game payoffs of the learning game satisfy the assumptions of the one-shot, large-noise game of section 7. Hence, we can exactly implement the first-best outcome in the learning game using the techniques of that section. Formally:

Theorem 7 *In the above learning game:*

1. *There always exists an efficient period-1 PSS* $\left(\tau_\theta^{\ell_1}\right)_{(\ell_1,\theta)\in[0,1]^2}$.
2. *Assume LCED. Then a floor-based PSS with floor* $\kappa^* = \zeta_1^{-1}(p_m^1)$ *is a least-cost efficient PSS. It has the following property.*
   (a) *If* $r^1_{\left(k_1^*-\sigma/2\right)-} \geq p_m^1$*, no subsidy is ever paid in period 1 when all agents invest:* τ_θ^1 *is identically zero.*
   (b) *If* $r^1_{\left(k_1^*-\sigma/2\right)-} < p_m^1$*, full insurance plus a subsidy is given when agents miscoordinate. Moreover, a positive subsidy* τ_θ^1 *is paid when all agents invest in period 1 for states* θ *in a nonempty interval* $[\theta_3, k_1^* - \sigma/2]$.

Proof. Follows immediately from Claim 10 and Theorem 5. ■

9 Duopoly Competition

In this section we formally study the duopoly model that is discussed in Frankel (2017, section 3.7). The model is constructed from our base model (Frankel 2017, section 2) by assuming that the outside option consists of investing in a competing firm. Let us refer to the original firm as firm 1 and to the outside option as firm 2. The firms have the same marginal cost which, as above, is normalized to zero. Firms 1 and 2 first simultaneously set their prices p_1 and p_2, respectively. The agents then see their signals and choose in which firm to invest.[68] If a proportion ℓ invest in firm 1 at state θ, investors in firm 1 (resp., 2) get the payoffs $v_\theta^\ell - p_1$ (resp., $o_\theta^\ell - p_2$). Thus, the relative payoff from investing in firm 1 vs. firm 2 is $r_\theta^\ell - (p_1 - p_2)$ rather than, in the monopoly case, $r_\theta^\ell - p$: from the agents' point of view, the price differential $p_1 - p_2$ now plays the role of the monopoly price p.

[68] We assume an agent must use one platform or the other.

Now assume r satisfies the primitive assumptions AM, PMC, and SM. Since each firm's price lies in $[0, \overline{p}]$, the price differential $p_1 - p_2$ lies in the compact set $[-\overline{p}, \overline{p}]$. Thus, by Claim 1 and Theorem 2 in Frankel (2017), in the limit as the private noise scale factor σ shrinks, each agent invests in firm 1 if her signal is less than the threshold $k = \theta_R^{p_1 - p_2}$; else she invests in firm 2. By (15) and the continuity of the mean payoff function R (which is implied by SM), the threshold k is uniquely defined by $R_k = p_1 - p_2$. Thus, by SM, k is a monotonic function of both p_1 and p_2: it is jointly controlled by the two firms. In equilibrium, they must prefer the same threshold k.

Because the threshold k must be optimal for both firms, the APSS of the base model will not implement the social optimum. For suppose a scheme is in place that yields the augmented demand curve AIMOQ in Figure 2 of Frankel (2017). Suppose the firms choose the efficient point, M. Firm 2's investment probability is $1 - \Phi(\theta)$, which equals the horizontal distance between point M and the right boundary of the box. By the preceding discussion, the height of the augmented demand curve $\widetilde{R}$ equals the difference in prices $p_1 - p_2$. Suppose now that firm 2 lowers its price slightly. The outcome will move from point M to point I. This raises firm 2's investment probability by the length of segment IM. As long as p_2 was initially positive, firm 2's profits will rise. Since firm 2 has a profitable deviation, point M is not an equilibrium under the given scheme.

In the remainder of this section we construct an alternative scheme, which is asymptotically efficient and revenue-neutral. The scheme consists of a simple state-dependent "miscoordination tax" on each firm's agents. That is, in order to raise an agent's relative willingness to invest in, say, firm 1 at a given state, we impose a tax on agents who choose firm 2. In order to preserve Action Monotonicity, this tax is decreasing in the proportion of agents who choose firm 2.[69]

We first solve for the laissez-faire equilibrium. We begin with an imaginary world in which each firm maximizes, not its actual payoff, but the limit of its payoffs as the noise scale factor σ shrinks to zero. We give sufficient conditions for the existence of a unique equilibrium (p_1^*, p_2^*) of this (imaginary) limit game. For any $\sigma > 0$, let (p_1^σ, p_2^σ) be an equilibrium of the actual game with noise scale factor σ. Let $(\sigma_n)_{n=1}^\infty$ be any decreasing sequence that converges to zero. We then show that the chosen equilibria $(p_1^{\sigma_n}, p_2^{\sigma_n})$ of the actual game must converge to the unique equilibrium (p_1^*, p_2^*) of the limit game.

Letting $k = \theta_R^{p_1 - p_2}$, firm 1's payoff in the limit game is $p_1 \Phi(k)$ which equals $(R_k + p_2) \Phi(k)$

[69] While one may also attain the first-best while relying only on subsidies, this is only so for some parameters and the schemes are more complex. In contrast, there is always a simple and efficient tax-based scheme.

(which we denote $\Pi^1(k, p_2)$) by (25). Its first order condition for an optimal probability $\Phi(k)$ of investment can thus be written

$$0 = \frac{\partial \Pi^1(k, p_2)}{\partial \Phi(k)} = \mu_k^1 + p_2 \tag{60}$$

where μ_k^1 denotes firm 1's marginal profit $\partial \Pi^1(k, 0) / \partial \Phi(k) = R_k + \Phi(k) R_k' / \Phi'(k)$ from raising its investment probability $\Phi(k)$ when the other firm's price p_2 is zero. Condition (60) suffices for an optimum for firm 1 as long as the marginal profit function μ_k^1 is decreasing in the investment probability $\Phi(k)$ or, equivalently, in the threshold k, which we assume.

Likewise, firm 2 gets $p_2[1 - \Phi(k)]$ which equals $(p_1 - R_k)[1 - \Phi(k)]$ (which we denote $\Pi^2(k, p_1)$) by (25). Firm 2's first order condition for an optimal probability $1 - \Phi(k)$ of investment can thus be written

$$0 = \frac{\partial \Pi^2(k, p_1)}{\partial [1 - \Phi(k)]} = \mu_k^2 + p_1 \tag{61}$$

where μ_k^2 denotes firm 2's marginal profit $\partial \Pi^2(k, 0) / \partial [1 - \Phi(k)] = -R_k + [1 - \Phi(k)] R_k' / \Phi'(k)$ from raising its investment probability $1 - \Phi(k)$ when the other firm's price p_1 is zero. Condition (61) suffices for an optimum for firm 2 as long as the firm's marginal profit function μ_k^2 is decreasing in its probability $1 - \Phi(k)$ of investment or, equivalently, increasing in the threshold k, which we assume. Summarizing, the second order conditions we assume are:

Duopoly Second Order Conditions (DOSC). Both $\mu_k^1 = R_k + \Phi(k) R_k' / \Phi'(k)$ and $-\mu_k^2 = R_k - [1 - \Phi(k)] R_k' / \Phi'(k)$ are continuous and decreasing in the threshold k, and satisfy $\lim_{k \to -\infty} \mu_k^1 = \lim_{k \to -\infty} (-\mu_k^2) = \infty$ and $\lim_{k \to \infty} \mu_k^1 = \lim_{k \to \infty} (-\mu_k^2) = -\infty$.

This condition is mild since R_k is decreasing by MSM and $\Phi(k)$ is increasing.

Now, the price difference $p_1 - p_2$ equals R_k by (25) and also equals $\mu_k^1 - \mu_k^2$ by (60) and (61). Hence, k must be a root of the function

$$M_k \stackrel{d}{=} \mu_k^1 - \mu_k^2 - R_k = R_k + [2\Phi(k) - 1] R_k' / \Phi'(k) . \tag{62}$$

The following property, which we assume, ensures that M has a unique root:

Unique Duopoly Equilibrium (UDE). The function $M_k = R_k + [2\Phi(k) - 1] R_k' / \Phi'(k)$ is continuous and decreasing in the threshold k, and satisfies $\lim_{k \to -\infty} M_k = \infty$ and $\lim_{k \to \infty} M_k = -\infty$.

Part 1 of the following theorem states that the limit game has a unique equilibrium (p_1^*, p_2^*). Part 2 states that any equilibrium of the game with positive noise must be close to this unique equilibrium when the agents' signal errors are small.

Theorem 8 *Assume r satisfies AM, PMC, SM, and that r and Φ jointly satisfy DOSC and UDE. Let $\widehat{\theta}$ be the unique (by UDE) root of the function M defined in (62). Then*

1. *the limit game has a unique equilibrium (p_1^*, p_2^*), which is given by $p_1^* = -\mu^2_{\widehat{\theta}}$ and $p_2^* = -\mu^1_{\widehat{\theta}}$;*
2. *for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any equilibrium (p_1^σ, p_2^σ) of the duopoly game with any private signal scale factor σ in $(0, \delta)$:*
   - *(a) the absolute gaps $|p_1^\sigma - p_1^*|$ and $|p_2^\sigma - p_2^*|$ in prices are each less than ε, and*
   - *(b) each agent invests in firm 1 if her signal is less than $\widehat{\theta} - \varepsilon$ and in firm 2 if her signal exceeds $\widehat{\theta} + \varepsilon$.*

Proof. See p. 38 below. ■

We now give sufficient conditions for agents to invest in a given firm inefficiently often. The marginal social benefit of the agents switching en masse from firm 2 to firm 1 at the state θ is just s_θ, defined in (2). Hence, by DMSB, there is a unique threshold θ^*, given by $s_{\theta^*} = 0$, such that it is efficient for the agents to choose firm 1 (resp., firm 2) when the state is below (above) θ^*.

