Managerial Entrenchment, Stakeholders, and Capital Structure*

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Abstract

We study the effects of combining managerial entrenchment (Harris and Raviv 1990) with liquidation-averse stakeholders (Titman 1984). This can give rise to a declining marginal cost of debt. As a result, the manager's objective function can be a nonconcave function of debt and thus have multiple local maxima. In this setting, small shocks can cause large jumps in debt even in the absence of any issuance costs.

J.E.L. Codes: G32, G33, G34.

Keywords: Managerial Entrenchment, Stakeholders, Capital Structure, Debt, Leveraged Buyouts, Swaps.

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1 Introduction

We study a dynamic model of optimal capital structure in which a manager's objective function can be nonconcave in a firm's debt level. Thus, small changes in a firm's characteristics or environment can cause large changes in the manager's chosen debt level. This may help explain large changes in capital structure as occur, e.g., in leveraged buyouts (LBOs) and debt-for-equity swaps. Importantly, the results are obtained without a fixed cost of debt issuance (e.g., underwriting costs) and without any equity issuance cost (e.g., due to asymmetric information).

The model combines ingredients from Harris and Raviv [47] and Titman [93]. As in Harris and Raviv [47] (but not in Titman [93]), a firm has an entrenched manager who prefers not to liquidate the firm.[1] As in Titman [93] (but not in Harris and Raviv [47]), liquidation harms the firm's small stakeholders: its customers, suppliers, and workers.[2] Hence the risk of liquidation can discourage them from doing business with the firm.[3]

[1]The idea that debt shifts control rights away from corporate insiders in bad states appears also in Aghion and Bolton [1] and Hart and Moore [48]. More generally, the notion that debt helps align the incentives of managers and firm owners is due to Jensen and Meckling [57].

[2]Empirical evidence for Titman's hypothesis appears in Titman and Wessels [94].

[3]Empirical evidence comes from Graham *et al* [43], who find that the filing of a reorganization bankruptcy raises the employee quit rate by 10-17%, and Babina [7], who finds that unexpected industry shocks raise a worker's likelihood of leaving a more levered firm by about 25%. Similarly, Brown and Matsa [15] find that distressed firms attract fewer job applicants than nondistressed firms. Andrade and Kaplan [4, p. 1475] find that about a third of distressed firms report trouble retaining key customers and suppliers. Hortaçsu *et al* [56] find that the used cars of distressed automakers fetch lower prices at auction, particularly for cars that have more time left on their warranties. In J.D. Power's 2009 Avoider Study, 18% of new car buyers who avoided a particular vehicle model cited concerns about the model's future as a reason (Hortaçsu *et al* [56]). Indeed, General Motors eliminated its Pontiac, Saturn, and Hummer brands during its most recent reorganization bankruptcy, leading many owners of these brands to switch to non-GM vehicles (Terlep [91]).

We also incorporate several other innovations relative to these early papers. Harris and Raviv [47] assume the entrenched manager never liquidates and defaults only if she must. We endogenize this decision: the entrenched manager receives control rents and chooses optimally whether to default, to liquidate, or to continue to run the firm. While Titman considers only liquidation costs, we also include costs of reorganization bankruptcies for a firm's small stakeholders.[4] While these two papers study two-period models, we study a dynamic model so as to derive predictions about jumps in debt. We incorporate techniques from global games (Carlson and van Damme [18]) to reduce indeterminacy. Finally, we give a recursive algorithm to quickly find the best and worst equilibrium. In many cases, these coincide: the equilibrium is unique.

A rough intuition for the nonconcavity of the manager's objective function is as follows. For this function to be concave, the manager's net marginal benefit of debt must be decreasing in the debt level. This net marginal benefit is simply the marginal benefit (say, due to higher interest deductions) less the marginal cost (due, e.g., to driving away the firm's small stakeholders). If, as seems plausible, the marginal benefit of debt is rising in the debt level,[5] then the marginal cost of debt must also have this property. However, this is not true in practice: the latter cost sometimes falls in the debt level. Where this happens, the manager's objective function is locally convex, so jumps in debt can result from small shocks.

How can the marginal cost of debt fall in the debt level? In our setting, this marginal cost has two sources. First, an increase in debt raises the chance of a reorganization bankruptcy, which the small stakeholders dislike. Second, a rise in

[4]Maksimovic and Titman [67] argue that even purely financial distress tends to drive away a firm's stakeholders as it lowers the firm's incentive to honor its implicit contracts with them and to protect its reputation.

[5]The reason is that as debt rises, so does the interest rate (because of the higher default risk). This raises the proportion of the debt repayment that is deductible interest rather than nondeductible principal.

debt can raise the chance of liquidation, which the small stakeholders abhor. We show by example that both effects can be lower at higher debt levels.

Our model makes several predictions that are supported in the empirical literature. We find that if a firm's stakeholders have more to lose in a reorganization bankruptcy, a firm will choose lower debt in order to maintain their loyalty. Indeed, Shivdasani and Stefanescu [87] find that firms with defined-benefit pension plans tend to have lower debt. We also find that managerial entrenchment helps a firm avoid bankruptcy, thus eliciting more participation from stakeholders. Indeed, Cen, Dasgupta, and Sen [20] and Johnson, Karpoff, and Yi [58] find that antitakeover provisions can enhance firm value by strengthening stakeholder relationships. We also find that firms with entrenched management tend to choose lower debt levels, as found empirically by Berger, Ofek, and Yermack [11] Garvey and Hanka [39], and Lundstrum [66].

Our model has strategic complementarities, which can give rise to multiple self-fulfilling prophecies. In order to reduce the risk of indeterminacy, we use tools from the theory of global games. Such games were first studied by Carlsson and van Damme [18] in the context of 2-player, 2-action games with two pure Nash equilibria. They showed that if, instead of the game's payoffs being common knowledge, each player receives a slightly noisy signal of these payoffs, there is a unique equilibrium. This result has been generalized to multiple players and actions, and to more general information and payoff structures (e.g., Frankel, Morris and Pauzner [36], Morris and Shin [72, 76]). Similar findings are obtained in dynamic games with frictions and shocks under common knowledge of payoffs (Burdzy, Frankel, and Pauzner [17]; Frankel and Pauzner [37]).[6]

In a global game, as the fundamental crosses a given threshold, aggregate behavior changes abruptly. This property makes global games useful for studying aggregate fluctuations and crises. Applications include bank runs and international contagion

[6] For limitations on the uniqueness result, see Angeletos, Hellwig, and Pavan [5], Angeletos and Werning [6], Chassang [24], Hellwig, Mukherji, and Tsyvinski [53], and Morris and Shin [73].

(Goldstein and Pauzner [42, 41]), currency crises, debt pricing, and market crashes (Morris and Shin [71, 74, 73]), search-driven business cycles (Burdzy and Frankel [16]), investment cycles (Chamley [22], Oyama [82]), neighborhood tipping (Frankel and Pauzner [38]), merger waves (Toxvaerd [95]), and recurring crises (Frankel [34]).

In our model, the benefit of debt is to shield income from taxation as in Modigliani and Miller [70]. Other possible benefits of debt include completing markets (Allen and Gale [3], Stiglitz [88]), limiting rent extraction (Bronars and Deere [14])[7], signalling firm quality (Ross [86]), and minimizing informational asymmetries (Myers and Majluf [78])[8].

2 Model

The model takes place in an infinite sequence of periods $t = 0, 1, ...$ There are two types of goods: a perishable consumption good and a durable capital good ("capital"). The price of each is normalized to one.[9] There are three types of players; all are risk-neutral and fully rational:

1. A single infinite-lived founder. In each period $t = 0, 1, ...$, the founder is endowed with a large amount of the consumption good. The firm is initially managed by the founder.

2. An infinite set of identical infinite-lived creditors $i = 1, 2, ...$, each of whom is also endowed with a large amount of the consumption good in each period. The

[7] See also Sengupta [27] and Perotti and Spear [83].

[8] See also DeMarzo and Duffie [28], DeMarzo, Frankel, and Jin [29], Leland and Pyle [65], and Nachman and Noe [79].

[9] There will be only one firm so there is no market for used capital. Rather, when the firm is liquidated, a fraction α of the capital is converted back into the consumption good and the remainder is lost.

firm borrows from at most one creditor in each period, whom we will refer to as "the creditor".

3. An infinite sequence $t = 0, 1, ...$ of generations of agents. Each generation consists of a unit measure $[0, 1]$ of agents. The agents in generation t are born at the beginning of period t and die at the end of this period. The agents will decide whether or not to do business with the firm (to "invest"); they can be interpreted as either workers, customers, or small suppliers.

Let the "manager" denote the player - either the founder or one of the creditors - who currently controls the firm. For simplicity, we assume there are no other shareholders: the manager holds all equity in the firm. Initially, the manager is the founder. If and when the founder defaults, the firm's creditor takes over as manager and then has the option of issuing debt to another creditor, and so on.

We will refer to the founder and creditors as "large players" (in contrast to the agents, who are infinitesimal). The large players discount the future at the rate $\beta \in (0, 1)$: if a large player consumes c_t in each period t, her consumption payoff is $\sum_{t=0}^{\infty} \beta^t c_t$. In addition, the founder receives a private control benefit b equal to a positive constant $b_0 > 0$ in each period in which she is in control of the firm: if she loses control in period $T+1$, her payoff in the game is $\sum_{t=0}^{\infty} \beta^t c_t + \sum_{t=0}^{T} \beta^t b_0$. As this control benefit will bias the founder against loss of control, we will also refer to the founder as the entrenched manager. Unlike the founder, a creditor who takes over as manager is not entrenched: her control benefit b is zero.

There are two types of one-period debt: corporate and personal. Corporate debt is issued by the firm to a large player other than the firm's current manager. Interest payments on corporate debt are deductible against the firm's corporate profits tax. However, there is default risk as the manager is not personally liable for the firm's corporate debt. Personal debt is risk-free and consists of one-period loans made by one large large player to another at a market interest rate ρ. Interest income on personal debt is untaxed (a normalization). As a player must be indifferent between

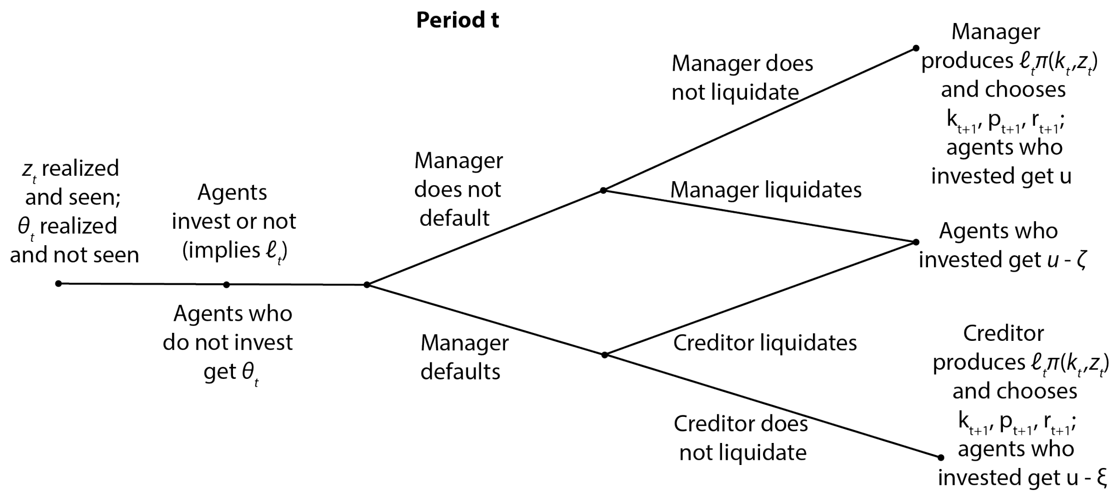

Figure 1: Timing of events in period t.

consuming a unit of the consumption good today and consuming $1+\rho$ units tomorrow, the risk-free rate must therefore equal

$$\rho = \frac{1}{\beta} - 1. \tag{1}$$

The events in period t are illustrated in Figure 1. The firm enters each period t with capital k_t, corporate debt p_t issued in period $t-1$ and due in period t, and a promised interest rate r_t on this debt. A new generation of agents is then born. Next a productivity shock $z_t \in [\underline{z}, \overline{z}] \subset \Re_{++}$ is drawn from a distribution $G(\cdot|z_{t-1})$ and commonly observed by all players. The agents' common outside option $\theta_t \sim H$ is then realized but not observed. Each agent $i \in [0,1]$ then sees a slightly noisy signal $x_i = \theta_t + \sigma\varepsilon_i$ of the outside option θ_t where $\sigma > 0$ is a constant and the random error ε_i has a smooth distribution with support contained in $[-1/2, 1/2]$. The agents then decide simultaneously whether or not to invest (to do business with the firm). An agent who does not invest gets her outside option θ_t. An agent who invests gets u if the manager neither defaults nor liquidates; $u - \xi$ if the manager defaults but the firm is not liquidated; and $u - \zeta$ if the firm is liquidated (either by the manager or the creditor). We assume $u > \zeta > \xi > 0$ are constants. We assume moreover that the outside option θ_t is uniformly distributed on $[\underline{\theta}, \overline{\theta}]$ where the lower bound $\underline{\theta}$ is

$u - \zeta - \varepsilon$, the upper bound $\overline{\theta}$ is $u + \varepsilon$, and

$$\varepsilon \in (0,1) \tag{2}$$

is a small fixed constant. We assume the density of H is bounded: there is an $\overline{h} < \infty$ such that, for each $\theta < \theta'$,

$$H(\theta') - H(\theta) \leq (\theta' - \theta)\,\overline{h}. \tag{3}$$

Let $\ell_t \in [0,1]$ be the proportion of agents in period t who invest, which we will call the "participation rate".

The incumbent manager (the large player who is in control of the firm at the beginning of period t), having observed the shock z_t and the participate rate ℓ_t, then decides whether or not to default. If she does not default, she repays the creditor $p_t(1 + r_t)$ of which the interest payment $r_t p_t$ is deductible against the firm's corporate income taxes.[10] If she defaults, she pays nothing to the creditor and gets a continuation payoff of zero; the creditor then takes over the firm (as the new manager) and forgives the debt p_t that he now owes to himself.

After the default decision has been made, the manager (who is either the incumbent or, following a default, the creditor) decides whether or not to liquidate the firm. Liquidation produces αk_t in revenue. If the manager does not liquidate, she receives $\ell_t \pi(k_t, z_t)$ in operating revenue where, for any shock z, per-agent revenue $\pi(k, z)$ is positive (resp., nonnegative) for all $k > 0$ (resp., $k \geq 0$) and nondecreasing in z (and increasing in z if $k > 0$). An example is the functional form $\pi(k, z) = zk^{\nu}$. She then chooses new capital and debt levels k_{t+1} and p_{t+1} respectively, as well as an interest rate r_{t+1} to offer on her new debt. She can also freely inject the consumption good from her endowment in order to cover any cash-flow shortfall of the firm.[11]

[10]The firm cannot deduct personal interest payments made by its manager from its corporate taxes.

[11]As the manager is the sole owner of the firm, this is equivalent to issuing new shares to herself. As debt is tax-preferred, we assume the manager cannot lend to her own firm.

If the firm's short-term debt p is negative, the cash balance $-p > 0$ can be lent (tax-free) at the risk-free rate ρ but the firm receives only a proportion $\eta \in (0,1)$ of the gross repayment, which captures the agency cost of free cash flow.[12] Thus,

$$\phi(p,r) = p\left[(1 + r(1-\tau_c))\, 1_{p>0} + \eta(1+\rho)\, 1_{p<0}\right] \tag{4}$$

is the firm's after-tax cost of repaying short-term debt with face value p and interest rate r where

$$\tau_c > 0 \tag{5}$$

is the corporate tax rate. Let

$$e(k_t, k_{t+1}, p_t, p_{t+1}, \ell_t, z_t, r_t) = \begin{cases} (1-\tau_c)\,\ell_t \pi(k_t, z_t) + p_{t+1} - \phi(p_t, r_t) \\ +\delta k_t \tau_c - \beta\left[(k_{t+1} - (1-\delta)k_t) + A(k_t, k_{t+1})\right] \end{cases} \tag{6}$$

denote the period-t cash flow plus the discounted cost of installing new capital $k_{t+1} - (1-\delta)k_t$, where

$$A(k_t, k_{t+1}) = \frac{a_1}{2}\left[\frac{k_{t+1} - (1-\delta)k_t}{k_t}\right]^2 k_t + (1-\alpha)\left[(1-\delta)k_t - k_{t+1}\right] 1_{k_{t+1} < (1-\delta)k_t} \tag{7}$$

is the capital adjustment cost where $a_0, a_1 > 0$.[13,14] We assume no cost of equity issuance: if the cash flow e is negative, the manager simply makes up the difference

[12] The role of this assumption is merely to ensure that the manager pays any positive cash balance of the firm to herself as a dividend: in equilibrium, p is never negative. If η were one, the manager would be indifferent between holding cash personally versus within the firm so the model would be indeterminate.

[13] The cost $(k_{t+1} - (1-\delta)k_t) + A(k_t, k_{t+1})$ of installing new capital is assumed to be incurred at the beginning of period $t+1$.

[14] The role of the second term is to avoid a bias against liquidation: in the limit as $k_{t+1} \to 0$ it becomes $(1-\alpha)(1-\delta)k_t$ which equals the capital lost, $(1-\alpha)k_{t+1}$, if the firm invests nothing (i.e., sets $k_{t+1} = (1-\delta)k_t$) and then liquidates in period $t+1$.

from her endowment of the consumption good. By (6), the corporate tax payment from the manager to the tax authority in period t is

$$T(k_t, p_t, \ell_t, z_t, r_t) = \tau_c \left[\ell_t \pi(k_t, z_t) - \delta k_t - r_t p_t 1_{p_t > 0}\right]$$

which we assume to be bounded below by a finite constant $\underline{\tau} \in (-\infty, 0]$; for instance, $-\underline{\tau} \geq 0$ might be the maximum tax refund per year.

We assume the interest rate r_t, capital k_t, and debt p_t are selected from compact sets $[\underline{r}, \overline{r}]$, $[\underline{k}, \overline{k}]$, and $[\underline{p}, \overline{p}]$, respectively, where[15]

$$\underline{r} \leq 0 < \rho < \overline{r},$$

$$0 < \underline{k} < \overline{k}, \text{ and}$$

$$\underline{p} \leq 0 < \overline{p}.$$

Choices that lie in these sets are called "admissible". In period t, a manager with control benefit $b_t \in \{0, b_0\}$ maximizes the value function

$$V(k_t, p_t, \ell_t, z_t, r_t, b_t) = \max \left\{ \underbrace{0}_{\text{default}}, \overbrace{\alpha k_t - \phi(p_t, r_t)}^{\text{liquidate}}, \overbrace{V^c(k_t, p_t, \ell_t, z_t, r_t, b_t)}^{\text{continue}} \right\} \tag{8}$$

where

$$V^c(k_t, p_t, \ell_t, z_t, r_t, b_t) = \max_{(k_{t+1}, p_{t+1}, r_{t+1}) \in \Gamma(z_t, b_t)} \left\{ \begin{array}{c} b_t + e(k_t, k_{t+1}, p_t, p_{t+1}, \ell_t, z_t, r_t) \\ +\beta E_t V(k_{t+1}, p_{t+1}, \ell_{t+1}, z_{t+1}, r_{t+1}, b_t) \end{array} \right\} \tag{9}$$

is her payoff from continuing to operate the firm as a going concern and $\Gamma(z_t, b_t)$, given formally below in (12), is the set of capital-debt-interest choices $(k_{t+1}, p_{t+1}, r_{t+1})$ that

[15]Compactness is needed for technical reasons; one can select the bounds to be far enough apart that the constraints never bind. In particular, $\underline{r}$ and $\underline{p}$ can be large and negative, while $\overline{r}$, $\overline{k}$, and $\overline{p}$ are large and positive. The lower bound $\underline{k}$ on capital is also needed for technical reasons and can be arbitrarily close to zero. Finally, we assume $\underline{r} \leq 0$ so that we can treat the interest rate as zero when there is no debt; the results would be the same with $\underline{r}$ equal to the risk-free rate ρ.

are feasible for the manager if she does not default or liquidate in period t.[16]

If the firm defaults in period t, the creditor takes over the firm. If the creditor does not liquidate, he gets a continuation payoff of $V^c(k_t, 0, \ell_t, z_t, 0, 0)$. If he does liquidate, he gets the liquidation proceeds αk_t. Hence, the creditor's expected payoff in period $t+1$ from lending p_{t+1} at the rate r_{t+1} in period t is

$$\Psi(k_{t+1}, p_{t+1}, r_{t+1}, z_t, b_t) \stackrel{d}{=} E\left[\begin{array}{c} (1 - d_{t+1})\, p_{t+1}(1 + r_{t+1}) \\ +d_{t+1} V(k_{t+1}, 0, \ell_{t+1}, z_{t+1}, 0, 0) \end{array} \middle| k_{t+1}, p_{t+1}, r_{t+1}, z_t, b_t \right] \tag{10}$$

where d_{t+1} is one if the manager defaults in period $t+1$ and zero otherwise. As the creditor's outside option is to lend p_{t+1} at the risk-free rate , the creditor will lend the manager p_{t+1} as long as

$$\Psi(k_{t+1}, p_{t+1}, r_{t+1}, z_t, b_t) \geq p_{t+1}(1 + \rho). \tag{11}$$

Accordingly, the creditor's set of acceptable capital-debt-interest triplets, mentioned above, is given formally by

$$\Gamma(z_t, b_t) = \{(k_{t+1}, p_{t+1}, r_{t+1}) \in B^c : \Psi(k_{t+1}, p_{t+1}, r_{t+1}, z_t, b_t) \geq p_{t+1}(1 + \rho)\} \tag{12}$$

where we define

$$B^c = [\underline{k}, \overline{k}] \times [\underline{p}, \overline{p}] \times [\underline{r}, \overline{r}] \tag{13}$$

to be the set of feasible capital-debt-interest triplets.[17]

[16] The set $\Gamma(z, b)$ is never empty as it includes $(k, 0, r)$ for all k, r: the manager can always borrow nothing. On the other hand, a triplet (k, p, r) with debt $p > 0$ is in $\Gamma(z, b)$ only if a creditor is willing to lend p to a manager with control parameter b at the interest rate r if the manager's planned capital is k and the current shock is z.

[17] The superscript "c" is used to distinguish the set from a discrete version of the set B^d, which we will define below.

3 Defining Equilibrium

Our solution concept is Perfect Bayesian Equilibrium (PBE) together with a mild monotonicity property, defined below in Definition 1. Moreover, we focus on the limit as the agents' signal errors shrinks to zero: as $\sigma \to 0$. In this limit, given the strategies of the manager and creditors, the agents' behavior is uniquely determined. Our simulations suggest that this crucial simplification leads to a unique outcome of the game as a whole.

We first show that in any PBE, the optimal default and liquidation decisions are given by threshold strategies.

Claim 1. In any PBE, there is a threshold function $D(k_t, p_t, z_t, r_t, b_t) \in \Re$ such that the manager defaults (resp., does not default) if $\ell_t < (\geq) D(k_t, p_t, z_t, r_t, b_t)$.[18] The function D is given by[19]

$$D(k_t, p_t, z_t, r_t, b_t) = \min \left\{ \{1+\varepsilon\} \cup \left\{ \ell \in [0,1] : \ \phi(p_t, r_t) \leq V(k_t, 0, \ell_t, z_t, 0, b_t) \ \right\} \right\} \in [0,1] \cup \{1+\varepsilon\} \quad (14)$$

Proof. By (8), a default threshold strategy is optimal as $e(k_t, k_{t+1}, p_t, p_{t+1}, \ell_t, z_t, r_t)$ is increasing in ℓ_t and by the envelope theorem. The optimal threshold is the least ℓ for which the manager is willing not to default in period t if she believes that, given a decision not to default or to liquidate, in period $t+1$ she will choose a triplet $(k_{t+1}, p_{t+1}, r_{t+1})$ in the constraint set $\Gamma(z_t, b_t)$. This implies that the manager's

[18]If $\ell_t = D(k_t, p_t, z_t, r_t, b_t)$, the manager is indifferent; we assume in this case that she does not default.

[19]The constant ε is defined in (2). The presence of $\{1+\varepsilon\}$ ensures that, if the second set in the min is empty, then $D(k_t, p_t, z_t, r_t, b_t)$ is set to $1+\varepsilon$. This ensures that $\ell_t \in [0,1]$ is always less than $D(k_t, p_t, z_t, r_t, b_t)$ and thus the manager never continues.

optimal default threshold is given by the function

$$
\begin{aligned}
&D(k_t, p_t, z_t, r_t, b_t) \\
&= \min \left\{ \{1+\varepsilon\} \cup \left\{ \begin{array}{l} \ell_t \in [0,1] \text{ such that} \\ \underbrace{\text{default}}_{0} \le \max \left\{ \overbrace{\alpha k_t - \phi(p_t, r_t)}^{\text{liquidate}}, \overbrace{V^c(k_t, p_t, \ell_t, z_t, r_t, b_t)}^{\text{continue}} \right\} \end{array} \right\} \right\} \\
&\qquad = \min \left\{ \{1+\varepsilon\} \cup \left\{ \ell_t \in [0,1] : \ \phi(p_t, r_t) \le V(k_t, 0, \ell_t, z_t, 0, b_t) \right\} \right\} \\
&\qquad\qquad \in [0,1] \cup \{1+\varepsilon\}. \quad (15)
\end{aligned}
$$

as claimed. □

Recall that the incumbent manager first decides whether or not to default. If she does not default, then she repays her debt p_t and makes an optimal liquidation decision. Since the debt has been repaid, it is not relevant to the liquidation decision. We now show, moreover, that it is optimal for the incumbent manager to default if and only if the participation rate ℓ_t is not less than some threshold $L(k_t, z_t, b_t)$.