Claim 11 *Assume r satisfies AM, PMC, SM, that r and Φ jointly satisfy DOSC and UDE, and that s satisfies DMSB. Let $\widehat{\theta}$ be the unique equilibrium threshold identified in Theorem 8, and let θ^* be the efficient threshold given by $s_{\theta^*} = 0$.*

1. *If (4) holds and the equilibrium proportion $\Phi\left(\widehat{\theta}\right)$ who invest in firm 1 is at least one half, then firm 2 is chosen too often: $\widehat{\theta} < \theta^*$.*
2. *If the opposite of (4) holds and $\Phi\left(\widehat{\theta}\right)$ is at most one half, then the agents invest in firm 1 too often: $\widehat{\theta} > \theta^*$.*

Proof. By Theorem 8, the limiting equilibrium threshold satisfies $M_{\widehat{\theta}} = 0$. For part 1, assume (4) holds: firm 1 displays stronger spillovers than firm 2.[70] Then, as noted in the discussion of that inequality, $s > R$. By (62), for $\widehat{\theta} < \theta^*$ it then suffices to assume that $\left[2\Phi\left(\widehat{\theta}\right) - 1\right] R'_{\widehat{\theta}}/\Phi'\left(\widehat{\theta}\right)$ is nonpositive which, by SM, holds if the equilibrium measure $\Phi\left(\widehat{\theta}\right)$ who invest in firm 1 is at least 1/2. Part 2 is analogous. ■

An intuition is as follows. First consider the competitive outcome: when firm 1 (resp., 2) believes it can raise its investment probability $\Phi(k)$ (resp., $1 - \Phi(k)$) without affecting prices. Firm 1 (2) will then always want to raise this probability unless its price were already equal to zero. In equilibrium, both prices - and thus, by (25), the mean relative payoff R - must equal zero. The competitive threshold k^C is thus given by $R_{k^C} = 0$. Assuming (4), s_{k^C} must then exceed zero: welfare would rise if, in a neighborhood of the threshold k^C, all agents were to invest in firm 1 rather than firm 2. Intuitively, (4) implies that positive externalities are stronger among firm 1's investors. Under competition there is no other distortion, so the agents must invest in firm 1 too infrequently in equilibrium: the competitive threshold k^C lies below the efficient threshold θ^*.

Under duopoly behavior, there is an additional distortion that can alter this conclusion. Suppose, taking firm 2's price p_2 as given, firm 1 raises its investment probability $\Phi(k)$ by one small unit. As in the competitive case, the increase in this probability raises its expected revenue by p_1. But now, by (25), firm 1 realizes that it can accomplish this only by changing its price p_1 by $R'_k/\Phi'(k)$, which is negative by SM. This decrease lowers firm 1's expected revenue by $\Phi(k) R'_k/\Phi'(k)$, which lowers 1's incentive to make the change. Similarly, if - taking firm 1's price as given - firm 2 raises its investment probability $1 - \Phi(k)$ by one small unit, this increase raises its expected revenue by p_2 as in the competitive case. But firm 2 now realizes that, in order to attain its goal, it must change its price by $R'_k/\Phi'(k)$, which is negative: the decrease lowers the firm's expected revenue by $[1 - \Phi(k)] R'_k/\Phi'(k)$, which lowers 2's incentive to make the change.

While the price change $R'_k/\Phi'(k)$ is the same for the two firms, the payoff penalty is not: it is greater for the firm that has a higher investment probability, as the penalty equals the price change multiplied by this probability. Thus, firm 1 faces a larger payoff penalty than firm 2 when firm 1's investment probability $\Phi(k)$ at the threshold k exceeds one half. If this holds at the competitive threshold k^C, then price-setting behavior pushes the threshold down relative to the competitive one: $\widehat{\theta}$ is less than k^C which, we already saw, is less than

[70] If the opposite inequality holds, simply swap the firms' indices so that (4) holds.

the efficient threshold θ^*. Thus, agents invest in firm 1 too infrequently in equilibrium: $\widehat{\theta} < \theta^*$. If instead $\Phi(k)$ is less than one half, the price penalty is larger for firm 2 than firm 1, so the laissez-faire duopoly threshold exceeds the competitive threshold: without further information, we cannot be sure that $\widehat{\theta}$ is less than θ^*.

In the small-noise limit, the welfare analysis of our base model (Frankel 2017, section 2) applies unchanged to the duopoly case: it is efficient for the agents to invest in firm 1 (resp., 2) if the state θ is less (greater) than the socially optimal threshold θ^* defined by $s_{\theta^*} = 0$. We now consider a scheme that induces the firms to choose this efficient threshold. The idea is simple: the scheme changes the mean relative payoff by the constant $-M_{\theta^*}$. At any state θ, the augmented mean relative payoff $\widetilde{R}_\theta$ then equals $R_\theta - M_{\theta^*}$. As this does not affect the slope (i.e., $\widetilde{R}'_\theta = R'_\theta$ for any θ), the equilibrium condition becomes $M_k - M_{\theta^*} = 0$. By UDE, this equals zero uniquely at $k = \theta^*$: the outcome is efficient.

We focus on the natural generalization of an APRS (Frankel 2017, section 3.3): a scheme in which investors in either firm can be taxed or subsidized.[71] Such a scheme consists of a pair $\left(\tau_\theta^{1,\ell}, \tau_\theta^{2,\ell}\right)$ of transfer functions, where an agent who invests in firm $i = 1, 2$ receives the (possibly negative) transfer $\tau_\theta^{i,\ell}$. We permit schemes in the following class.

APRDS. An *Asymptotically Predictable, Revenue-Neutral Duopoly Scheme* (APRDS) is a pair $\left(\tau_\theta^{1,\ell}, \tau_\theta^{2,\ell}\right)$ of transfer functions with the following two properties.

1. Asymptotic Predictability. The augmented relative payoff function $\widetilde{r}_\theta^\ell = r_\theta^\ell + \tau_\theta^{1,\ell} - \tau_\theta^{2,\ell}$ satisfies the sufficient conditions for an asymptotically unique equilibrium in the agent subgame: AM, MSM, and OSL.
2. No Transfers Under Coordination: for all θ, $\tau_\theta^{1,1} = \tau_\theta^{2,0} = 0$.

Property 2 states that in the small-noise limit, if all agents invest in the same firm then no transfers are paid.

An efficient APRDS is as follows. If $M_{\theta^*} \geq 0$, the transfer $\tau_\theta^{2,\ell}$ to investors in firm 2 is identically zero. The transfer $\tau_\theta^{1,\ell}$ to investors in firm 1 is the nonpositive amount $-2(1-\ell) M_{\theta^*}$. Intuitively, we lower the agents' willingness to invest in firm 1 by imposing a tax on firm 1 investors that is increasing (actually, linear) in the proportion who do not invest in firm 1. Hence, it is zero if all invest in firm 1: $\tau_\theta^{1,1} = 0$. Thus, when $M_{\theta^*} \geq 0$ the

[71] One can also raise augmented demand by $-M_{\theta^*}$ with subsidies alone. However, such a scheme is asymptotically costless only if, at each state θ, augmented demand $\widetilde{R}_\theta = R_\theta - M_{\theta^*}$ lies between the minimum and maximum relative payoffs, r_θ^0 and r_θ^1. This does not hold in general.

scheme satisfies No Transfers under Coordination. Clearly, the integral $\int_{\ell=0}^{1}\left(\tau_{\theta}^{1,\ell}-\tau_{\theta}^{2,\ell}\right)d\ell$ of the relative transfer to firm 1's investors equals the desired increment $-M_{\theta^*}$, whence the augmented mean relative payoff $\widetilde{R}_\theta$ equals $R_\theta - M_{\theta^*}$ as desired. An analogous policy when $M_{\theta^*} < 0$ is described in the proof below. The formal result is as follows.

Theorem 9 *Assume r satisfies AM, PMC, SM, and that r and Φ jointly satisfy DOSC and UDE. Let θ^* be the efficient threshold, given by $s_{\theta^*} = 0$. Then there is an APRDS $\left(\tau_{\theta}^{1,\ell},\tau_{\theta}^{2,\ell}\right)$ that asymptotically implements the efficient outcome: in the limit as the noise scale factor σ shrinks to zero, the agents invest in firm 1 (resp., 2) whenever the state θ is less (greater) than θ^*.*

Proof. The case $M_{\theta^*} \geq 0$ is described in the text. When $M_{\theta^*} < 0$, the transfer $\tau_{\theta}^{1,\ell}$ to firm 1's investors is identically zero while the transfer $\tau_{\theta}^{2,\ell}$ to firm 2's investors users is the negative amount $2\ell M_{\theta^*}$. As the transfer to firm 2's investors is zero when all invest in firm 2 ($\tau_{\theta}^{2,0} = 0$), the scheme satisfies No Transfers Under Coordination when $M_{\theta^*} < 0$ as well. Again, $\int_{\ell=0}^{1}\left(\tau_{\theta}^{1,\ell}-\tau_{\theta}^{2,\ell}\right)d\ell$ equals $-M_{\theta^*}$, whence $\widetilde{R}_\theta$ equals $R_\theta - M_{\theta^*}$ as desired.