Claim 2. In any PBE, if the incumbent manager does not default, she will liquidate (resp., will not liquidate) if $\ell_t < (\ge) L(k_t, z_t, b_t)$ where[20]

$$
L(k_t, z_t, b_t) = \min \left\{ \{1+\varepsilon\} \cup \left\{ \ell_t \in [0,1] : \ \alpha k_t \le V^c(k_t, 0, \ell_t, z_t, 0, b_t) \right\} \right\} \in [0,1] \cup \{1+\varepsilon\}. \quad (16)
$$

Proof. By (8) and (9), continuing yields a higher payoff than liquidating if and only if

$$
\alpha k_t - \phi(p_t, r_t) < \max_{(k_{t+1}, p_{t+1}, r_{t+1}) \in \Gamma(z_t, b_t)} \left\{ \begin{array}{c} b + e(k_t, k_{t+1}, p_t, p_{t+1}, \ell_t, z_t, r_t) \\ +\beta E_t V(k_{t+1}, p_{t+1}, \ell_{t+1}, z_{t+1}, r_{t+1}, b_t) \end{array} \right\}
$$

[20]If $\ell_t = L(k_t, z_t, b_t)$, the manager is indifferent; we assume in this case that she does not liquidate.

which, adding $\phi(p_t, r_t)$ to both sides, is equivalent to

$$\alpha k_t < \max_{(k_{t+1},p_{t+1},r_{t+1})\in\Gamma(z_t,b_t)} \left\{ \begin{array}{c} b + e(k_t, k_{t+1}, 0, p_{t+1}, \ell_t, z_t, 0) \\ +\beta E_t V(k_{t+1}, p_{t+1}, \ell_{t+1}, z_{t+1}, r_{t+1}, b_t) \end{array} \right\}$$
$$= V^c(k_t, 0, \ell_t, z_t, 0, b_t)$$

by (9). The claim follows as $e(k_t, k_{t+1}, 0, p_{t+1}, \ell_t, z_t, 0)$ is increasing in ℓ_t and by the envelope theorem. □

If the incumbent manager defaults, the creditor takes over, forgives the debt p_t that the firm owes him, and makes an optimal liquidation decision. He thus faces the same liquidation decision as the incumbent manager would if her control parameter b_t were 0. This implies the following result.

Corollary 1. *In any PBE, if the incumbent manager defaults, the creditor will liquidate (resp., will not liquidate) if* $\ell_t < (\geq) L(k_t, z_t, 0)$ *where the threshold function* L *is given by (16).*

We now introduct our first restriction on the set of PBE: a mild monotonicity property.

Definition 1. A PBE is monotone if the manager's payoff from continuing to operate the firm is nondecreasing in her control rent parameter b: for any (k, p, ℓ, z, r),

$$V^c(k, p, \ell, z, r, b_0) \geq V^c(k, p, \ell, z, r, 0). \tag{17}$$

Claim 3. In any monotone PBE, D and L are nonincreasing in b and V is nondecreasing in b.

Proof. By (6), $e(k_t, k_{t+1}, p_t, p_{t+1}, \ell_t, z_t, r_t)$ is increasing in ℓ_t. Hence, by the envelope theorem and (9), V^c is increasing in ℓ_t. Accordingly, by (15) and (16), D and L are nonincreasing in b. Finally, (8) implies that V is nondecreasing in b whenever V^c is. □

We now show that in any monotone PBE, there is a threshold

$$\Lambda_t = \Lambda(k_t, p_t, z_t, r_t, b_t)$$

such that a firm, whose manager has parameter b, is liquidated if and only if $\ell_t < \Lambda_t$. Define the state at period t to be $s_t = (k_t, p_t, \ell_t, z_t, r_t, z_{t-1}, b_t)$ where b_t is the control parameter of the incumbent manager. Moreover, if a function f depends only on a subset of the arguments in s_t, we may either explicitly list this subset as arguments or simply write $f(s_t)$. For instance, we may write the function D defined in (15) either as $D(k_t, p_t, z_t, r_t, b_t)$ or simply as $D(s_t)$. Let

$$\Lambda(s_t) = \min\{L(k_t, z_t, 0), \max\{D(s_t), L(k_t, z_t, b_t)\}\}. \tag{18}$$

Claim 4. In any monotone PBE, the firm is liquidated if and only if the participation rate ℓ_t is less than $\Lambda(s_t)$. The manager defaults and the firm is not liquidated if ℓ_t is at least $\Lambda(s_t)$ but less than $\max\{D(s_t), \Lambda(s_t)\}$.

Proof. By Claim 3, L is nonincreasing in b. Hence, there are three cases.[21]

1. If $D(s_t) < L(k_t, z_t, b_t)$ then the firm will be liquidated if and only if ℓ_t is less than $L(k_t, z_t, b_t)$ which, in this case, equals $\Lambda(s_t)$.

2. If $L(k_t, z_t, b_t) \leq D(s_t) < L(k_t, z_t, 0)$ then the firm will be liquidated if and only if ℓ_t is less than $D(s_t)$ which, in this case, equals $\Lambda(s_t)$.

3. If $D(s_t) \geq L(k_t, z_t, 0)$ then the firm will be liquidated if and only if ℓ_t is less than $L(k_t, z_t, 0)$ which, in this case, equals $\Lambda(s_t)$.

□

We now face our second and final restriction on the set of PBEs. It is needed as the agents face a coordination problem: if more of them invest, this may make

[21]See diagram17Jul2020.jpg in mvs folder.

the firm less likely to be liquidated, which makes investing more worthwhile for the agents. Thus, there can be multiple self-fulfilling prophecies. In order to obtain a unique prediction for the agents' behavior, we focus hencefroth on the limit as the agents' signals become precise. In this limit, the agents care only about the incumbent manager's default threshold $D(s_t)$ and the firm's liquidation threshold $\Lambda(s_t)$:

Claim 5. In any monotone PBE in the limit as the agents' signals become precise (as $\sigma \to 0$), the agents invest if and only if their outside option θ_t does not exceed the threshold

$$\Omega(s_t) = u - (\zeta - \xi)\min\{1, \Lambda(s_t)\} - \xi \min\{1, \max\{D(s_t), \Lambda(s_t)\}\}. \tag{19}$$

Proof. An agent's realized net payoff from investing is $u - \theta_t$ if

$$\ell_t \geq \max\{D(s_t), \Lambda(s_t)\}.$$

It is $u - \xi - \theta_t$ if $\max\{D(s_t), \Lambda(s_t)\} > \ell_t \geq \Lambda(s_t)$ and $u - \zeta - \theta_t$ if $\ell_t < \Lambda(s_t)$. There are dominance regions: it is strictly dominant (not) to invest if $\theta_t < u - \zeta$ (resp., $\theta_t > u$) and such realizations have positive probability since $\theta_t \sim U[\underline{\theta}, \overline{\theta}]$ where $\underline{\theta} < u - \zeta$ and $\overline{\theta} > u$. By standard global games results, in the limit as the signal errors shrink to zero ($\sigma \to 0$), a new agent invests if and only if doing so is optimal on the belief that $\ell_t \sim U[0,1]$: if and only if $u - \theta_t - \int_{\ell=0}^{\Lambda(s_t)} \zeta d\ell - \int_{\ell=\Lambda(s_t)}^{\max\{D(s_t),\Lambda(s_t)\}} \xi d\ell > 0$ or, equivalently, $\theta_t < u - \zeta \min\{1, \Lambda(s_t)\} - \xi[\min\{1, \max\{D(s_t), \Lambda(s_t)\}\} - \min\{1, \Lambda(s_t)\}]$ which, reorganized, yields (19). □

By Claim 5, in any monotone PBE in the limit as the signal errors vanish, the proportion ℓ_t of agents who invest in period t is zero (one) if $\theta_t > (<)\, \Omega(s_t)$. Thus, written in terms of the distribution H of the agents' outside option θ_t, the expected period-t continuation payoff of a manager with control parameter b can be written as

$$E_{t-1}V(k_t, p_t, \ell_t, z_t, r_t, b_t) = W(k_t, p_t, r_t, z_{t-1}, b_t) \tag{20}$$

where we define

$$W\left(k_t, p_t, r_t, z_{t-1}, b_t\right) \stackrel{d}{=} \int_{\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} V\left(k_t, p_t, 1, z_t, r_t, b_t\right) H\left(\Omega\left(k_t, p_t, z_t, r_t, b_t\right)\right) \\ +V\left(k_t, p_t, 0, z_t, r_t, b_t\right)\left[1 - H\left(\Omega\left(k_t, p_t, z_t, r_t, b_t\right)\right)\right] \end{array} \right\} dG\left(z_t | z_{t-1}\right). \quad (21)$$

Intuitively, with probability $H\left(\Omega\left(k_t, p_t, z_t, r_t, b_t\right)\right)$, the agents' outside option θ_t is less than their investment threshold $\Omega\left(k_t, p_t, z_t, r_t, b_t\right)$: the proportion ℓ_t of agents who invest is 1, so the manager gets $V\left(k_t, p_t, 1, z_t, r_t, b_t\right)$. With complementary probability $1 - H\left(\Omega\left(k_t, p_t, z_t, r_t, b_t\right)\right)$, the agents' outside option θ_t exceeds their investment threshold $\Omega\left(k_t, p_t, z_t, r_t, b_t\right)$: the proportion ℓ_t of agents who invest is zero, so the manager gets $V\left(k_t, p_t, 0, z_t, r_t, b_t\right)$.

By identical reasoning, the creditors' expected continuation payoff from lending in period $t-1$ (equation (10)) can be rewritten as

$$\Psi\left(k_t, p_t, r_t, z_{t-1}, b_t\right) \stackrel{d}{=} \int_{z_t=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \Upsilon\left(k_t, p_t, 1, z_t, r_t, b_t\right) H\left(\Omega\left(k_t, p_t, z_t, r_t, b_t\right)\right) \\ +\Upsilon\left(k_t, p_t, 0, z_t, r_t, b_t\right)\left[1 - H\left(\Omega\left(k_t, p_t, z_t, r_t, b_t\right)\right)\right] \end{array} \right\} dG\left(z_t | z_{t-1}\right) \quad (22)$$

where, by Claim 1,

$$\Upsilon\left(k_t, p_t, \ell_t, z_t, r_t, b_t\right) = 1_{\ell_t \geq D\left(k_t, p_t, z_t, r_t, b_t\right)} p_t\left(1 + r_t\right) + 1_{\ell_t < D\left(k_t, p_t, z_t, r_t, b_t\right)} V\left(k_t, 0, \ell_t, z_t, 0, 0\right). \quad (23)$$

is the creditor's continuation payoff given both the shock z_t and the proportion ℓ_t of agents who invest.

Having defined the above functions, we now formally define our solution concept.

Definition 2. A monotone PBE in the limit as the agents' signal errors vanish is a vector $\left(\Gamma, V, V^c, D, L, \Lambda, \Omega, W, \Psi, \Upsilon\right)$ of functions satisfying the definitions and equilibrium conditions (12), (8), (9), (14), (16), (18), (19), (21), (22), and (23), respectively, as well as the monotonicity condition (17).

For brevity, we will use “monotone PBE” to refer to a monotone PBE in the limit as the agents’ signal errors vanish.

4 Characterizing and Finding the Equilibria

Our global games approach yields a unique outcome in the agents’ investment game in every period. However, it does not rule out the possibility of multiple monotone PBEs in the full dynamic model. Intuitively, there are strategic complementarities between the agents (viewed as a group) and the manager and creditors. The agents are more willing to invest in period t if they expect the firm not to be liquidated in period t. But if the agents are more willing to invest in period $t+1$, this makes the firm less likely to be liquidated in period t since its continuation value is higher. In this way, one may have high-agent-investment, low-liquidation equilibria coexisting with low-agent-investment, high-liquidation equilibria.[22]

In this section, we derive some features of the most and least optimistic such PBEs and provide a recursive algorithm for finding these extreme equilibria. When the extreme equilibria concide, they are the unique monotone PBE. This appears to be common in practice.

By the above construction, in each period t, the state $s_t = (k_t, p_t, \ell_t, z_t, r_t, z_{t-1}, b_t)$ lies in the compact state space

$$S^c = S^c_k \times S^c_p \times S^c_\ell \times S^c_z \times S^c_r \times S^c_z \times \{0, b_0\} \tag{24}$$

where “c” denotes “continuous” and the components of S^c are

$$S^c_k = \left[\underline{k}, \overline{k}\right], \; S^c_p = \left[\underline{p}, \overline{p}\right], \; S^c_\ell = \left[\underline{\ell}, \overline{\ell}\right], \; S^c_z = \left[\underline{z}, \overline{z}\right], \text{ and } S^c_r = \left[\underline{r}, \overline{r}\right].$$

[22] There are also strategic complementarities between the manager and the creditors: the creditors will lend at a lower interest rate if they expect the manager not to default in the next period which, in turn, is more likely if the manager expects to be able to refinance at better terms.

Some of our technical results require a discrete state space. Thus, we will also consider alternative state spaces of the form

$$S^d = S_k^d \times S_p^d \times S_\ell^d \times S_z^d \times S_r^d \times S_z^d \times \{0, b_0\}$$

where S_ℓ^d is the evenly spaced grid $\left\{0, \frac{1}{N}, ..., \frac{N}{N}\right\}$ (a subset of S_ℓ^c) for some integer $N > 1$; the other components of S^d are discrete and nonempty, and satisfy

$$S_k^d \subset S_k^c,\ 0 \in S_p^d \subset S_p^c,\ S_z^d \subset S_z^c, \text{ and } \{0, \rho\} \subseteq S_r^d \subset S_r^c.$$

Finally, let B^d denote the set $S_k^d \times S_p^d \times S_r^d$ of all permissible triplets (k, p, r); it is the discrete analogue of the set B^c defined in (13).

Restricting the participation rate ℓ to $\left\{0, \frac{1}{N}, ..., \frac{N}{N}\right\}$ corresponds to assuming that there is a finite number N of agents. In this case, $D(s)$ and $L(s)$ will each be an integer multiple of $1/N$ or will equal $1 + \varepsilon$. By standard global games results, as the agents' signal errors shrink to zero, an agent invests if and only if doing so is optimal if she believes that each number m of other agents who invest is equally likely to take each value from zero to $N - 1$. Since the agent's payoff from investing depends on ℓ_t only if she also invests, she invests if and only if doing so is optimal under the belief that, conditional on herself investing, the participation rate ℓ_t is equally likely to take each of the values in $\left\{\frac{1}{N}, ..., \frac{N}{N}\right\}$. An agent with such beliefs will invest if and only if her outside option θ_t is less than $\Omega^N(s_t)$ which converges to $\Omega(s_t)$, uniformly on S, in the limit as $N \to \infty$.[23] We can thus use (19) as an approximation of the agents' investment threshold.

[23] Why? Let $c_1 = \min\{1, \Lambda(s_t)\}$ and $c_2 = \min\{1, \max\{D(s_t), \Lambda(s_t)\}\}$. The agent will invest if her outside option θ_t is less than

$$\begin{aligned} u - \frac{1}{N-1}\sum_{m=1}^{Nc_1-1}\zeta - \frac{1}{N-1}\sum_{m=Nc_1}^{Nc_2-1}\xi &= u - \frac{Nc_1 - 1}{N-1}\zeta - \frac{N(c_2 - c_1)}{N-1}\xi \\ &= u - \frac{(N-1)c_1 + c_1 - 1}{N-1}\zeta - \frac{(N-1)(c_2 - c_1) + c_2 - c_1}{N-1}\xi \end{aligned}$$

which, by (19), equals $\Omega(s_t) + \varepsilon^N$ where $\varepsilon^N = \frac{1-c_1}{N-1}\zeta - \frac{c_2-c_1}{N-1}\xi$ converges to zero as N grows. Moreover, the convergence is clearly uniformly in $c_1, c_2 \in [0, 1]$ and thus in s_t.

We will sometimes drop time subscripts, writing simply $s = (k, p, \ell, z, r, z_{-1}, b)$ where z_{-1} refers to the shock from the prior period. Moreover, if a function f depends only on a subset of the arguments in s, we may either explicitly list this subset as arguments or simply write $f(s)$. For instance, we may write the function Υ defined in (23) either as $\Upsilon(k, p, \ell, z, r, b)$ or simply as $\Upsilon(s)$. Real-valued functions are ordered in the usual way; e.g., "Ω is lower than Ω'" means "$\Omega(s) \leq \Omega'(s)$ at every state s in S". Set-valued functions are ordered by inclusion: "Γ is lower than Γ'" is equivalent to "$\Gamma(s) \subseteq \Gamma'(s)$ for each state s in S".

A problem that arises is that while greater optimism raises some equilibrium functions,[24] it lowers the chance of default and liquidation, which entails a *decline* in the default threshold D and the liquidation thresholds L and Λ. In our iterations, it will be more convenient for all functions to be rising in players' optimism. To obtain this feature, we simply multiply these exceptional functions by -1: we will work with $-D$, $-L$, and $-\Lambda$.

Let $(S, B, S_k, S_p, S_\ell, S_z, S_r, S_z)$, denote either the continuous state spaces

$$\left(S^c, B^c, S^c_k, S^c_p, S^c_\ell, S^c_z, S^c_r, S^c_z\right)$$

or their discrete counterpart $\left(S^d, B^d, S^d_k, S^d_p, S^d_\ell, S^d_z, S^d_r, S^d_z\right)$. We next combine the functions D, L, Λ, Ω, V^c, V, W, Υ, Ψ, and Γ, into a single vector-valued function F:

[24] In particular, greater optimism about the willingness of the manager and creditor not to liquidate the firm tends to raise the threshold Ω below which the agents will invest. Similarly, greater optimism about the agents' willingness to invest and the creditors' willingness to lend makes it more worthwhile for the manager to continue to operate the firm, thus raising her continuation payoff V and its expectation W. Similarly, greater optimism about the agents' willingness to invest and the manager's willingness to repay her loan makes it more worthwhile for the creditor to lend, thus raising his expected payoff Ψ from lending and expanding the set Γ of capital-debt-interest pairs that the is willing to finance as well as the range Γ of interest rates that he would accept.

for any state s in S, let

$$F(s) = (-D(s), -L(s), -\Lambda(s), \Omega(s), V^c(s), V(s), W(s), \Upsilon(s), \Psi(s), \Gamma(s)) \tag{25}$$

which lies in

$$\Sigma = \Re^9 \times 2^B. \tag{26}$$

The function

$$F = (-D, -L, -\Lambda, \Omega, V^c, V, W, \Upsilon, \Psi, \Gamma) \tag{27}$$

is a function from the state space S into Σ. Let Φ be the set of all functions $F : S \to \Sigma$.

While we permit the functions in F (other than Γ) to take any real values, in our iterations we will need to bound them. We can bound the continuation payoffs V and Ψ of the manager and creditor, respectively. Let welfare be the sum of these payoffs. Let $\overline{S}$ be the highest such welfare that is attainable in period t at any state (k_t, p_t, r_t, z_{t-1}) in a world without adjustment costs under the most optimistic possible assumptions: (a) all agents invest in every period, (b) capital k is at its maximum $\overline{k}$ and does not depreciate, (c) the shock z always equals its maximum value $\overline{z}$, (d) the control benefit b equals its maximum value b_0, and (e) the firm gets its maximum tax subsidy of $-\underline{\tau}$ in each period. There are two cases. First, if welfare is maximized by liquidation, $\overline{S} = \alpha\overline{k}$. Second, if welfare is maximized by continuing, $\overline{S} = b_0 + \pi\left(\overline{k}, \overline{z}\right) - \underline{\tau} + \beta\overline{S}$. Accordingly,

$$\overline{S} = \max\left\{\alpha\overline{k}, b_0 + \pi\left(\overline{k}, \overline{z}\right) - \underline{\tau} + \beta\overline{S}\right\}.$$

Solving, we obtain an upper bound on welfare:

$$\overline{S} \overset{d}{=} \max\left\{\alpha\overline{k}, \frac{1}{1-\beta}\left[b_0 + \pi\left(\overline{k}, \overline{z}\right) - \underline{\tau}\right]\right\}. \tag{28}$$

By limited liability, the continuation payoffs V and Ψ of the manager and creditor, respectively, must both lie in $\left[0, \overline{S}\right]$.

We next define two best-response functions from Φ to Φ. When iterating rom above, we will use the optimistic best response function $F \to \overline{b}_F$. When iterating from below, we will use its pessimistic counterpart $F \to \underline{b}_F$. The functions differ only in what players do when they have more than one optimal action. Under the optimistic (resp., pessimistic) best response function, in such a situation the player takes her highest (lowest) acceptable action. This is the action that most (least) benefits the other players and thus that encourages them to take their highest (lowest) acceptable actions as well.

Let F in Φ be any function vector of the form (27) and let $s = (k, p, \ell, z, r, z_{-1}, b)$ be any state in S.[25] The optimistic best response function is the row vector

$$\overline{b}_F(s) = \left(\begin{array}{l} -\underline{d}_F(s), -\underline{l}_F(s), -\underline{\lambda}_F(s), \overline{\omega}_F(s), \\ v_F^c(s), v_F(s), w_F(s), \overline{\upsilon}_F(s), \overline{\psi}_F(s), \overline{\gamma}_F(s) \end{array} \right)$$

while the pessimistic best response function is the row vector

$$\underline{b}_F(s) = \left(\begin{array}{l} -\overline{d}_F(s), -\overline{l}_F(s), -\overline{\lambda}_F(s), \underline{\omega}_F(s), \\ v_F^c(s), v_F(s), w_F(s), \underline{\upsilon}_F(s), \underline{\psi}_F(s), \underline{\gamma}_F(s) \end{array} \right).$$

(As indicated, the components in places 5, 6, and 7 do not differ between the two vectors.)

The components of $\overline{b}_F(s)$ and $\underline{b}_F(s)$ are as follows. In each definition, any occurrence of k, p, ℓ, z, r, z_{-1}, or b on the right hand side refers to that component of the argument s while any occurrence of D, L, Λ, Ω, V^c, V, W, Υ, Ψ, or Γ on the right hand side refers to the given component of the argument F. The idea of these functions is that we assume players have beliefs about how they and others will behave in period $t+1$. We then solve for the players' optimal actions in period t via backwards induction until we reach the beginning of period t.

In particular, the last event in period t is the creditor's acceptance or rejection of the manager's proposal $(k_{t+1}, p_{t+1}, r_{t+1})$. If the creditor accepts, his realized payoff

[25] As a mnemonic, we have used the lower case version of each function on the right hand side of (25) to denote the best response to that function.

function (corresponding to Υ) in period $t+1$, if the other equilibrium functions in period $t+1$ are given by F and if he expects the manager to repay the debt whenever she is willing to do so, is

$$\overline{\upsilon}_F(s) = \ 1_{\ell \geq D(s)} p (1+r) + 1_{\ell < D(s)} V(k, 0, \ell, z, 0, 0) \tag{29}$$

by (23). In this definition, the variables ℓ, p, r, k, and z on the right hand side are components of the argument s of $\upsilon_F(s)$ while the functions D and V that appear on the right hand side are components of the argument F of $\upsilon_F(s)$. On the other hand, if the creditor expects the manager to default whenever she is willing to do so, the creditor's realized payoff function from accepting is instead

$$\underline{\upsilon}_F(s) = \ 1_{\ell > D(s)} p (1+r) + 1_{\ell \leq D(s)} V(k, 0, \ell, z, 0, 0) \tag{30}$$

by (23).

The creditor's best-response *expected* payoff function from lending (corresponding to Ψ) in period t, given the next period's equilibrium functions D, V, and Ω (which are components of F), is thus

$$\overline{\psi}_F(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{\upsilon}_F(k, p, 1, z', r, z_{-1}, b) H(\Omega(k, p, z', r, b)) \\ +\overline{\upsilon}_F(k, p, 0, z', r, z_{-1}, b) [1 - H(\Omega(k, p, z', r, b))] \end{array} \right\} dG(z'|z_{-1}) \in [0, \overline{S}] \tag{31}$$

by (22) if she is optimistic about the manager's default decision and

$$\underline{\psi}_F(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \underline{\upsilon}_F(k, p, 1, z', r, z_{-1}, b) H(\Omega(k, p, z', r, b)) \\ +\underline{\upsilon}_F(k, p, 0, z', r, z_{-1}, b) [1 - H(\Omega(k, p, z', r, b))] \end{array} \right\} dG(z'|z_{-1}) \in [0, \overline{S}] \tag{32}$$

if she is pessimistic. Accordingly, the creditor's acceptance function (corresponding to Γ) in period t, if the other equilibrium functions in period $t+1$ are given by F, is

$$\overline{\gamma}_F(s) = \left\{(k', p', r') \in B : \overline{\psi}_F(k', p', r', z, b) \geq p'(1+\rho)\right\} \subseteq B \tag{33}$$

if he is optimistic (by (12)) and

$$\underline{\gamma}_F(s) = \left\{ (k', p', r') \in B : \underline{\psi}_F(k', p', r', z, b) \geq p'(1+\rho) \right\} \subseteq B \tag{34}$$

if pessimistic. Working backwards in period t, the manager's best-response *expected* continuation payoff function (corresponding to W) in the present period, if the equilibrium functions V and Ω in the next period are as specified in F, is

$$w_F(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} V(k, p, 1, z', r, b) H(\Omega(k, p, z', r, b)) \\ +V(k, p, 0, z', r, b)[1 - H(\Omega(k, p, z', r, b))] \end{array} \right\} dG(z'|z_{-1}) \in [0, \overline{S}] \tag{35}$$

by (21).[26] The manager's realized payoff function from continuing (corresponding to V^c) in the present period, given the creditor's acceptance set γ_F in the present period and the manager's expected continuation payoff w_F in the present period, is thus

$$v_F^c(s) = \max_{(k',p',r') \in \gamma_F(z,b)} \left\{ \begin{array}{c} b + e(k, k', p, p', \ell, z, r) \\ +\beta w_F(k', p', r', z, b) \end{array} \right\} \in [0, \overline{S}]. \tag{36}$$

by (9). This implies that the manager's realized payoff function (corresponding to V) in the present period, given γ_F and w_F, is

$$v_F(s) = \max \left\{ \underbrace{0}_{\text{default}}, \overbrace{\alpha k - \phi(p, r)}^{\text{liquidate}}, \overbrace{v_F^c(s)}^{\text{continue}} \right\} \in [0, \overline{S}] \tag{37}$$

using (8). Accordingly, the manager's default threshold function (corresponding to D) in the present period, given her realized payoff function in the present period v_F, and assuming she does *not* default when indifferent, is

$$\underline{d}_F(s) = \min\{\{1+\varepsilon\} \cup \{\ell' \in [0,1] : \phi(p, r) \leq v_F(k, 0, \ell', z, 0, b)\}\} \in [0,1] \cup \{1+\varepsilon\} \tag{38}$$