For any M_{θ^*}, the slopes $\widetilde{R}'_\theta$ and R'_θ are equal, so $\widetilde{R}$ and Φ satisfy DOSC and UDE as R and Φ do. When $M_{\theta^*} \geq 0$, $\widetilde{r}_\theta^\ell$ equals $r_\theta^\ell - 2(1-\ell)M_{\theta^*}$; when $M_{\theta^*} < 0$, it equals $r_\theta^\ell - 2\ell M_{\theta^*}$. Since r satisfies AM, $\widetilde{r}$ does as well (replacing the constant k_1 by $k_1 + 2|M_{\theta^*}|$). Since r satisfies SM, so does $\widetilde{r}$ as the transfers $\left(\tau_{\theta}^{1,\ell},\tau_{\theta}^{2,\ell}\right)$ do not vary with the state θ. Finally, for any state θ, $\widetilde{r}_\theta^1 - \widetilde{R}_\theta$ equals $r_\theta^\ell - R_\theta + |M_{\theta^*}|$ which exceeds k_2 since r satisfies PMC: $\widetilde{r}$ satisfies PMC. We have confirmed that $\widetilde{r}$ satisfies AM, PMC, and SM (whence by Claim 1 the scheme is Asymptotically Predictable and thus is an APRDS), and that $\widetilde{r}$ and Φ jointly satisfy DOSC and UDE. By Theorem 8, then, the efficient outcome θ^* is selected in the limit as the noise vanishes. ■

10 Proofs

Proof of Theorem 4. Since $0 = s_{\theta^*} < r_{\theta^*}^1$, there is a

$$\theta_0 \leq \underline{\theta}_r^{\overline{p}} \tag{63}$$

that is low enough that (22) holds. Let $\underline{\phi} > 0$ be a lower bound for $\phi(\theta)$ on $\theta \in [\theta_0, \theta^*]$. Now let

$$k_3' \in \left(0, \min\left\{k_3, \frac{R_{\underline{\theta}_r^{\overline{p}}} - r_{\theta^*}^1}{\theta^* - \underline{\theta}_r^{\overline{p}}}, \underline{\phi} r_{\theta^*}^1\right\}\right). \tag{64}$$

To show the interval is nonempty, it suffices to prove that

$$\frac{R_{\underline{\theta}_r^{\overline{p}}} - r^1_{\theta^*}}{\theta^* - \underline{\theta}_r^{\overline{p}}} > 0. \tag{65}$$

Now, $m_{\widehat{\theta}} = 0 = s_{\theta^*}$ by DMSB and DMR, so $\theta^* > \widehat{\theta}$ by (9). And

$$r^1_{\widehat{\theta}} < \overline{p} = r^0_{\underline{\theta}_r^{\overline{p}}} \leq r^1_{\underline{\theta}_r^{\overline{p}}} \tag{66}$$

by (8), (26) and SM(r), and AM(r), respectively; thus, $\widehat{\theta} > \underline{\theta}_r^{\overline{p}}$ by SM(r). It follows that $\theta^* > \widehat{\theta} > \underline{\theta}_r^{\overline{p}}$. Accordingly, the denominator in (65) is is positive. The numerator is positive as $r^1_{\theta^*} < r^1_{\widehat{\theta}} < \overline{p} = r^0_{\underline{\theta}_r^{\overline{p}}} \leq R_{\underline{\theta}_r^{\overline{p}}}$ which holds, respectively, by SM(r) since $\theta^* > \widehat{\theta}$ as shown above; equation (8); (66); and AM(r). Hence (65) holds as claimed.

By (64), $R^*_\theta = r^1_{\theta^*} + k_3'(\theta^* - \theta) < R_\theta$ when $\theta = \underline{\theta}_r^{\overline{p}}$, whence $\theta^* > \theta_1 > \underline{\theta}_r^{\overline{p}} \geq \theta_0$ by (21) and (63). Let the mean transfer T_θ equal $r^1_{\theta^*} - R_\theta + k_3'(\theta^* - \theta)$ on $\theta \in (\theta_0, \theta^*]$ and zero elsewhere. The transfer τ^ℓ_θ is given by (20) on $\theta \in [\theta_1, \theta^*]$ and by $\tau^\ell_\theta = 2(1-\ell)T_\theta < 0$ (a tax) on $\theta \in (\theta_0, \theta_1)$. One can verify that $\int_{\ell=0}^1 \tau^\ell_\theta d\ell$ always equals T_θ as required. Moreover, τ^1_θ is identically zero: the scheme satisfies No Transfers Under Coordination. The scheme also satisfies Asymptotic Predictability:

Lemma 9 *Assume AM(r), SM(r), and PMC(r). Then*

1. *for θ in $[\theta_1, \theta^*]$, the weight α_θ lies in $[0,1]$;*
2. *AM($\widetilde{r}$), MSM($\widetilde{r}$), and OSL($\widetilde{r}$) hold for all θ.*

Proof.

1. The weight α_θ is nonnegative as $T_\theta \geq 0$ and, by PMC(r), $r^1_\theta > R_\theta$. By SM(r), $r^1_\theta \geq r^1_{\theta^*} + k_3(\theta^* - \theta)$ for $\theta < \theta^*$, so since $k_3' < k_3$, $T_\theta + R_\theta = r^1_{\theta^*} + k_3'(\theta^* - \theta) \leq r^1_\theta$, whence $\alpha_\theta \leq 1$.

2. The function $\widetilde{R}$ is left-continuous by (23) and continuity of R^*. It falls at a rate between k_3 and k_4 at states $\theta \notin (\theta_0, \theta^*]$; jumps downwards at θ^* and at θ_0; and falls at the rate $k_3' \in (0, k_3)$ at states in (θ_0, θ^*). Hence, MSM($\widetilde{r}$) holds with k_3 replaced by k_3'. AM and OSL are proved for different intervals of θ as follows.

   (a) The intervals $(-\infty, \theta_0]$ and (θ^*, ∞). Here $\widetilde{r}$ coincides with r and thus satisfies AM and OSL by Claim 1.

(b) The interval $(\theta_0 - \varepsilon, \theta_1]$ for any small $\varepsilon > 0$. Here $T_\theta \leq 0$, so $\widetilde{r}^\ell_\theta = r^\ell_\theta + 2(1-\ell)T_\theta$ is nondecreasing in ℓ as r^ℓ_θ is. Also for such θ, $\widetilde{r}^1_\theta - \widetilde{r}^0_\theta = r^1_\theta - r^0_\theta - 2T_\theta \leq k'_1$: AM$(\widetilde{r})$ holds with k'_1 replacing k_1. Moreover, since τ^ℓ_θ and thus $\widetilde{r}^\ell_\theta$ jumps *downwards* at $\theta = \theta_0$, it suffices to check OSL$(\widetilde{r})$ for $\theta_1 \geq \theta' > \theta \geq \theta_0$, where $\left|\widetilde{r}^\ell_{\theta'} - \widetilde{r}^\ell_\theta\right| \leq \left|r^\ell_{\theta'} - r^\ell_\theta\right| + 2(1-\ell)\left|T_{\theta'} - T_\theta\right| \leq (3k_4 + 2k'_3)(\theta' - \theta)$ by the triangle inequality and MSM(r): $\widetilde{r}$ satisfies OSL with $k_5 = 3k_4 + 2k'_3$.

(c) The interval $[\theta_1, \theta^*]$. The proof is identical to that of part 3 of the proof of Lemma 8.

■

We have shown that τ is an APRS. By Lemma 9, Theorems 2 and 3 remain valid in the presence of τ, where R in those results is replaced by $\widetilde{R}_\theta = R_\theta + T_\theta$. Hence, by Theorem 3, in the limit as σ goes to zero the firm chooses an investment threshold θ that maximizes the augmented limiting payoff function $\Pi_{\widetilde{r}}(\theta) = \widetilde{R}^-_\theta \Phi(\theta)$ if there is a unique such maximizer.

Now, $\Pi_{\widetilde{r}}(\theta^*) = r^1_{\theta^*}\Phi(\theta^*)$ is positive by (3) and since $s_{\theta^*} = 0$ by DMSB. An upper bound on the firm's profit from a threshold $\theta \leq \theta_0$ is its maximum price $\overline{p}$ times the probability $\Phi(\theta_0)$ that the state does not exceed θ_0. By (22), then, no threshold $\theta \leq \theta_0$ maximizes $\Pi_{\widetilde{r}}$. As $\theta^* > \widehat{\theta}$, $m_\theta < 0$ for all $\theta \geq \theta^*$ by DMR and hence $\Pi_{\widetilde{r}}(\theta) < \Pi_{\widetilde{r}}(\theta^*)$ for all $\theta > \theta^*$. Finally, for any θ in (θ_0, θ^*), $\Phi(\theta^*) - \Phi(\theta) \geq (\theta^* - \theta)\underline{\phi}$ whence, since $\Phi(\theta) \leq 1$,

$$\begin{aligned}\Pi_{\widetilde{r}}(\theta^*) - \Pi_{\widetilde{r}}(\theta) &= r^1_{\theta^*}\left[\Phi(\theta^*) - \Phi(\theta)\right] - k'_3(\theta^* - \theta)\Phi(\theta) \\ &\geq r^1_{\theta^*}(\theta^* - \theta)\underline{\phi} - k'_3(\theta^* - \theta)\end{aligned}$$

which is positive by (64). We have shown that for any $\theta \neq \theta^*$, $\Pi_{\widetilde{r}}(\theta^*) > \Pi_{\widetilde{r}}(\theta)$ as claimed. Q.E.D.$_{\text{Theorem 4}}$

Proof of Theorem 6. As in the proof of Theorem 1, the expected relative payoff $\pi_\sigma(x,k)$ from investing of an agent i whose signal x_i is x, when each other player j invests if and only if her signal x_j lies below a common threshold k, is given by (27).