[26] This best response does not depend on the manager's optimism or pessimism as it takes the functions V and Ω as given. The same is true of the best response functions v_F^c and v_F that follow.

by (14) and the corresponding threshold if she *does* default when indifferent is

$$\overline{d}_F(s) = \min\{\{1+\varepsilon\} \cup \{\ell' \in [0,1] : \phi(p,r) < v_F(k,0,\ell',z,0,b)\}\} \in [0,1] \cup \{1+\varepsilon\}. \tag{39}$$

Likewise, the manager's best-response liquidation threshold (corresponding to L) in the present period, if she does *not* liquidate when indifferent, is

$$\underline{l}_F(s) = \min\{\{1+\varepsilon\} \cup \{\ell' \in [0,1] : \alpha k \leq v_F^c(k,0,\ell',z,0,b)\}\} \in [0,1] \cup \{1+\varepsilon\} \tag{40}$$

by (16) while the corresponding threshold if she *does* liquidate when indifferent is

$$\bar{l}_F(s) = \min\{\{1+\varepsilon\} \cup \{\ell' \in [0,1] : \alpha k < v_F^c(k,0,\ell',z,0,b)\}\} \in [0,1] \cup \{1+\varepsilon\}. \tag{41}$$

By (18), the "optimistic" firm liquidation threshold function, which corresponds to (38) and (40), is

$$\underline{\lambda}_F(s) = \min\{\underline{l}_F(k,z,0), \max\{\underline{d}_F(s), \underline{l}_F(k,z,b)\}\} \in [0,1] \cup \{1+\varepsilon\} \tag{42}$$

while the corresponding "pessimistic" threshold is

$$\overline{\lambda}_F(s) = \min\left\{\bar{l}_F(k,z,0), \max\left\{\overline{d}_F(s), \bar{l}_F(k,z,b)\right\}\right\} \in [0,1] \cup \{1+\varepsilon\}. \tag{43}$$

Finally, at the beginning of period t, the agents decide whether or not to invest. By (19), their optimal investment threshold under optimistic beliefs is

$$\overline{\omega}_F(s) = u - (\zeta - \xi)\min\{1, \underline{\lambda}_F(s)\} - \xi \min\{1, \max\{\underline{d}_F(s), \underline{\lambda}_F(s)\}\} \in [u-\zeta, u] \tag{44}$$

while their optimal pessimistic threshold is

$$\underline{\omega}_F(s) = u - (\zeta - \xi)\min\left\{1, \overline{\lambda}_F(s)\right\} - \xi \min\left\{1, \max\left\{\overline{d}_F(s), \overline{\lambda}_F(s)\right\}\right\} \in [u-\zeta, u]. \tag{45}$$

This completes the process of backwards induction within period t.

An important feature of the best-response functions defined above is that their fixed points are monotone PBE's.[27]

Claim 6. If the vector F defined in (25) solves either $F = \overline{b}_F$ or $F = \underline{b}_F$, it is a monotone PBE.

Proof. Suppose $F = \overline{b}_F$. Then (38), (40), (42), (44), (36), (37), (35), (29), (31), and (33) imply the equilibrium conditions (14), (16), (18), (19), (9), (8), (21), (23), (22), and (12), respectively. The proof for $F = \underline{b}_F$ is analogous but uses corresponding "pessimistic" versions of the equilibrium conditions (14), (16), (18), (19), (9), (8), (21), (23), (22), and (12). □

For each n, we will now construct a sequence of vectors

$$\overline{F}_n = \left(-\underline{D}_n, -\underline{L}_n, -\underline{\Lambda}_n, \overline{\Omega}_n, \overline{V}^c_n, \overline{V}_n, \overline{W}_n, \overline{\Upsilon}_n, \overline{\Psi}_n, \overline{\Gamma}_n\right) \tag{46}$$

of upper bounds on any monotone PBE F and show that this sequence is monotonically decreasing and that its $\overline{F} = \lim_{n\to\infty} \overline{F}_n$ exists and is itself a monotone PBE. Further, we will construct a function vector

$$\underline{F}_n = \left(-\overline{D}_n, -\overline{L}_n, -\overline{\Lambda}_n, \underline{\Omega}_n, \underline{V}^c_n, \underline{V}_n, \underline{W}_n, \underline{\Upsilon}_n, \underline{\Psi}_n, \underline{\Gamma}_n\right) \tag{47}$$

of lower bounds on any monotone PBE vector F and show that this sequence is monotonically rising and that its limit $\underline{F} = \lim_{n\to\infty} \underline{F}_n$ exists and is itself a monotone PBE. In this sense, $\overline{F}$ and $\underline{F}$ are the most optimistic and most pessimistic PBE's. In practice, they often coincide: there is a unique monotone PBE.

[27] There may also exist equilibria that are not fixed points of either best-response function. Intuitively, these equilibria involve a *mixture* of optimistic and pessimistic best responses. We do not study these equilibria as they lie between the upper and lower bounds of our iterations, which typically coincide.

For the first two rounds $n_0 = -1, 0$, the components of $\overline{F}_{n_0}$ (resp., of $\underline{F}_{n_0}$) at each state s in S are simply the previously derived upper (resp., lower) bounds on the various components of the function vector F in any monotone PBE.[28] By (**??**), (16), and (18), this means that the initial bounds on the default and liquidation thresholds are

$$\underline{D}_{n_0}(s) = \underline{L}_{n_0}(s) = \underline{\Lambda}_{n_0}(s) = 0 \text{ and } \overline{D}_{n_0}(s) = \overline{L}_{n_0}(s) = \overline{\Lambda}_{n_0}(s) = 1 + \varepsilon. \tag{48}$$

By (19), the initial bounds on the agents' investment threshold function are

$$\overline{\Omega}_{n_0}(s) = u \text{ and } \underline{\Omega}_{n_0}(s) = u - \zeta. \tag{49}$$

By (8), (21), and (22), the initial bounds on the large players' continuation payoff functions are

$$\left(\overline{V}^c_{n_0}(s), \overline{V}_{n_0}(s), \overline{W}_{n_0}(s), \overline{\Upsilon}_{n_0}(s), \overline{\Psi}_{n_0}(s)\right) = \left(\overline{S}, \overline{S}, \overline{S}, \overline{S}, \overline{S}\right)$$
$$\text{and } \left(\underline{V}^c_{n_0}(s), \underline{V}_{n_0}(s), \underline{W}_{n_0}(s), \underline{\Upsilon}_{n_0}(s), \underline{\Psi}_{n_0}(s)\right) = (0,0,0,0,0). \tag{50}$$

Finally, by (12), the initial bounds on the set of permissible capital-debt-interest triplets are

$$\overline{\Gamma}_{n_0}(s) = \left[\underline{k}, \overline{k}\right] \times \left[\underline{p}, \overline{p}\right] \times \left[\underline{r}, \overline{r}\right] \text{ and } \underline{\Gamma}_{n_0}(s) = \left[\underline{k}, \overline{k}\right] \times \{0\} \times \left[\underline{r}, \overline{r}\right]. \tag{51}$$

For subsequent rounds, we iterate using the best-response function defined above: for each round $n > 0$ and each state s in S,

$$\overline{F}_n(s) = \overline{b}_{\overline{F}_{n-1}}(s) \text{ and } \underline{F}_n(s) = \underline{b}_{\underline{F}_{n-1}}(s). \tag{52}$$

In Theorem 1 below we show that the iterations are bounded (part 1) and monotone (part 2), and thus converge by the monotone convergence theorem. In parts 3-5 we prove some comparative statics properties. Finally, in part 6 we show that the

[28] Having the first two rounds be identical eases the proof of the base case in our induction argument below.

functions in the iterations bound their counterparts in any monotone PBE (defined above). After this claim we will show that if the continuous state space S is replaced with an arbitrarily fine grid, the limits from above and below are each monotone PBEs (which may coincide).

Theorem 1.

1. Bounded Sequences: for each n, (a) $\overline{F}_n \geq \underline{F}_0$ and (b) $\underline{F}_n \leq \overline{F}_0$.

2. Monotone Sequences: for each n, (a) $\overline{F}_{n-1} \geq \overline{F}_n$ and (b) $\underline{F}_{n-1} \leq \underline{F}_n$.

3. Comparative Statics #1: in each iteration round, higher agent participation does not harm the manager or creditor. More precisely, let $\underline{\kappa} = (1 - \tau_c)\, \pi\, (\underline{k}, \underline{z})$; for each n:

   (a) $\overline{V}^c_n$, $\overline{V}_n$, and $\overline{\Upsilon}_n$ are nondecreasing in ℓ;

   (b) $\underline{V}^c_n$, $\underline{V}_n$, and $\underline{\Upsilon}_n$ are nondecreasing in ℓ.

4. Comparative Statics #2: in each iteration round, a higher control benefit b of the incumbent manager encourages investment by the agents, discourages default and liquidation, and does not harm any player. More precisely, for each n, (a) each component of $\overline{F}_n$ is nondecreasing in b and (b) each component of $\underline{F}_n$ is nondecreasing in b.

5. Comparative Statics #3: in each iteration round, a higher interest rate r on old debt does not reduce the chance of default and liquidation, and does not benefit the agents and incumbent manager. More precisely, for each n, (a) $(\underline{D}_n, \underline{L}_n, \underline{\Lambda}_n)$ is nondecreasing in r while $\left(\overline{\Omega}_n, \overline{W}_n, \overline{V}^c_n, \overline{V}_n\right)$ is nonincreasing in r;[29] and (b)

[29]It's not possible to sign the effect of r on $\overline{\Upsilon}_n$ and $\overline{\Psi}_n$ since there are two effects: the creditor gets more when the manager repays, but the manager is more likely to default, thus transferring the firm to the creditor who values it less.

$\left(\overline{D}_n, \overline{L}_n, \overline{\Lambda}_n\right)$ is nondecreasing in r while $(\underline{\Omega}_n, \underline{W}_n, \underline{V}^c_n, \underline{V}_n)$ is nonincreasing in r.

6. The sequences are bounds on the set of monotone PBEs: for any monotone PBE F and any round n, (a) $\overline{F}_n \geq F$ and (b) $\underline{F}_n \leq F$.

For each state s in S, we define the limits of the above sequences as follows. For the declining series $\left(\overline{F}_n\right)$ the limit is the infimum while for the rising series $(\underline{F}_n)$ it is the supremum:

$$\overline{F} = \inf_n \overline{F}_n \text{ and } \underline{F} = \sup_n \underline{F}_n. \tag{53}$$

For the set-valued component Γ, "infimum" means intersection while "supremum" means union:

$$\overline{\Gamma}(s) \stackrel{d}{=} \bigcap_{n=0}^{\infty} \overline{\Gamma}_n(s) \text{ and } \underline{\Gamma}(s) \stackrel{d}{=} \bigcup_{n=0}^{\infty} \underline{\Gamma}_n(s). \tag{54}$$

We now show that if the state space S is discrete, then the upper limit $\overline{F}$ (resp., the lower limit $\underline{F}$) is the highest (resp., lowest) monotone PBE. Accordingly, if they coincide, they are the unique monotone PBE.

Theorem 2. *Assume S is discrete. $\underline{F}$ and $\overline{F}$ are each monotone PBEs (and may coincide), and any monotone PBE F lies between them: $\underline{F} \leq F \leq \overline{F}$.*

5 Manager's Objective Function

The expected payoff $W(k_t, p_t, r_t, z_{t-1}, b_t)$ of the manager is the maximum that the manager would pay, just before the start of period t, to retain control of her firm. Similarly, the creditor's expected payoff $\Psi(k_t, p_t, r_t, z_{t-1}, b_t)$ from lending in period $t-1$ is the most any creditor would pay, just before the start of period t, in order to obtain the existing creditor's debt claim on the firm: it is the market value of the firm's debt. We refer to the sum

$$O(k_t, p_t, r_t, z_{t-1}, b_t) = W(k_t, p_t, r_t, z_{t-1}, b_t) + \Psi(k_t, p_t, r_t, z_{t-1}, b_t)$$

of these two functions as the firm's *private value*.[30] We will show that if the manager can always choose the interest rate $r_t = R(k_t, p_t)$ that leaves the creditor indifferent between lending and not, then the manager's optimal capital-debt pair (k_t, p_t) maximizes the objective function

$$O(k_t, p_t, R(k_t, p_t), z_{t-1}, b_t) - [k_t + A(k_{t-1}, k_t)], \tag{55}$$

which is just the firm's private value evaluated at this optimal interest rate, less the capital cost $k_t + A(k_{t-1}, k_t)$: the cost k_t of acquiring the capital plus the cost $A(k_{t-1}, k_t)$ of installing it. As the capital cost is unaffected by the firm's debt choice p_t, the firm's private value O alone is the manager's objective function with respect to debt. The formal result is as follows.

Claim 7. Assume a continuous state space ($S = S^c$) and that the creditors' continuation payoff function $\Psi(k, p, r, z_{-1}, b)$ is continuous in the interest rate r. Let $R(k_t, p_t)$ denote the interest rate r_t at which a creditor is just willing to fund the choice (k_t, p_t). The manager's optimal capital-debt pair (k_t, p_t) in period $t-1$ maximizes (55).