Lemma 10 *If h satisfies TM, $\pi_\sigma(x,k)$ is continuous and nonincreasing in $x \in I$.*

Proof. Since $\phi(\theta') = 1$ at all states θ' in $[0,1]$, the denominator of (28) equals one whenever x is in I, whence $\omega_\sigma(\theta|x)$ equals $\frac{1}{\sigma}f\left(\frac{x-\theta}{\sigma}\right)$ so $\pi_\sigma(x,k)$ equals $\frac{1}{\sigma}\int_{\theta=x-\sigma/2}^{x+\sigma/2} f\left(\frac{x-\theta}{\sigma}\right) h_\theta^{F\left(\frac{k-\theta}{\sigma}\right)} d\theta$. For any threshold k, let the signals $x' > x$ both be in I. Applying the change of variables $\theta' = x - x' + \theta$ to $\pi_\sigma(x',k)$ and then renaming θ' to θ, $\pi_\sigma(x',k) - \pi_\sigma(x,k)$ equals

$\frac{1}{\sigma}\int_{\theta=x-\sigma/2}^{x+\sigma/2} f\left(\frac{x-\theta}{\sigma}\right)\left[h_{\theta+x'-x}^{F\left(\frac{k-\theta+x-x'}{\sigma}\right)} - h_{\theta}^{F\left(\frac{k-\theta}{\sigma}\right)}\right] d\theta$. The integrand is nonpositive by TM(h). Since f is continuous with compact support, it has a finite upper bound $\overline{f}$. Hence, for $x'-x \in (0,\sigma)$,

$$\begin{aligned} 0 \;\geq\; & \pi_\sigma(x',k) - \pi_\sigma(x,k) \geq \frac{\overline{f}}{\sigma}\int_{\theta=x-\sigma/2}^{x+\sigma/2}\left[h_{\theta+x'-x}^{F\left(\frac{k-\theta+x-x'}{\sigma}\right)} - h_{\theta}^{F\left(\frac{k-\theta}{\sigma}\right)}\right] d\theta \\ = \; & \frac{\overline{f}}{\sigma}\left(\int_{\theta=x+\sigma/2}^{x'+\sigma/2} h_{\theta}^{F\left(\frac{k-\theta}{\sigma}\right)} d\theta - \int_{\theta=x-\sigma/2}^{x'-\sigma/2} h_{\theta}^{F\left(\frac{k-\theta}{\sigma}\right)} d\theta\right) \end{aligned}$$

which rises to zero as $x' \downarrow x$ since $h_{\theta}^{F\left(\frac{k-\theta}{\sigma}\right)}$ is monotone in θ by TM(h) and thus bounded on compact sets. Hence, $\pi_\sigma(x,k)$ is continuous and nonincreasing in $x \in I$. ■

Fix a price p in $[0,\overline{p}]$. Let k be any resulting equilibrium investment threshold. By DR$_1$, an agent (does not) invest for any $x \leq \sigma/2$ (resp., $x \geq 1-\sigma/2$). Hence, k must lie in I. Moreover, by continuity (Lemma 10), an agent must be indifferent when her signal x equals k: $\pi_\sigma(k,k)$ must equal p. Substituting k for x in (27) and using the fact that k lies in I, $\pi_\sigma(k,k)$ equals $\int_{\theta=k-\sigma/2}^{k+\sigma/2}\frac{1}{\sigma}f\left(\frac{k-\theta}{\sigma}\right)h_{\theta}^{F\left(\frac{k-\theta}{\sigma}\right)}d\theta$ which, by the change of variables $\ell = F\left(\frac{k-\theta}{\sigma}\right)$, equals H_k^σ.[72] We have shown that any equilibrium threshold k must lie in I and satisfy $H_k^\sigma = p$. But by LMSM, the set of such solutions is an interval $\left[\underline{k}_p^h, \overline{k}_p^h\right] \subset I$, which must be a singleton if H_k^σ is decreasing at any solution k to $H_k^\sigma = p$. Moreover, any k in the interval is an equilibrium given the price p: as an agent is indifferent when $x = k$, she must be willing (not) to invest if her signal x is less (greater) than k. This holds for $x \in I$ by Lemma 10 and for $x \notin I$ by DR$_1$. Finally, if $H_k^\sigma = p < p' = H_{k'}^\sigma$, we must have $k > k'$ as H^σ is continuous and nonincreasing, so $\underline{k}_p^h > \overline{k}_{p'}^h$ as claimed. Q.E.D.$_{\text{Theorem 6}}$

Proof of Theorem 5. We first produce an augmented mean relative payoff function $\widetilde{R}_k^\sigma$ that satisfies LMSM, induces the firm to choose the threshold k^*, and equals the minimum price at this threshold: $\widetilde{R}_{k^*}^\sigma = p_m^\sigma$. We then produce a floor-based PSS τ^* whose augmented relative payoff function $\widetilde{r} = r + \tau$ integrates to this $\widetilde{R}_k^\sigma$ and satisfies DR$_1$ and TM. Finally, we show that when LCED holds, the PSS τ^* minimizes the cost of implementing the efficient

[72] The first equality is proved as follows. Since k lies in I, the prior density $\phi(\theta)$ of the state equals one for all states in the interval $[k-\sigma/2, k+\sigma/2]$. Thus, substituting k for x, $\phi(\theta')$ (resp., $\phi(\theta)$) equals one for each state in the integral in (28) (resp., (27)). Hence the denominator of (28) - the integral of the posterior density $\frac{1}{\sigma}f\left(\frac{x-\theta'}{\sigma}\right)$ of the state θ' over its support - equals one and the posterior density $w_\sigma(\theta|k)$ in (27) is simply $\frac{1}{\sigma}f\left(\frac{k-\theta}{\sigma}\right)$.

threshold k^* among all PSS's τ that do so.

By (40) and the fact, from DLMB and DLMR, that $m^\sigma_{\widehat{k}} = 0 = s^\sigma_{k^*}$, we have

$$\sigma/2 \leq \widehat{k} < k^* \leq 1 - \sigma/2. \tag{67}$$

(This also relies on the fact, from DLMB and DLMR, that both thresholds k^* and $\widehat{k}$ lie in the interval $I = [\sigma/2, 1-\sigma/2]$.) Moreover, the minimum price p^σ_m (defined in (41)) satisfies

$$R^\sigma_{k^*} < 0 < p^\sigma_m < \overline{p}. \tag{68}$$

The first inequality in (68) holds as $R^\sigma_{k^*} < s^\sigma_{k^*} = 0$ by (40) and DLMB. The third holds by (41) since $G\left(\widehat{k}\right) < G(k^*)$ by (67) and since $R^\sigma_{\widehat{k}} < \overline{p}$ by (39), AM(r), and WSM(r). As for the second inequality, it suffices to show that $\Pi^\sigma_r\left(\widehat{k}\right) > 0$ by (41). Suppose there is no transfer scheme. If the firm charges $\overline{p}$ then, by Corollary 4, an agent invests if her signal x_i is less than that threshold k given by $R^\sigma_k = \overline{p}$. Since $R^\sigma_{\sigma/2} > \overline{p}$, this threshold k exceeds $\sigma/2$ by WSM(r), so $\Pi^\sigma_r\left(\widehat{k}\right) \geq \overline{p}G(\sigma/2)$. And by (35), $G(\sigma/2)$ equals $\int_{\theta=0}^{\sigma} F\left(\frac{1}{2} - \frac{\theta}{\sigma}\right) d\theta$ which becomes $\sigma \int_{\varepsilon=-1/2}^{1/2} F(\varepsilon)\, d\varepsilon > 0$ by the change of variables $\varepsilon = \frac{1}{2} - \frac{\theta}{\sigma}$: $\Pi^\sigma_r\left(\widehat{k}\right) > 0$ as claimed.

Now fix any constant

$$k_3' \in \left(0, \min\left\{1/k_4, \frac{1 - 3\sigma/2 - k^*}{p^\sigma_m - R^\sigma_{1-3\sigma/2}}, \frac{k^* - \sigma/2}{p^\sigma_m}\right\}\right). \tag{69}$$

Why must this interval be nonempty? First,

$$1 - 3\sigma/2 > k^* > 3\sigma/2. \tag{70}$$

The first inequality holds since $0 = s^\sigma_{k^*} < r^1_{k^* - \sigma/2}$ by DLMB and (40) and $r^1_{1-2\sigma} < 0$ by $\mathrm{DR}_2(r)$ whence by WSM(r), $k^* < 1 - 3\sigma/2$. The second holds as

$$R^\sigma_{3\sigma/2} > \overline{p} > p^\sigma_m > 0 > R^\sigma_{k^*} > R^\sigma_{1-3\sigma/2} \tag{71}$$

by, respectively, $\mathrm{DR}_2(r)$, (68) thrice, and (70) and WSM(r); therefore, $k^* > 3\sigma/2$ by WSM(r). Together these inequalities imply that the interval in (69) is nonempty.

Define $\psi_1(k) = R^\sigma_k$ and $\psi_2(k) = R^\sigma_k + (k - k^*)/k_3'$. Both are continuous in k by WSM(r). And $\psi_1(k)$ is decreasing in k by WSM(r) while $\psi_2(k)$ is increasing in k as $1/k_3' > k_4$ by (69) and by WSM(r). Hence, for $n = 1, 2$, there is a unique threshold $k = k^n$ defined by

$$\psi_n(k^n) = p_m^\sigma. \tag{72}$$

Moreover,

$$\psi_1(k) \gtreqless p_m^\sigma \text{ as } k \lesseqgtr k^1 \text{ and } \psi_2(k) \lesseqgtr p_m^\sigma \text{ as } k \lesseqgtr k^2. \tag{73}$$

For k_3' satisfying (69),

$$1 - 3\sigma/2 > k^2 > k^* > k^1 > 3\sigma/2. \tag{74}$$

All inequalities rely on (73). The first inequality then holds as $\psi_2(1 - 3\sigma/2) > p_m^\sigma$ since $k_3' < \frac{1-3\sigma/2-k^*}{p_m^\sigma - R_{1-3\sigma/2}^\sigma}$ by (69). The second and third hold since $\psi_2(k^*) = \psi_1(k^*) = R_{k^*}^\sigma < p_m^\sigma$ by (68). The fourth holds as $\psi_1(3\sigma/2) = R_{3\sigma/2}^\sigma > p_m^\sigma$ by (71).

We now verify that the target augmented demand function $\widetilde{R}_k^*$, defined in (42), satisfies LMSM. As both expressions equal p_m^σ at k^*, $\widetilde{R}_k^*$ is continuous at k^*. At thresholds k not in $[k^1, k^2]$, $\widetilde{R}_k^*$ equals R_k^σ. By (72), $\lim_{k \downarrow k^1} \widetilde{R}_k^* = R_{k^1}^\sigma$ and $\lim_{k \uparrow k^2} \widetilde{R}_k^* = R_{k^2}^\sigma$, so $\widetilde{R}_k^*$ is continuous at k^1 and k^2. Hence $\widetilde{R}_k^*$ is continuous in $k \in I$. It is nonincreasing in k since R_k^σ is decreasing by WSM(r), so it satisfies LMSM.

Let τ^* be the scheme defined in (43). We now verify that $\widetilde{r}^* = r + \tau^*$ satisfies DR$_1$ and TM. As for DR$_1$, the transfer $\tau_{k-\sigma F^{-1}(\ell)}^{*\ell}$ affects $\widetilde{r}_\theta^{*\ell}$ only for θ in $[k - \sigma/2, k + \sigma/2]$. Hence, since T_k^* and thus $\tau_{k-\sigma F^{-1}(\ell)}^{*\ell}$ is zero for any k not in (k^1, k^2), which is a subinterval of $(3\sigma/2, 1 - 3\sigma/2)$ by (74), $\widetilde{r}_\theta^{*\ell}$ equals r_θ^ℓ for all $\theta \notin (\sigma, 1 - \sigma)$. Thus, since r satisfies DR$_2$, $\widetilde{r}^*$ satisfies DR$_1$.

To check TM($\widetilde{r}^*$), we first decompose TM into two simpler conditions:

Lemma 11 *TM(h) holds if and only if (a) the minimum and maximum relative payoffs $h_{\theta'}^0$ and $h_{\theta'}^1$ are both nonincreasing in the state θ and (b) for any constant $k \in \Re$, $h_{k-\sigma F^{-1}(\ell)}^\ell$ is nondecreasing in $\ell \in [0, 1]$.*

Proof. First, for any given threshold k there are three intervals of states θ (including those that have zero prior probability):

$$\text{If } \left\{ \begin{array}{c} \theta \in I_1 = (-\infty, k - \sigma/2] \\ \theta \in I_2 = [k - \sigma/2, k + \sigma/2] \\ \theta \in I_3 = [k + \sigma/2, \infty) \end{array} \right\} \text{ then } \left\{ \begin{array}{c} F\left(\frac{k-\theta}{\sigma}\right) = 1 \\ F\left(\frac{k-\theta}{\sigma}\right) \in [0, 1] \\ F\left(\frac{k-\theta}{\sigma}\right) = 0 \end{array} \right\} \tag{75}$$

1. Only If. Assume TM. For any states $\theta'' > \theta'$, by taking the threshold k arbitrarily high or low so that the investment rate $F\left(\frac{k-\theta}{\sigma}\right)$ corresponding to both states $\theta = \theta', \theta''$ is identically zero or one, respectively, one can ensure that $h^0_{\theta''} \leq h^0_{\theta'}$ and $h^1_{\theta''} \leq h^1_{\theta'}$. This shows (a). As for (b), for any two investment rates $\ell' > \ell$ in $[0,1]$, let $\theta' = k - \sigma F^{-1}(\ell')$ and $\theta = k - \sigma F^{-1}(\ell)$. As F is increasing on $[-1/2, 1/2]$, F^{-1} is increasing on $[0,1]$; hence, $\theta' > \theta$. Moreover, $F\left(\frac{\theta'-k}{\sigma}\right) = \ell'$ and $F\left(\frac{k-\theta}{\sigma}\right) = \ell$. Thus, $h^{\ell'}_{k-\sigma F^{-1}(\ell')} = h^{F\left(\frac{k-\theta'}{\sigma}\right)}_{\theta'} \leq h^{F\left(\frac{k-\theta}{\sigma}\right)}_{\theta} = h^{\ell}_{k-\sigma F^{-1}(\ell)}$, where the inequality holds by TM.

2. If. Within I_1 and I_3, TM follows from property (a), while within I_2 TM follows from property (b). And as each pair of adjacent intervals has a common endpoint, the result holds for all θ by transitivity.

■

We now check the conditions. As for condition (a), $\widetilde{r}^{*0}_{\theta} = r^0_{\theta}$ is increasing in θ by WSM(r). Moreover, $\widetilde{r}^{*1}_{k-\sigma/2}$ equals $\max\left\{r^1_{k-\sigma/2}, \widetilde{R}^*_k\right\}$ and both components are increasing in k by WSM(r) and since $\widetilde{R}^*_k$ satisfies LMSM. As for condition (b), for $\ell \in (0,1]$, $\widetilde{r}^{*\ell}_{k-\sigma F^{-1}(\ell)} = \max\left\{r^{\ell}_{k-\sigma F^{-1}(\ell)}, \kappa^*_k\right\}$ is nondecreasing in ℓ by AM(r) and WSM(r). And for any $\ell > 0$, $\widetilde{r}^{*\ell}_{k-\sigma F^{-1}(\ell)} \geq r^{\ell}_{k-\sigma F^{-1}(\ell)} \geq \widetilde{r}^{*0}_{k+\sigma/2}$ where the 2nd inequality follows from AM(r) and WSM(r). This verifies condition (b).

We have shown that the PSS τ^* induces an augmented relative payoff function $\widetilde{r}^*$ that satisfies TM, LMSM, and DR$_1$. Thus, by Theorem 6, each price p in $[0, \overline{p}]$ leads the agents to select some threshold k that satisfies $\widetilde{R}^*_k = p$. We now show efficiency: that under the PSS τ^* there is a unique threshold PBE, in which the agents choose k^*.

Suppose the firm chooses the price p^{σ}_m. The set of thresholds k that satisfy $\widetilde{R}^*_k = p^{\sigma}_m$ is the nonempty interval $[k^1, k^*]$. By assumption, the agents must choose one of these. But in a PBE, the agents must respond with the threshold k^*. Why? If they were instead expected to choose a threshold $k < k^*$, the firm would deviate: it would lower its price by an arbitrarily small amount, inducing them to choose a threshold $k \geq k^*$. We conclude that the price p^{σ}_m leads the agents to choose the threshold k^* in any threshold PBE. Moreover, the firm's profit from the price p^{σ}_m is $p^{\sigma}_m G(k^*)$ which is positive by (71).

It remains to show that the price p^{σ}_m is optimal for the firm. If instead it chooses a price $p \in (0, p^{\sigma}_m)$, then the resulting threshold k must lie in (k^*, k^2) since thresholds $k \geq k^2$ result only from negative prices.[73] By (74), $[k^*, k^2]$ is a subset of I. And by (35), $G(\sigma/2)$ equals

[73] At any threshold $k \geq k^2$, the agents' willingness to pay $\widetilde{R}^*_k$ equals R^{σ}_k which is less than $R^{\sigma}_{k^*}$ by WSM(r);

$\int_{\theta=0}^{\sigma} F\left(\frac{1}{2}-\frac{\theta}{\sigma}\right) d\theta$ which lies in $(0,\sigma)$. For k in I, $G'(k)=1$ and thus

$$G(k)=G(\sigma/2)+k-\sigma/2\in(k-\sigma/2,k+\sigma/2). \tag{76}$$

From a threshold $k\in(k^*,k^2)$, the firm gets $\left(p_m^\sigma-(k-k^*)/k_3'\right)G(k)$ which is less than $p_m^\sigma G(k^*)$ if $k_3'<\frac{G(k)(k-k^*)}{p_m^\sigma[G(k)-G(k^*)]}=\frac{G(k)}{p_m^\sigma}$ since $G'(k)=1$ for k in I. This inequality holds as $G(k)>G(k^*)>k^*-\sigma/2$ by (76) and since $k_3'<(k^*-\sigma/2)/p_m^\sigma$ by (69). Finally, if instead the firm chooses a price $p>p_m^\sigma$, the resulting threshold k lies below k^1. Thus $\widetilde{R}_k^*=R_k^\sigma$, whence the firm's profit is $\Pi_r^\sigma(k)$ which cannot exceed its payoff $p_m^\sigma G(k^*)$ from the efficient threshold k^* by definition of p_m^σ: the firm will not choose such a price either.[74]

We have now show that the PSS τ^* leads the agents to select the efficient threshold k^*. However, there are other efficient PSS's. We now show that τ^* is cost-minimizing in the set of efficient PSS's. This part relies on the additional assumption LCED.