Proof. By (9), in period $t-1$ the manager chooses a pair (k_t, p_t) that maximizes

$$b + e(k_{t-1}, k_t, p_{t-1}, p_t, \ell_{t-1}, z_{t-1}, r_{t-1}) + \beta E_{t-1} V(k_t, p_t, \ell_t, z_t, R(k_t, p_t), b_t) \tag{56}$$

By assumption, the interest rate $R(k_t, p_t)$ satisfies the creditor's participation constraint (11) with equality whence, by (1), $p_t = \beta\Psi(k_t, p_t, R(k_t, p_t), z_{t-1}, b_t)$. Substituting this into the maximand in (56) and using (6) and (20), the firm's choice (k_t, p_t) must maximize $C_{t-1} + \beta\{O(k_t, p_t, R(k_t, p_t), z_{t-1}, b_t) - [k_t - A(k_{t-1}, k_t)]\}$ where the terms in $C_{t-1} = b + (1-\tau_c)\ell_{t-1}\pi(k_{t-1}, z_{t-1}) - \phi(p_{t-1}, r_{t-1}) + \delta k_{t-1}\tau_c + \beta(1-\delta)k_{t-1}$ are independent of the manager's choice (k_t, p_t). The result follows. □

[30]The firm's private value exceeds its market value since W includes the manager's control benefit b_t. However, we must divide Ψ by O if we wish to obtain a debt-to-value measure that lies in $[0, 1]$.

By limited liability, the expected payoffs of both manager and creditor must be nonnegative. Accordingly, the ratio

$$\frac{\Psi\left(k_t, p_t, R\left(k_t, p_t\right), z_{t-1}, b_t\right)}{O\left(k_t, p_t, R\left(k_t, p_t\right), z_{t-1}, b_t\right)}$$

must lie in $[0,1]$. This is thus a convenient index of the firm's indebtedness. Abusing terminology as above, we will refer to this as the firm's debt-to-value ratio.

A Proofs

Proof of Theorem 1. We first show some useful properties of two of the components of $\overline{b}_F$.

Claim 8. The best-response functions $\overline{d}_F\left(s\right)$ and $\overline{l}_F\left(s\right)$ have the following properties.

1. They are nondecreasing in r.
2. They are nonincreasing in V and V^c, respectively.

Proof. Part 1 holds as $\phi\left(p, r\right)$ is nondecreasing in r. Part 2 holds since an increase in V loosens the inequality in (38) and a rise in V^cloosens the inequality in (38) □

Part 1(a). For $\underline{L}_n, \underline{D}_n, \underline{\Lambda}_n$, this part holds by (48), (40), (38), and (42). For $\overline{V}_n, \overline{W}_n, \overline{\Upsilon}_n, \overline{\Psi}_n$, it holds by (50), (37), (35), (29), and (31). For $\overline{\Omega}_n$ it holds by (49) and (44). For $\overline{\Gamma}_n$ it holds by (51). This completes the proof of part 1(a).

Part 1(b). For $\overline{L}_n, \overline{D}_n, \overline{\Lambda}_n$, this part holds by (48), (40), (38), and (42). For $\underline{V}_n, \underline{W}_n, \underline{\Upsilon}_n, \underline{\Psi}_n$, it holds by (50), (37), (35), (29), and (31). For $\underline{\Omega}_n$ it holds by (49) and (44). For $\underline{\Gamma}_n$ it holds by (51) and (33). This completes the proof of part 1(b).

The proof of parts 2-6 is by induction.

Base case ($n = 0$). Part 2(a-b) holds because the iterations for $n = -1$ and $n = 0$ are the same functions. Parts 3(a-b) and 5(a-b) hold because the functions for $n = 0$ do not depend on ℓ or r. Part 4(a-b) holds by construction and (28). For $n = 0$, part 6 holds by construction.

Induction step: $n \geq 1$. We consider each function in turn.

We begin with $\overline{\Upsilon}_n$ and $\underline{\Upsilon}_n$. By (29) and (52),

$$\overline{\Upsilon}_n(s) = 1_{\ell \geq \underline{D}_{n-1}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-1}(s)} \overline{V}_{n-1}(k, 0, \ell, z, 0, 0)$$

and

$$\overline{\Upsilon}_{n-1}(s) = 1_{\ell \geq \underline{D}_{n-2}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-2}(s)} \overline{V}_{n-2}(k, 0, \ell, z, 0, 0)$$

We begin with part 2(a). Define

$$\widehat{\Upsilon}(s) = \quad 1_{\ell \geq \underline{D}_{n-2}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-2}(s)} \overline{V}_{n-1}(k, 0, \ell, z, 0, 0).$$

We have

$$\begin{aligned}
\begin{bmatrix} \overline{\Upsilon}_n(s) \\ -\overline{\Upsilon}_{n-1}(s) \end{bmatrix} &= \left[\overline{\Upsilon}_n(s) - \widehat{\Upsilon}(s)\right] + \left[\widehat{\Upsilon}(s) - \overline{\Upsilon}_{n-1}(s)\right] \\
&= \begin{bmatrix} 1_{\ell \geq \underline{D}_{n-1}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-1}(s)} \overline{V}_{n-1}(k, 0, \ell, z, 0, 0) \\ -1_{\ell \geq \underline{D}_{n-2}(s)} p(1+r) - 1_{\ell < \underline{D}_{n-2}(s)} \overline{V}_{n-1}(k, 0, \ell, z, 0, 0) \end{bmatrix} \\
&\quad + \begin{bmatrix} 1_{\ell \geq \underline{D}_{n-2}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-2}(s)} \overline{V}_{n-1}(k, 0, \ell, z, 0, 0) \\ -\ 1_{\ell \geq \underline{D}_{n-2}(s)} p(1+r) - 1_{\ell < \underline{D}_{n-2}(s)} \overline{V}_{n-2}(k, 0, \ell, z, 0, 0) \end{bmatrix} \\
&= \begin{bmatrix} 1_{\ell \geq \underline{D}_{n-1}(s)} \\ -1_{\ell \geq \underline{D}_{n-2}(s)} \end{bmatrix} p(1+r) + \begin{bmatrix} 1_{\ell < \underline{D}_{n-1}(s)} \\ -1_{\ell < \underline{D}_{n-2}(s)} \end{bmatrix} \overline{V}_{n-1}(k, 0, \ell, z, 0, 0) \qquad (57) \\
&\quad + 1_{\ell < \underline{D}_{n-2}(s)} \left[\overline{V}_{n-1}(k, 0, \ell, z, 0, 0) - \overline{V}_{n-2}(k, 0, \ell, z, 0, 0)\right] \qquad (58)
\end{aligned}$$

Now, line (58) is nonpositive since $\overline{V}_{n-1} \leq \overline{V}_{n-2}$ by induction. And we can rewrite line (57) as

$$\begin{aligned}
&-\left[1_{\ell \geq \underline{D}_{n-2}(s)} - 1_{\ell \geq \underline{D}_{n-1}(s)}\right] p(1+r) + \left[1_{\ell \geq \underline{D}_{n-2}(s)} - 1_{\ell \geq \underline{D}_{n-1}(s)}\right] \overline{V}_{n-1}(k, 0, \ell, z, 0, 0) \\
&\qquad = \left[1_{\underline{D}_{n-1}(s) > \ell \geq \underline{D}_{n-2}(s)}\right] \left[\overline{V}_{n-1}(k, 0, \ell, z, 0, 0) - p(1+r)\right].
\end{aligned}$$

The second line is nonpositive by part 1 of the following claim, which shows part 2(a); the proof of part 2(b) is analogous but uses part 2 of Claim 9.

Claim 9. In round $n-1$, if the participation rate ℓ is less than the manager's default threshold, then the creditor is harmed by default. More precisely:

1. for any $\ell < \underline{D}_{n-1}(s)$, $\overline{V}_{n-1}(k,0,\ell,z,0,0) \leq p(1+r)$.

2. for any $\ell < \overline{D}_{n-1}(s)$, $\underline{V}_{n-1}(k,0,\ell,z,0,0) \leq p(1+r)$.

Proof. Part 1. First, whenever $\ell < \underline{D}_{n-1}(s)$, (38) implies that $\overline{V}_{n-1}(k,0,\ell,z,0,b)$ is less than $\phi(p,r)$ which equals $p(1+r(1-\tau_c))$ by (4) (as the manager will never default if $p \leq 0$). Moreover, $\overline{V}_{n-1}(k,0,\ell,z,0,b) \geq \overline{V}_{n-1}(k,0,\ell,z,0,0)$ by part 4(a) and the induction hypothesis. Finally, $p(1+r(1-\tau_c)) \leq p(1+r)$ by (5). Part 2 is analogous. □

For parts 3(a) and 4(a), we must show that $\overline{\Upsilon}_n(s)$ is nondecreasing in ℓ and b. Let $(\ell',b') \geq (\ell,b)$. Defining

$$\widetilde{\Upsilon} = 1_{\ell' \geq \underline{D}_{n-1}(k,p,z,r,b')} p(1+r) + 1_{\ell' < \underline{D}_{n-1}(k,p,z,r,b')} \overline{V}_{n-1}(k,0,\ell,z,0,0),$$

we have

$$\begin{aligned}
\left[\begin{array}{c} \overline{\Upsilon}_n(k,p,\ell',z,r,b') \\ -\overline{\Upsilon}_n(k,p,\ell,z,r,b) \end{array}\right] &= \left[\overline{\Upsilon}_n(k,p,\ell',z,r,b') - \widetilde{\Upsilon}\right] + \left[\widetilde{\Upsilon} - \overline{\Upsilon}_n(k,p,\ell,z,r,b)\right] \\
&= \left[\begin{array}{c} 1_{\ell' \geq \underline{D}_{n-1}(k,p,z,r,b')} p(1+r) + 1_{\ell' < \underline{D}_{n-1}(k,p,z,r,b')} \overline{V}_{n-1}(k,0,\ell',z,0,0) \\ -1_{\ell' \geq \underline{D}_{n-1}(k,p,z,r,b')} p(1+r) - 1_{\ell' < \underline{D}_{n-1}(k,p,z,r,b')} \overline{V}_{n-1}(k,0,\ell,z,0,0) \end{array}\right] \\
&+ \left[\begin{array}{c} 1_{\ell' \geq \underline{D}_{n-1}(k,p,z,r,b')} p(1+r) + 1_{\ell' < \underline{D}_{n-1}(k,p,z,r,b')} \overline{V}_{n-1}(k,0,\ell,z,0,0) \\ -1_{\ell \geq \underline{D}_{n-1}(k,p,z,r,b)} p(1+r) - 1_{\ell < \underline{D}_{n-1}(k,p,z,r,b)} \overline{V}_{n-1}(k,0,\ell,z,0,0) \end{array}\right] \\
&= 1_{\ell' < \underline{D}_{n-1}(k,p,z,r,b')} \left[\overline{V}_{n-1}(k,0,\ell',z,0,0) - \overline{V}_{n-1}(k,0,\ell,z,0,0)\right] \qquad (59) \\
+ &\left[\begin{array}{c} 1_{\ell' \geq \underline{D}_{n-1}(k,p,z,r,b')} \\ -1_{\ell \geq \underline{D}_{n-1}(k,p,z,r,b)} \end{array}\right] p(1+r) + \left[\begin{array}{c} 1_{\ell' < \underline{D}_{n-1}(k,p,z,r,b')} \\ -1_{\ell < \underline{D}_{n-1}(k,p,z,r,b)} \end{array}\right] \overline{V}_{n-1}(k,0,\ell,z,0,0) \qquad (60)
\end{aligned}$$

Line (59) is nonnegative since, by induction, $\overline{V}_{n-1}$ is nondecreasing in ℓ. As for line (60), suppose first that $b' = b$ and $\ell' > \ell$. Then line (60) can be rewritten as

$1_{\ell' \geq \underline{D}_{n-1}(k,p,z,r,b) > \ell} \left[p(1+r) - \overline{V}_{n-1}(k,0,\ell,z,0,0) \right]$ which is nonnegative by part 1 of Claim 9. Now suppose that $\ell' = \ell$ and $b' > b$. By induction, $\underline{D}_{n-1}$ is nonincreasing in b so $\underline{D}_{n-1}(s) \geq \underline{D}_{n-1}(k,p,z,r,b')$, whence line (60) can be rewritten as

$$\left[1_{\underline{D}_{n-1}(s) > \ell \geq \underline{D}_{n-1}(k,p,z,r,b')} \right] \left[p(1+r) - \overline{V}_{n-1}(k,0,\ell,z,0,0) \right]$$

which is nonnegative by part 1 of Claim 9: we have shown parts 3(a) and 4(a). The proof of parts 3(b) and 4(b) is analogous but uses part 2 of Claim 9.