Consider an arbitrary efficient PSS τ (which may not be cost-minimizing). Its cost $C(\tau)$ is the sum of $C_0(\tau)$ and $C_1(\tau)$, defined on p. 8 of this document. Let p^τ denote the agents' augmented demand $\widetilde{R}_{k^*}^\sigma$ at the efficient threshold that results from τ; efficiency implies $p^\tau\geq p_m^\sigma$. Let $S(\ell)$ denote the relative payoff $\widetilde{r}_{k^*-\sigma F^{-1}(\ell)}^{\ell}$ resulting from τ and let $s(\ell)$ denote $r_{k^*-\sigma F^{-1}(\ell)}^{\ell}$. For all $k<k^*$, $\widetilde{r}_{k-\sigma/2}^{1}\geq\max\left\{r_{k-\sigma/2}^1,S(1)\right\}$ since $\tau_{k-\sigma/2}^1\geq 0$ and by TM. Hence, $\tau_{k-\sigma/2}^1\geq\max\left\{S(1)-r_{k-\sigma/2}^1,0\right\}$. In order to minimize $C(\tau)$, the latter condition must hold with equality: $\tau_{k-\sigma/2}^1=\max\left\{S(1)-r_{k-\sigma/2}^1,0\right\}$. The coordination cost $C_0(\tau)$ then equals $\psi(S(1))\overset{d}{=}\int_{k=\sigma/2}^{k^*}\max\left\{S(1)-r_{k-\sigma/2}^1,0\right\}dk$. The function $\psi(S(1))$ is increasing in $S(1)$ and equals zero when $S(1)$ equals $r_{(k^*-\sigma/2)-}^1$ which, in turn, is not less than $s(1)$.

Adding the constant $\int_{\theta=k^*-\sigma/2}^{k^*+\sigma/2}\left[\ell r_\theta^\ell\right]_{\ell=F\left(\frac{k^*-\theta}{\sigma}\right)}d\theta$ to $C_1(\tau)$ (which does not change the cost-minimizing τ) and using the change of variables $\ell=F\left(\frac{k^*-\theta}{\sigma}\right)$, the cost of the efficient PSS τ becomes $C(\tau)=\psi(S(1))+\int_{\ell=0}^{1}\phi(\ell)S(\ell)d\ell$ where $\phi(\ell)=\frac{\sigma\ell}{F'(F^{-1}(\ell))}$ is an increasing function of ℓ.[75] This function $C(\tau)$ is minimized subject to $\int_{\ell=0}^1 S(\ell)d\ell\geq p_m^\sigma$ and, for all ℓ in $[0,1]$, $S(\ell)\geq s(\ell)$ and $S'(\ell)\geq 0$.

Now consider two efficient PSS's τ_1 and τ_2 that give rise to nondecreasing functions

by (40), $R_{k^*}^\sigma$ in turn is less than $\widehat{s}_{k^*}$ which equals zero by definition of k^*.

[74] Recall that we assume the firm chooses the efficient threshold whenever it is willing to do so.

[75] By LCED, $d\ln F(\varepsilon)/d\varepsilon=F'(\varepsilon)/F(\varepsilon)$ is decreasing in ε. Thus, letting $\varepsilon=F^{-1}(\ell)$, $\phi(F(\varepsilon))$ becomes $\sigma F(\varepsilon)/F'(\varepsilon)$ which is increasing in ε. So $\phi(F(\varepsilon))$ is increasing in ε whence, as F is increasing, so must be ϕ.

S_1 and S_2 (where $S_i(\ell) = r^{\ell}_{k^*-\sigma F^{-1}(\ell)} + (\tau_i)^{\ell}_{k^*-\sigma F^{-1}(\ell)}$ for $i = 1, 2$) that each integrates (over $\ell \in [0,1]$) to the same value $p^\tau \geq p^\sigma_m$. Assume that τ_1 is a floor-based PSS, whence $S_1(\ell) = \max\{s(\ell), \kappa\}$ for some floor κ. Define $H(\ell) = S_2(\ell) - S_1(\ell)$. Since $S_2(\ell) \geq s(\ell)$, $S_2(\ell)$ is nonincreasing, and $\int_{\ell=0}^{1} H(\ell)\, d\ell = 0$, there is an $\ell^* \in [0,1]$ such that $H(\ell) < 0$ if and only if $\ell < \ell^*$. And since ϕ is increasing, $\int_{\ell=0}^{1} [\phi(\ell) - \phi(\ell^*)] H(\ell)\, d\ell$ equals $\int_{\ell=0}^{\ell^*} [\phi(\ell) - \phi(\ell^*)] H(\ell)\, d\ell + \int_{\ell=\ell^*}^{1} [\phi(\ell) - \phi(\ell^*)] H(\ell)\, d\ell$. The first integral in this sum is nonnegative since, for all $\ell < \ell^*$, $\phi(\ell) < \phi(\ell^*)$ and $H(\ell) < 0$. The second integral is nonnegative since, for all $\ell > \ell^*$, $\phi(\ell) > \phi(\ell^*)$ and $H(\ell) \geq 0$. Accordingly, $\int_{\ell=0}^{1} \phi(\ell) H(\ell)\, d\ell$ is not less than $\phi(\ell^*) \int_{\ell=0}^{1} H(\ell)\, d\ell$ which is zero. This shows that the cost of S_2 during a partial run is at least that of S_1. As for the cost at states at which all invest, since ψ is an increasing function it suffices that $S_2(1) \geq S_1(1)$. If $S_1(1) = s(1)$ we are done since $S_2(1) \geq s(1)$. If instead $S_1(1) > s(1)$ then $\kappa > s(1)$ whence $S_1(\ell)$ is constant and equal to p^τ. So since S_2 is nondecreasing and integrates (over $\ell \in [0,1]$) to p^τ, we must have $S_2(1) \geq p^\tau$. This shows that $S_2(1) \geq S_1(1)$ as claimed.

We have shown that the cheapest efficient PSS that yields a given demand $p^\tau \geq p^\sigma_m$ is floor-based. It thus is of the form $S_{p^\tau}(\ell) = \max\{s(\ell), \kappa_{p^\tau}\}$ where κ_{p^τ} is the unique solution to $\int_{\ell=0}^{1} \max\{s(\ell), \kappa_{p^\tau}\}\, d\ell = p^\tau$. As κ_{p^τ} is increasing in p^τ, the miscoordination cost $\int_{\ell=0}^{1} \phi(\ell) S_{p^\tau}(\ell)\, d\ell = \int_{\ell=0}^{1} \phi(\ell) \max\{s(\ell), \kappa_{p^\tau}\}\, d\ell$ is increasing in p^τ. Similarly, the coordination cost is nondecreasing in p^τ: it is a nondecreasing function ψ of $S_{p^\tau}(1)$ which, in turn, is nondecreasing in p^τ since κ_{p^τ} is. It follows that the total cost $C(\tau)$ of an efficient floor-based PSS τ is strictly minimized when $p^\tau = p^\sigma_m$. Hence the floor-based PSS τ^* defined in (43) is cost-minimizing for the planner. Q.E.D.$_{\text{Theorem 5}}$

Proof of Claim 7: Let $\overline{\phi} < \infty$ be an upper bound of the prior density ϕ of θ. By Theorem 2, for $\varepsilon' = \varepsilon\left(2\overline{p}\overline{\phi}\right)^{-1}$ there exists a $\delta > 0$ such that for any price $p \in [0, \overline{p}]$ and any private noise scale factor σ in the interval $(0, \delta)$, the firm's profit $P_\sigma(p)$ from the price p lies in $\left[\underline{P}(p), \overline{P}(p)\right]$ where $\underline{P}(p) = p\Phi(\theta^p_R - \varepsilon')$ and $\overline{P}(p) = p\Phi(\theta^p_R + \varepsilon')$. But this interval also contains $P(p)$. Hence $|P(p) - P_\sigma(p)| \leq \left|\overline{P}(p) - \underline{P}(p)\right| \leq 2\overline{p}\overline{\phi}\varepsilon' = \varepsilon$ as claimed. Q.E.D.$_{\text{Lemma 3}}$Part 1. As a continuous function on a compact set is bounded and

$\partial m^{\ell}_{2,\theta}/\partial\theta < 0$ by MMSB, there is a $\overline{m} < \infty$ such that

$$\left|m^{\ell}_{2,\theta}\right| < \overline{m} \text{ and } \partial m^{\ell}_{2,\theta}/\partial\theta \in [-\overline{m}, 0) \text{ for all } (\ell, \theta) \in [0,1]^2. \tag{77}$$

Thus, by (46) and MMSB, $\frac{\partial s^{\ell_1}_{2,\theta}}{\partial \ell_1} = \left(\frac{1}{1-\ell_1}\right)^2 \int_{\ell=\ell_1}^{1} \left(m^{\ell}_{2,\theta} - m^{\ell_1}_{2,\theta}\right) d\ell \in \left(0, \frac{2\overline{m}}{1-\ell_1}\right]$ for $\ell_1 < 1$ and

$s^1_{2,\theta} = \lim_{\ell_1 \uparrow 1} s^{\ell_1}_{2,\theta}$ by definition of $s^1_{2,\theta}$ (equation (46)): $s^{\ell_1}_{2,\theta}$ is increasing and differentiable in $\ell_1 \in [0,1]$ for any θ in $[0,1]$. As for θ, for $\ell_1 < 1$ we have $\partial s^{\ell_1}_{2,\theta}/\partial\theta = \frac{1}{1-\ell_1}\int_{\ell=\ell_1}^{1}\left(\partial m^{\ell}_{2,\theta}/\partial\theta\right) d\ell$ which lies in$[-\overline{m}, 0)$ by (77). A similar argument holds at $\ell_1 = 1$, so $s^{\ell_1}_{2,\theta}$ is decreasing and differentiable in θ for any ℓ_1 in $[0,1]$. Part 2 holds by MMSB and (46). As for part 3, as F is continuous, $s^{F\left(\frac{k_1-\theta}{\sigma}\right)}_{2,\theta}$ is continuous in θ by part 1 of Claim 7. By part 2 of Claim 7, it is positive (negative) for $\theta < 2\sigma$ (resp., $\theta > 1-2\sigma$). So by the intermediate value theorem, it has a root θ in $(2\sigma, 1-2\sigma)$. This root must also be unique. Why? Consider any two states $\theta' > \theta$, both in $(2\sigma, 1-2\sigma)$; let $\ell_1' = F\left(\frac{k_1-\theta'}{\sigma}\right)$ and $\ell_1 = F\left(\frac{k_1-\theta}{\sigma}\right)$. As F is nondecreasing, $\ell_1 \geq \ell_1'$. Hence, if $s^{\ell_1}_{2,\theta} = 0$, then $s^{\ell_1'}_{2,\theta'} < 0$ by part 1 of Claim 7. Similarly, if $s^{\ell_1'}_{2,\theta'} = 0$, then $s^{\ell_1}_{2,\theta} > 0$. Letting $\ell_1 = F\left(\frac{k_1-\theta}{\sigma}\right)$ and totally differentiating $s^{\ell_1}_{2,\theta} \equiv 0$ (which must hold at the period-2 efficient threshold $\theta = k_2^*(k_1)$), we obtain

$$\left[\partial s^{\ell_1}_{2,\theta}/\partial\ell_1\right]\frac{1}{\sigma}f\left(\frac{k_1-\theta}{\sigma}\right)d\left(k_1-\theta\right) + \left[\partial s^{\ell_1}_{2,\theta}/\partial\theta\right]d\theta = 0. \tag{78}$$

The first (second) derivative in square brackets is positive (negative) by Claim 7. Hence $d\theta/d\left(k_1-\theta\right) \in \Re_+$. Grouping the terms involving $d\theta$, we obtain

$$\frac{d\theta}{dk_1} = \frac{\left[\partial s^{\ell_1}_{2,\theta}/\partial\ell_1\right]\frac{1}{\sigma}f\left(\frac{k_1-\theta}{\sigma}\right)}{\left(\left[\partial s^{\ell_1}_{2,\theta}/\partial\ell_1\right]\frac{1}{\sigma}f\left(\frac{k_1-\theta}{\sigma}\right) - \partial s^{\ell_1}_{2,\theta}/\partial\theta\right)} \geq 0 \tag{79}$$

so $k_2^*(k_1)$ is nondecreasing and differentiable in k_1. By (79), an increase in k_1 leads to a finite increase or no change in θ: $k_2^*(k_1)$ is continuous and nondecreasing. If the increase in k_1 causes θ to rise then, by (78), we must have $f\left(\frac{k_1-\theta}{\sigma}\right) > 0$ and $k_1 - \theta$ must rise by a finite amount. If it leads to no change in θ, then $k_1 - \theta$ trivially rises by $dk_1 > 0$. Hence $k_1 - k_2^*(k_1)$ is increasing and differentiable in k_1. Q.E.D.$_{\text{Claim 7}}$

Proof of Claim 8: Using part 3 of Claim 7, we can rewrite social welfare in (45) as follows:

$$SW\left(k_1\right) = \int_{\theta=0}^{1} w^{F\left(\frac{k_1-\theta}{\sigma}\right)}_{1,\theta}d\theta + \int_{\theta=0}^{k_2^*(k_1)} w^1_{2,\theta}d\theta + \int_{\theta=k_2^*(k_1)}^{1} w^{F\left(\frac{k_1-\theta}{\sigma}\right)}_{2,\theta}d\theta.$$

By part 3 of Claim 7, $w^1_{2,\theta}$ equals $w^{F\left(\frac{k_1-\theta}{\sigma}\right)}_{2,\theta}$ when θ equals $k_2^*(k_1)$ which, in turn, is continuous in k_1. Hence, the effect of k_1 on $SW(k_1)$ that occurs via the change in $k_2^*(k_1)$ is zero to first order. It follows that the marginal social benefit of raising the first-period threshold k_1 is

$$SW'(k_1) = \int_{\theta=0}^{1} m_{1,\theta}^{F\left(\frac{k_1-\theta}{\sigma}\right)} \frac{1}{\sigma} f\left(\frac{k_1-\theta}{\sigma}\right) d\theta + \int_{\theta=k_2^*(k_1)}^{1} m_{2,\theta}^{F\left(\frac{k_1-\theta}{\sigma}\right)} \frac{1}{\sigma} f\left(\frac{k_1-\theta}{\sigma}\right) d\theta. \tag{80}$$

The density $f\left(\frac{k_1-\theta}{\sigma}\right)$ in (80) is nonzero only for θ in $[k_1 - \sigma/2, k_1 + \sigma/2]$. Hence, by MMSB, (80) is positive (negative) for $k_1 \leq 3\sigma/2$ (resp., for $k_1 \geq 1 - 3\sigma/2$) as claimed. For k_1 in $[3\sigma/2, 1 - 3\sigma/2]$, the change of variables $\ell_1 = F\left(\frac{k_1-\theta}{\sigma}\right)$ yields $SW'(k_1) = s_{k_1}^{\sigma}$. The effect of an increase in k_1 on $s_{k_1}^{\sigma}$ while holding $\ell_1^*(k_1)$ fixed is negative and finite by MMSB and (47). Now consider the effect on $s_{k_1}^{\sigma}$ that is mediated through a change in $\ell_1^*(k_1)$. This change is $\frac{d\ell_1^*(k_1)}{dk_1} = \frac{1}{\sigma} f\left(\frac{k_1-k_2^*(k_1)}{\sigma}\right) \frac{d\left[k_1-k_2^*(k_1)\right]}{dk_1}$ which, by part 3 of Claim 7, is nonnegative and finite. If $\frac{d\ell_1^*(k_1)}{dk_1}$ is zero we are done. If instead it is positive, then $f\left(\frac{k_1-k_2^*(k_1)}{\sigma}\right) > 0$. In this case, the effect of the increase in $\ell_1^*(k_1)$ on $s_{k_1}^{\sigma}$ is of the same sign as $m_{2,k_2^*(k_1)}^{F\left(\frac{k_1-k_2^*(k_1)}{\sigma}\right)}$, which is nonpositive. Why? The welfare gain $w_{2,\theta}^1 - w_{2,\theta}^{F\left(\frac{k_1-\theta}{\sigma}\right)}$ from the rest investing is identically zero at $\theta = k_2^*(k_1)$. Totally differentiating this identity w.r.t. k_1 and θ, we obtain

$$\frac{d\theta}{d(k_1-\theta)} = \frac{m_{2,\theta}^{F\left(\frac{k_1-\theta}{\sigma}\right)} \frac{1}{\sigma} f\left(\frac{k_1-\theta}{\sigma}\right)}{\frac{\partial}{\partial\theta}\left[w_{2,\theta}^1 - w_{2,\theta}^{\ell_1}\right]_{\ell_1=F\left(\frac{k_1-\theta}{\sigma}\right)}}$$

which is nonnegative by (79). The denominator is proportional to $\frac{\partial}{\partial\theta} s_{\theta}^{\ell_1}$ and thus is negative by part 1 of Claim 7. Thus, since $f\left(\frac{k_1-\theta}{\sigma}\right) > 0$, $m_{2,\theta}^{F\left(\frac{k_1-\theta}{\sigma}\right)}$ is nonpositive as claimed. We conclude that $s_{k_1}^{\sigma}$ is decreasing and continuous in $k_1 \in [3\sigma/2, 1-3\sigma/2]$ as claimed. Q.E.D.$_{\text{Claim 8}}$

Proof of Claim 10:

- AM(r): Given DAM, we need only verify that for any state θ and investment rates $\ell_1' > \ell_1$, $s_{2,\theta}^{\ell_1'} - s_{2,\theta}^{\ell_1}$ is nonnegative and bounded. By (53), $s_{2,\theta}^{\ell_1'}$ and $s_{2,\theta}^{\ell_1}$ are both equal-weighted means of $m_{2,\theta}^{\ell}$ over the intervals $\ell \in [\ell_1', 1]$ and $\ell \in [\ell_1, 1]$, respectively. By MMSB, $m_{2,\theta}^{\ell}$ is increasing in ℓ, so $s_{2,\theta}^{\ell_1'} - s_{2,\theta}^{\ell_1} > 0$. Finally, $s_{2,\theta}^{\ell_1'} - s_{2,\theta}^{\ell_1}$ is bounded above by $m_{2,\theta}^1 - m_{2,\theta}^0$ which, by MMSB, is continuous in θ and thus bounded on $\theta \in [0,1]$. This verifies AM(r).
- WSM(r): By (57), $r_{\theta}^{\ell_1}$ satisfies WSM for $\theta \neq k_2^*(k_1)$ by DSM and part 1 of Claim 7. It equals $r_{1,\theta}^{\ell_1} + r_{2,\theta}^{\ell_1}$ when $\theta > k_2^*(k_1)$ and $r_{1,\theta}^{\ell_1} + \min\left\{s_{2,\theta}^{\ell_1}, r_{2,\theta}^1\right\}$ when $\theta < k_2^*(k_1)$. Hence,

$r_\theta^{\ell_1}$ jumps downwards when θ crosses $k_2^*(k_1)$ from below.[76] Since its discontinuities are isolated and involve only downwards jumps, WSM(r) holds.