As for part 6(a), we know by induction that $\underline{D}_{n-1} \leq D$ and $\overline{V}_{n-1} \geq V$ for any monotone PBE functions D and V. Define

$$\widetilde{\Upsilon}(s) = \quad 1_{\ell \geq \underline{D}_{n-1}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-1}(s)} V(k,0,\ell,z,0,0).$$

By (29),

$$\begin{aligned}
\begin{bmatrix} \overline{\Upsilon}_n(s) \\ -\Upsilon(s) \end{bmatrix} &= \left[\overline{\Upsilon}_n(s) - \widetilde{\Upsilon}(s) \right] + \left[\widetilde{\Upsilon}(s) - \Upsilon(s) \right] \\
&= \begin{bmatrix} 1_{\ell \geq \underline{D}_{n-1}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-1}(s)} \overline{V}_{n-1}(k,0,\ell,z,0,0) \\ -1_{\ell \geq \underline{D}_{n-1}(s)} p(1+r) - 1_{\ell < \underline{D}_{n-1}(s)} V(k,0,\ell,z,0,0) \end{bmatrix} \\
&\quad + \begin{bmatrix} 1_{\ell \geq \underline{D}_{n-1}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-1}(s)} V(k,0,\ell,z,0,0) \\ -\ 1_{\ell \geq D(s)} p(1+r) - 1_{\ell < D(s)} V(k,0,\ell,z,0,0) \end{bmatrix} \\
&= 1_{\ell < \underline{D}_{n-1}(s)} \left[\overline{V}_{n-1}(k,0,\ell,z,0,0) - V(k,0,\ell,z,0,0) \right] && (61) \\
&\quad + \begin{bmatrix} 1_{\ell \geq \underline{D}_{n-1}(s)} \\ -1_{\ell \geq D(s)} \end{bmatrix} p(1+r) + \begin{bmatrix} 1_{\ell < \underline{D}_{n-1}(s)} \\ -1_{\ell < D(s)} \end{bmatrix} V(k,0,\ell,z,0,0) && (62)
\end{aligned}$$

Now, line (61) is nonnegative since $\overline{V}_{n-1} \geq V$ by induction. And since, by induction, $\underline{D}_{n-1}(s) \leq D(s)$, we can rewrite line (62) as

$$\begin{aligned}
&\left[1_{\ell \geq \underline{D}_{n-1}(s)} - 1_{\ell \geq D(s)} \right] p(1+r) - \left[1_{\ell \geq \underline{D}_{n-1}(s)} - 1_{\ell \geq D(s)} \right] V(k,0,\ell,z,0,0) \\
&\qquad = \left[1_{D(s) > \ell \geq \underline{D}_{n-1}(s)} \right] \left[p(1+r) - V(k,0,\ell,z,0,0) \right].
\end{aligned}$$

The second line is nonnegative by the following claim, so $\overline{\Upsilon}_n(s) \geq \Upsilon(s)$: part 6(a) holds; the proof of 6(b) is analogous.

Claim 10. In any monotone PBE, if the participation rate ℓ is less than the manager's default threshold $D(s)$ then the creditor is harmed by default: $V(k,0,\ell,z,0,0) \leq p(1+r)$.

Proof. First, whenever $\ell < D(s)$, (14) implies that $V(k,0,\ell,z,0,b)$ is less than $\phi(p,r)$ which equals
$p(1+r(1-\tau_c))$ by (4) (as the manager will never default if $p \leq 0$). Moreover, $V(k,0,\ell,z,0,b) \geq V(k,0,\ell,z,0,0)$ by Definition 1 and Claim 3. Finally,

$$p(1+r(1-\tau_c)) \leq p(1+r)$$

by (5). □

We now consider $\overline{\Psi}_n$ and $\underline{\Psi}_n$. By (31) and (52),

$$\overline{\Psi}_n = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{\Upsilon}_n(k,p,1,z',r,b)\, H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right) \\ +\overline{\Upsilon}_n(k,p,0,z',r,b)\left[1-H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1}) \quad (63)$$

lies nowhere above

$$\overline{\Psi}_{n-1} = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{\Upsilon}_{n-1}(k,p,1,z',r,b)\, H\left(\overline{\Omega}_{n-2}(k,p,z',r,b)\right) \\ +\overline{\Upsilon}_{n-1}(k,p,0,z',r,b)\left[1-H\left(\overline{\Omega}_{n-2}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1})$$

as $\overline{\Upsilon}_n \leq \overline{\Upsilon}_{n-1}$ (shown above), $\overline{\Omega}_{n-1} \leq \overline{\Omega}_{n-2}$ (by induction), and $\overline{\Upsilon}_{n-1}$ is nondecreasing in ℓ (by induction). This proves part 2(a); part 2(b) is analogous. Parts 3 and 5 are vacuous. As for part 4(a), $\overline{\Psi}_n$ is nondecreasing in b since $\overline{\Omega}_{n-1}$ is nondecreasing in b (by induction) and $\overline{\Upsilon}_n$ is nondecreasing in ℓ and b (shown above); part 4(b) is analogous. As for part 6(a), for any monotone PBE functions Υ and Ω, $\overline{\Upsilon}_n$ is nondecreasing in ℓ and $\overline{\Upsilon}_n \geq \Upsilon$ (both shown above) and $\overline{\Omega}_{n-1} \geq \Omega$ (by induction), so $\overline{\Psi}_n \geq \Psi$ by (22) and (63); the proof of 6(b) is analogous.

We now turn to $\overline{\Gamma}_n$ and $\underline{\Gamma}_n$. By (33) and (52),

$$\overline{\Gamma}_n(s) = \left\{(k',p',r') \in B : \overline{\Psi}_n(k',p',r',z,b) \geq p'(1+\rho)\right\} \quad (64)$$

is a subset of

$$\overline{\Gamma}_{n-1}(s) = \left\{(k', p', r') \in B : \overline{\Psi}_{n-1}(k', p', r', z, b) \geq p'(1+\rho)\right\}$$

since $\overline{\Psi}_n \leq \overline{\Psi}_{n-1}$ (shown above), proving part 2(a); part 2(b) is analogous. Parts 3 and 5 are vacuous. Part 4(a) holds since $\overline{\Psi}_n$ is nondecreasing in b (shown above); part 4(b) is analogous. As for part 6(a), for any monotone PBE functions Ψ and Γ, we have $\Psi \leq \overline{\Psi}_n$ (shown above) so $\Gamma \subseteq \overline{\Gamma}_n$ by (12) and (64); the proof of 6(b) is analogous.

We now consider $\overline{W}_n$ and $\underline{W}_n$. By (35) and (52),

$$\overline{W}_n = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{V}_{n-1}(k,p,1,z',r,b)\, H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right) \\ +\overline{V}_{n-1}(k,p,0,z',r,b)\left[1 - H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1})$$

and

$$\overline{W}_{n-1} = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{V}_{n-2}(k,p,1,z',r,b)\, H\left(\overline{\Omega}_{n-2}(k,p,z',r,b)\right) \\ +\overline{V}_{n-2}(k,p,0,z',r,b)\left[1 - H\left(\overline{\Omega}_{n-2}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1}).$$

Define

$$\widehat{W}(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{V}_{n-2}(k,p,1,z',r,b)\, H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right) \\ +\overline{V}_{n-2}(k,p,0,z',r,b)\left[1 - H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1}).$$

We have $\widehat{W} \leq \overline{W}_{n-1}$ since by induction, $\overline{\Omega}_{n-1}$ lies nowhere above $\overline{\Omega}_{n-2}$ and

$$\overline{V}_{n-2}(k,p,\ell,z,r,b)$$

is nondecreasing in ℓ. Moreover, $\widehat{W} \geq \overline{W}_n$since, by induction, $\overline{V}_{n-1}$ lies nowhere above $\overline{V}_{n-2}$. This establishes that $\overline{W}_n$ lies nowhere above $\overline{W}_{n-1}$, proving part 2(a); part 2(b) is analogous. Part 3 is vacuous. As for parts 4(a) and 5(a), we must show that $\overline{W}_n$ is nondecreasing in b and nonincreasing in r. This holds since, by induction, $\overline{V}_{n-1}$ and $\overline{\Omega}_{n-1}$ are nondecreasing in b nonincreasing in r, and since $\overline{V}_{n-1}$ is nondecreasing in ℓ. Parts 4(b) and 5(b) are analogous. As for part 6(a), we know by induction that $\overline{V}_{n-1}$ is nondecreasing in ℓ and that $\overline{V}_{n-1} \geq V$ and $\overline{\Omega}_{n-1} \geq \Omega$ for any monotone PBE functions V and Ω, whence $w_{\overline{F}_{n-1}} \geq w_F = W$ by (35), (52), and Claim 6; the proof of 6(b) is analogous.

We now turn to $\overline{V}_n^c$ and $\underline{V}_n^c$. By (36) and (52),

$$\overline{V}_n^c(s) = \max_{(k',p',r') \in \overline{\Gamma}_n(z,b)} \left\{ \begin{array}{c} b + e(k,k',p,p',\ell,z,r) \\ +\beta \overline{W}_n(k',p',r',z,b) \end{array} \right\} \tag{65}$$

lies nowhere above

$$\overline{V}_{n-1}^c(s) = \max_{(k',p',r') \in \overline{\Gamma}_{n-1}(z,b)} \left\{ \begin{array}{c} b + e(k,k',p,p',\ell,z,r) \\ +\beta \overline{W}_{n-1}(k',p',r',z,b) \end{array} \right\}$$

since $\overline{W}_n \leq \overline{W}_{n-1}$ and $\overline{\Gamma}_{n-1} \subseteq \overline{\Gamma}_{n-2}$ (both shown above). This shows part 2(a); part 2(b) is analogous. Moreover, by (6), e and thus $\overline{V}_n^c$ is nondecreasing in ℓ, proving part 3(a); part 3(b) is analogous. As for parts 4(a), we must show that $\overline{V}_n^c$ is nondecreasing in b. This holds since $\overline{\Gamma}_n$ and $\overline{W}_n$ have these properties (shown above); the proof of 4(b) is analogous. As for part 5(a), we must show that $\overline{V}_n^c$ is nonincreasing in r. This holds since, by (6), e is nonincreasing in r; the proof of 5(b) is analogous. As for part 6(a), for any monotone PBE functions W and Γ, we showed above that $W \leq \overline{W}_n$ and $\Gamma \subseteq \overline{\Gamma}_n$ so $\overline{V}_n^c(s) \geq V^c$ by (9), (20), and (65); the proof of 6(b) is analogous.

We now turn to $\overline{V}_n$ and $\underline{V}_n$. By (37) and (52),

$$\overline{V}_n(s) = \max\left\{0, \alpha k - \phi(p,r), \overline{V}_n^c(s)\right\} \tag{66}$$

lies nowhere above

$$\overline{V}_{n-1}(s) = \max\left\{0, \alpha k - \phi(p,r), \overline{V}_{n-1}^c(s)\right\}$$

since $\overline{V}_n^c \leq \overline{V}_{n-1}^c$ (shown above). This shows part 2(a); part 2(b) is analogous. Moreover, $\overline{V}_n^c$ is nondecreasing in ℓ (shown above) whence $\overline{V}_n$ also has this property, proving part 3(a); part 3(b) is analogous. As for parts 4(a) and 5(a), we must show that $\overline{V}_n$ is nondecreasing in b and nonincreasing in r. This holds since $\overline{V}_n^c$ has these properties (shown above) and $\phi(p,r)$ is nonincreasing in r by (4). The proof of

4(b) and 5(b) is analogous. As for part 6(a), for any monotone PBE function V^c, we showed above that $\overline{V}_n^c \geq V^c$ so $\overline{V}_n \geq V$ by (8) and (66); the proof of 6(b) is analogous.

We now turn to $\underline{D}_n$ and $\overline{D}_n$. As shown above, $\overline{V}_n \leq \overline{V}_{n-1}$. Hence, by (38) and (52),

$$\underline{D}_n (s) = \min \left\{ \{1 + \varepsilon\} \cup \left\{ \ell' \in [0,1] : \phi(p,r) \leq \overline{V}_n (k, 0, \ell', z, 0, b) \right\} \right\} \tag{67}$$

lies nowhere below

$$\underline{D}_{n-1} (s) = \min \left\{ \{1 + \varepsilon\} \cup \left\{ \ell' \in [0,1] : \phi(p,r) \leq \overline{V}_{n-1} (k, 0, \ell', z, 0, b) \right\} \right\},$$

showing part 2(a); the proof of part 2(b) is analogous. Part 3 is not relevant. As for part 4(a), we must show that $\underline{D}_n$ is nonincreasing in b. This holds as $\overline{V}_n$ is nondecreasing in b (shown above); the proof of 4(b) is analogous. By (4), $\phi(p,r)$ is nondecreasing in r; hence, by (67), $\underline{D}_n$ is nondecreasing in r, proving 5(a); the proof of 5(b) is analogous. Finally, for any monotone PBE function V, we showed above that $\overline{V}_n \geq V$ so $\underline{D}_n \leq D$ by (14) and (67), proving part 6(a); the proof of 6(b) is analogous.

We first consider $\underline{L}_n$ and $\overline{L}_n$. As shown above, $\overline{V}_n^c \leq \overline{V}_{n-1}^c$ whence, by (40) and (52),

$$\underline{L}_n (s) = \min \left\{ \{1 + \varepsilon\} \cup \left\{ \ell' \in [0,1] : \alpha k \leq \overline{V}_n^c (k, 0, \ell', z, 0, b) \right\} \right\} \tag{68}$$

lies nowhere below

$$\underline{L}_{n-1} (s) = \min \left\{ \{1 + \varepsilon\} \cup \left\{ \ell' \in [0,1] : \alpha k \leq \overline{V}_{n-1}^c (k, 0, \ell', z, 0, b) \right\} \right\},$$

showing part 2(a); the proof of part 2(b) is analogous. Part 3 is not relevant. As for part 4(a), we must show that $\underline{L}_n$ is nonincreasing in b. This holds since $\overline{V}_n^c$ is nondecreasing in b (shown above); the proof of 4(b) is analogous. By (68), $\underline{L}_n$ does not depend on r, proving 5(a); the proof of 5(b) is analogous. Finally, for the manager's payoff V^c from continuing in any monotone PBE F, $\overline{V}_n^c \geq V^c$ as shown above so $\underline{L}_n \leq L$ by (16) and (68), proving part 6(a); the proof of 6(b) is analogous.