- $\mathrm{DR}_2(r)$: follows from DDR, (46), and MMSB.
- DLMR(r): follows from DWMR.
- DLMB$\left(s_{k_1}^\sigma\right)$: implied by Claim 8.

Equations (40) and (39): follow from (58) and (59), respectively. Q.E.D.$_{\text{Claim 10}}$

Proof of Theorem 8.

Part 1. In the above text we showed that any equilibrium threshold k must be a root of M. Moreover, by UDE, there exists a unique such root; let us denote it k^*. And by (60) and (61) and DOSC, for each firm $i = 1, 2$, to prefer the threshold k^*, the price charged by firm $j \neq i$ must satisfy $p_j = -\mu_{k^*}^i$. Thus, (p_1^*, p_2^*) is the only possible equilibrium of the limit game. To show that this pair is indeed an equilibrium of the limit game, it suffices to show that each price is positive: the firms' limit payoffs are positive. (If a firm's price is nonpositive, the firm would prefer a small positive price as this would surely avoid negative profits and would yield positive profits with positive probability.) First, μ_k^1 can be written as $M_k + [1 - \Phi(k)]\, R_k'/\Phi'(k)$ while μ_k^2 can be expressed as $-M_k + \Phi(k)\, R_k'/\Phi'(k)$. Moreover, the state θ has full support, so $\Phi(k^*)$ lies in $(0, 1)$. Thus, as M_{k^*} equals zero, $\mu_{k^*}^1$ and $\mu_{k^*}^2$ must both be negative by SM: the prices p_1^* and p_2^* are positive as claimed.

Part 2. By Theorem 2, for each $n = 1, 2, ...$ there is a $\delta_n > 0$ such that for any prices $p_1, p_2 \in [0, \overline{p}]$ and any private noise scale factor σ in the interval $(0, \delta_n)$, in any strategy profile that survives iterated deletion of strictly dominated strategies, each agent invests if her signal is less than $\theta_R^{p_1 - p_2} - 1/n$ and does not invest if her signal exceeds $\theta_R^{p_1 - p_2} + 1/n$. For each n, fix any noise scale factor σ_n in $(0, \delta_n)$ and let (p_1^n, p_2^n) be any equilibrium of the duopoly game with noise scale factor σ_n. Let k^n be defined implicitly by $R_{k^n} = p_1^n - p_2^n$: it is the asymptotic threshold that corresponds to the price pair (p_1^n, p_2^n).

As $[0, \overline{p}]^2$ is compact, by taking subsequences if needed we may assume that the sequence $(p_1^n, p_2^n)_{n=1}^\infty$ converges to a limit (p_1^∞, p_2^∞) by the Bolzano–Weierstrass Theorem. Let k^∞ equal the limit of the thresholds k^n as n goes to infinity. We claim that $M_{k^\infty} = 0$, $p_1^\infty = \mu_{k^\infty}^2$, and

[76] Why? By (52) and (53), $s_\theta^{\ell_1} - R_{2,\theta}^{\ell_1}$ equals $E_{2,\theta}^{\ell_1}$ which is nonnegative, so $s_\theta^{\ell_1} \geq R_{2,\theta}^{\ell_1}$. Moreover, $R_{2,\theta}^{\ell_1} \geq r_{2,\theta}^{\ell_1}$ by DAM. Hence, $s_\theta^{\ell_1} \geq r_{2,\theta}^{\ell_1}$. But also by DAM, $r_{2,\theta}^1 \geq r_{2,\theta}^{\ell_1}$. Combining the last two inequalities yields $\min\left\{s_{2,\theta}^{\ell_1}, r_{2,\theta}^1\right\} \geq r_{2,\theta}^{\ell_1}$ which implies the result.

$p_2^\infty = \mu^1_{k^\infty}$. This will imply that $(p_1^\infty, p_2^\infty, k^\infty)$ equals (p_1^*, p_2^*, k^*) as claimed, since the latter triplet uniquely solves these three equations by DOSC, UDE, and part 1 of this theorem.

We first show that $\mu^1_{k^\infty} = -p_2^\infty$; an analogous proof (omitted) shows that $\mu^2_{k^\infty} = -p_1^\infty$. For any price vector (p_1, p_2), let k denote the asymptotic threshold given implicitly by $R_k = p_1 - p_2$. By SM, the relation between k and p_1 is one-to-one. Thus, given firm 2's price p_2, we may think of firm 1 as choosing k rather than p_1. In game n, Theorem 2 then implies that firm 1's payoff $U_1^n(k, p_2)$ lies in $[(R_k + p_2)\Phi(k - 1/n), (R_k + p_2)\Phi(k + 1/n)]$. Moreover, firm 1's asymptotic payoff $U_1^\infty(k, p_2) = (R_k + p_2)\Phi(k)$ also lies in this interval. Since p_2 lies in $[0, \overline{p}]$, the absolute difference $|U_1^n(k, p_2) - U_1^\infty(k, p_2)|$ between 1's payoff in game n and its limiting payoff is at most $2(|R_k| + \overline{p})\overline{\phi}/n$ where $\overline{\phi}$ is an upper bound on the prior density Φ' of the state θ.

Let $\widehat{k}$ denote the (by DOSC) unique solution to $\mu^1_{\widehat{k}} = -p_2^\infty$ and assume $\widehat{k}$ is not equal to k^∞; we will derive a contradiction. We can decompose $U_1^n\left(\widehat{k}, p_2^n\right) - U_1^n(k^n, p_2^n)$ as $a^n - b^n + c^n - d^n + f$ where $a^n = U_1^n\left(\widehat{k}, p_2^n\right) - U_1^\infty\left(\widehat{k}, p_2^n\right)$, $b^n = U_1^n(k^n, p_2^n) - U_1^\infty(k^n, p_2^n)$, $c^n = U_1^\infty\left(\widehat{k}, p_2^n\right) - U_1^\infty\left(\widehat{k}, p_2^\infty\right)$, $d^n = U_1^\infty(k^n, p_2^n) - U_1^\infty(k^\infty, p_2^\infty)$, and $f = U_1^\infty\left(\widehat{k}, p_2^\infty\right) - U_1^\infty(k^\infty, p_2^\infty)$. By DOSC, firm 1's limiting payoff function $U_1^\infty(k, p_2) = (R_k + p_2)\Phi(k)$ is strictly concave in k, so f is positive. As shown above, $|a^n| \leq 2\left(\left|R_{\widehat{k}}\right| + \overline{p}\right)\overline{\phi}/n$ and $|b^n| \leq 2(|R_{k^n}| + \overline{p})\overline{\phi}/n$. Hence, by continuity of R and the convergence of $(k^n)_{n=1}^\infty$, there is an $n_1 < \infty$ such that for all $n > n_1$, neither $|a^n|$ nor $|b^n|$ exceeds $f/8$. By the continuity of U_1^∞, there is also an $n_2 < \infty$ such that for all $n > n_2$, neither $|c^n|$ nor $|d^n|$ exceeds $f/8$. Hence, for $n > \max\{n_1, n_2\}$, $U_1^n\left(\widehat{k}, p_2^n\right) - U_1^n(k^n, p_2^n)$ is at least $f/2$ which is positive: for firm 1 in game n, $\widehat{k}$ is a better response to the price p_2^n than k^n is - a contradiction. We conclude that $\widehat{k}$ equals k^∞, whence $\mu^1_{k^\infty} = -p_2^\infty$. Finally, recall that k^∞ equals $\lim_{n\to\infty} k^n$ and, for each n, k^n satisfies $R_{k^n} = p_1^n - p_2^n$. Thus, by the continuity of R, R_{k^∞} equals $p_1^\infty - p_2^\infty$ which, we have already shown, equals $\mu^1_{k^\infty} - \mu^2_{k^\infty}$. Thus, M_{k^∞} equals zero as well.

We conclude that in the limit as n grows to infinity, (p_1^n, p_2^n, k^n) converges to (p_1^*, p_2^*, k^*). Hence for any $\varepsilon > 0$ there is an $n > 0$ such that $|p_1^n - p_1^*|$, $|p_2^n - p_2^*|$, and $|k^n - k^*| + 1/n$ are each less than ε. Let $\delta = \delta_n$ and consider any private signal scale factor σ_n in $(0, \delta)$. Since (p_1^n, p_2^n, k^n) is any equilibrium of the duopoly game with scale factor σ_n, part 2(a) holds. And in this game, we have shown that each agent invests platform 1 if her signal is less than $k^n - 1/n$ which exceeds $k^* - \varepsilon$; she invests platform 2 if her signal exceeds $k^n + 1/n$ which is less than $k^* + \varepsilon$. This confirms part 2(b). Q.E.D.$_{\text{Theorem 8}}$