We now turn to $\underline{\Lambda}_n$ and $\overline{\Lambda}_n$. By (42) and (52),

$$\underline{\Lambda}_n(s) = \min\{\underline{L}_n(k,z,0), \max\{\underline{D}_n(s), \underline{L}_n(k,z,b)\}\} \tag{69}$$

lies nowhere below

$$\underline{\Lambda}_{n-1}(s) = \min\left\{\underline{L}_{n-1}(k,z,0), \max\left\{\underline{D}_{n-1}(s), \underline{L}_{n-1}(k,z,b)\right\}\right\}$$

since $\underline{L}_n \leq \underline{L}_{n-1}$ and $\underline{D}_n \leq \underline{D}_{n-1}$ (shown above). This shows part 2(a); the proof of 2(b) is analogous. Part 3 is vacuous. As for parts 4(a) and 5(a), we showed above that $\underline{L}_n$ and $\underline{D}_n$ are nonincreasing in b and nondecreasing in r whence $\underline{\Lambda}_n$ inherits these properties; 4(b) and 5(b) are analogous. Finally, we showed above that $\underline{L}_n \leq L$ and $\underline{D}_n \leq D$ for any monotone PBE functions L and D, so $\underline{\Lambda}_n(s) \leq \Lambda$ by (18) and (69), proving part 6(a); the proof of 6(b) is analogous.

We finally consider $\overline{\Omega}_n$ and $\underline{\Omega}_n$. By (44) and (52),

$$\overline{\Omega}_n = u - (\zeta - \xi)\min\{1, \underline{\Lambda}_n(s)\} - \xi \min\{1, \max\{\underline{D}_n(s), \underline{\Lambda}_n(s)\}\} \tag{70}$$

lies nowhere above

$$\overline{\Omega}_{n-1} = u - (\zeta - \xi)\min\left\{1, \underline{\Lambda}_{n-1}(s)\right\} - \xi \min\left\{1, \max\left\{\underline{D}_{n-1}(s), \underline{\Lambda}_{n-1}(s)\right\}\right\}$$

since $\underline{\Lambda}_n \leq \underline{\Lambda}_{n-1}$ and $\underline{D}_n \leq \underline{D}_{n-1}$ (shown above), proving part 2(a); part 2(b) is analogous. Part 3 is vacuous. As for parts 4(a) and 5(a), $\overline{\Omega}_n$ is nondecreasing in b and nonincreasing in r since, as shown above, $\underline{\Lambda}_n$ and $\underline{D}_n$ have these properties; the proofs of 4(b) and 5(b) are analogous. Finally, we showed above that $\underline{\Lambda}_n \leq \Lambda$ and $\underline{D}_n \leq D$ for any monotone PBE functions Λ and D, so $\overline{\Omega}_n \geq \Omega$ by (19) and (70), proving part 6(a); the proof of 6(b) is analogous. **Q.E.D.$_{\text{Theorem 1}}$**

Proof of Theorem 2. Any monotone PBE F lies between $\underline{F}$ and $\overline{F}$ by part 6 of Theorem 1 and since weak inequalities and weak subset relations are preserved in the limit. By part 4 of Theorem 1, $\underline{V}^c$ and $\overline{V}^c$ are nondecreasing in b: by Definition 1, $\underline{F}$ and $\overline{F}$ are monotone. It remains to show that $\overline{F}$ is a PBE. By Claim 6 it suffices to show that $\overline{F}$ is a fixed point of the best-response function: that $\overline{F} = b_{\overline{F}}$. The proof will rely on notions of convergence and uniform continuity, which require metrics on both the domain S and the range $\Sigma = \Re^9 \times 2^B$ (equation (26)) of functions F given in (27). Our metric on S will be simply the sup norm. As for Σ, we define the metric Δ as follows. Each element of Σ is a vector $\sigma = (y, z)$ where $y \in \Re^9$ and z is a subset of B. We use the sup norm on any Euclidean space: for any $m = 1, 2, ...$, the distance $|y' - y|$ between any two vectors y, y' in $\Re^m$ is defined to be the maximum difference $\max_{i=1}^m |y'_i - y_i|$ between any respective components of the two vectors. For any $\sigma = (y, z)$ and $\sigma' = (y', z')$ in Σ, we define the distance $\Delta(\sigma, \sigma')$ between σ and σ' to be

$$\Delta(\sigma, \sigma') = \max\{|y - y'|, \Delta(z, z')\} \tag{71}$$

where (abusing notation slightly) $\Delta(z, z')$ denotes the Hausdorff distance between z and z':[31]

$$\Delta(z, z') = \max\left\{\sup_{x \in z} \inf_{x' \in z'} |x - x'|, \sup_{x' \in z'} \inf_{x \in z} |x - x'|\right\}. \tag{72}$$

When z is a subset of z', the first entry in the max in (72) is zero whence

$$\Delta(z, z') = \sup_{x' \in z'} \inf_{x \in z} |x - x'|. \tag{73}$$

We now prove the following useful result, where convergence, continuity, and uniform continuity are defined with respect to the metric Δ.

Lemma 1. *1. The convergence of $\underline{F}_n$ and $\overline{F}_n$ to $\underline{F}$ and $\overline{F}$, respectively, is uniform on S. Part 2. Each function mentioned in part 1 is uniformly continuous.*

Proof. Part 1. As S is finite, it contains no accumulation points; hence, any function on S is trivially continuous. As S is also compact and both S and Σ are metric spaces, any function from S to Σ is thus uniformly continuous by the Heine-Cantor therem.

[31] It may help to think of the Hausdorff metric as a two-step process. One first identifies the point of either set that is the greatest distance from the closest element of the other set. One then sets $d(\sigma, \sigma')$ equal to this greatest distance.

Part 1. By Dini's Theorem, if a monotone sequence of continuous functions converges pointwise to a continuous function on a compact space, then the convergence is uniform. □

Remark 1. Applying the Hausdorff metric (72), uniform continuity of $\Gamma \in \left\{\overline{\Gamma}, \overline{\Gamma}_n\right\}$ means simply that for each $\varepsilon > 0$, there is a $\delta > 0$ such that for any states $s', s'' \in S$ satisfying $|s' - s''| < \delta$: for any $(k', p', r') \in \Gamma(s')$ there is a $(k'', p'', r'') \in \Gamma(s'')$ such that $|(k'', p'', r'') - (k', p', r')| < \varepsilon$; and for any $(k'', p'', r'') \in \Gamma(s'')$ there is a $(k', p', r') \in \Gamma(s')$ such that $|(k'', p'', r'') - (k', p', r')| < \varepsilon$.

We now show that at each state s in S, $\overline{F}_n(s)$ converges to $b_{\overline{F}}(s)$ whence $\overline{F}(s) = b_{\overline{F}}(s)$. We consider each component in turn.

1. We claim that $\overline{\upsilon}_{\overline{F}}(s) = \overline{\Upsilon}(s)$ or, equivalently, that

$$\overline{\Upsilon}_n(s) = 1_{\ell \geq \underline{D}_{n-1}(s)} p(1+r) + 1_{\ell < \underline{D}_{n-1}(s)} \overline{V}_{n-1}(k, 0, \ell, z, 0, 0)$$

   converges to $\overline{\upsilon}_{\overline{F}}(s) = 1_{\ell \geq \underline{D}(s)} p(1+r) + 1_{\ell < \underline{D}(s)} \overline{V}(k, 0, \ell, z, 0, 0)$. There are two cases.

   (a) $\ell \geq \underline{D}(s)$, whence $\overline{\upsilon}_{\overline{F}}(s)$ equals $p(1+r)$. But $\underline{D}_{n-1}(s)$ is nondecreasing in n so for all n, we have $\ell \geq \underline{D}_{n-1}(s)$ and thus $\overline{\Upsilon}_n(s)$ equals $p(1+r)$ whose limit as $n \to \infty$ is trivially $p(1+r) = \overline{\upsilon}_{\overline{F}}(s)$.

   (b) $\ell < \underline{D}(s)$, whence $\overline{\upsilon}_{\overline{F}}(s)$ equals $\overline{V}(k, 0, \ell, z, 0, 0)$ Since

$$\underline{D}(s) = \lim_{n \to \infty} \underline{D}_{n-1}(s),$$

   there is an n^* such that for all $n > n^*$, $\ell < \underline{D}_{n-1}(s)$ and thus $\overline{\Upsilon}_n(s)$ equals $\overline{V}_{n-1}(k, 0, \ell, z, 0, 0)$ which converges to $\overline{V}(k, 0, \ell, z, 0, 0)$ as $n \to \infty$.

2. We claim that $\overline{\psi}_{\overline{F}}(s) = \overline{\Psi}(s)$ or, equivalently, that

$$\overline{\Psi}_n(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{\Upsilon}_n(k, p, 1, z', r, b) H\left(\overline{\Omega}_{n-1}(k, p, z', r, b)\right) \\ +\overline{\Upsilon}_n(k, p, 0, z', r, b)\left[1 - H\left(\overline{\Omega}_{n-1}(k, p, z', r, b)\right)\right] \end{array} \right\} dG(z'|z_{-1})$$

converges to

$$\overline{\psi}_{\overline{F}}(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{\Upsilon}(k,p,1,z',r,b)\, H\left(\overline{\Omega}(k,p,z',r,b)\right) \\ +\overline{\Upsilon}(k,p,0,z',r,b)\left[1 - H\left(\overline{\Omega}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1})$$

where we have used part 1 to substitute $\overline{\Upsilon}$ for $\overline{v}_{\overline{F}}$ in $\overline{\psi}_{\overline{F}}(s)$. By Lemma 1 and (3), for any $\varepsilon > 0$ there is an $n^* < \infty$ such that for $n > n^*$ and every z', $\left|\overline{\Upsilon}_n(k,p,\ell,z',r,b) - \overline{\Upsilon}(k,p,\ell,z',r,b)\right| < \varepsilon/2$ for $\ell \in \{0,1\}$ and

$$\left|H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right) - H\left(\overline{\Omega}(k,p,z',r,b)\right)\right| < \varepsilon/\left(2\overline{S}\right),$$

whence $\left|\overline{\Psi}_n(s) - \overline{\psi}_{\overline{F}}(s)\right| \leq \varepsilon$; accordingly, $\lim_{n\to\infty} \overline{\Psi}_n(s) = \overline{\psi}_{\overline{F}}(s)$.

3. We claim that $\overline{\gamma}_{\overline{F}}(s) = \overline{\Gamma}(s)$ or, equivalently, that

$$\overline{\Gamma}_n(s) = \left\{(k',p',r') \in B : \overline{\Psi}_n(k',p',r',z,b) \geq p'(1+\rho)\right\}$$

converges to $\overline{\gamma}_{\overline{F}}(s) = \left\{(k',p',r') \in B : \overline{\Psi}(k',p',r',z,b) \geq p'(1+\rho)\right\}$ (where we have used part 2 to substitute $\overline{\Psi}$ for $\overline{\psi}_{\overline{F}}$ in $\overline{\gamma}_{\overline{F}}(s)$). First suppose (k',p',r') is in $\overline{\Gamma}(s)$. Then for each n, (k',p',r') is in $\overline{\Gamma}_n(s)$ by (54) whence $\overline{\Psi}_n(k',p',r',z,b) \geq p'(1+\rho)$. And since weak inequalities are preserved in the limit, we have $\overline{\Psi}(k',p',r',z,b) \geq p'(1+\rho)$ as well: (k',p',r') is in $\overline{\gamma}_{\overline{F}}(s)$. Conversely, suppose (k',p',r') is in $\overline{\gamma}_{\overline{F}}(s)$, whence $\overline{\Psi}(k',p',r',z,b) \geq p'(1+\rho)$. For each n,

$$\overline{\Psi}_n(k',p',r',z,b) \geq p'(1+\rho)$$

by part 2 of Theorem 1 and thus (k',p',r') is in $\overline{\Gamma}_n(s)$; but then (k',p',r') is in $\overline{\Gamma}(s)$ by (54). We have shown that $\overline{\gamma}_{\overline{F}}(s) = \overline{\Gamma}(s)$ as claimed.

4. We claim that $w_{\overline{F}}(s) = \overline{W}(s)$ or, equivalently, that

$$\overline{W}_n = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{V}_{n-1}(k,p,1,z',r,b)\, H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right) \\ +\overline{V}_{n-1}(k,p,0,z',r,b)\left[1 - H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1})$$

converges to

$$w_{\overline{F}}(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \overline{V}(k,p,1,z',r,b)\, H\left(\overline{\Omega}(k,p,z',r,b)\right) \\ +\overline{V}(k,p,0,z',r,b)\left[1 - H\left(\overline{\Omega}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1}).$$

By Lemma 1 and (3), for any $\varepsilon > 0$ there is an $n^* < \infty$ such that for $n > n^*$ and every z', $\left|\overline{V}_n(k,p,\ell,z',r,b) - \overline{V}(k,p,\ell,z',r,b)\right| < \varepsilon/2$ for $\ell \in \{0,1\}$ and $\left|H\left(\overline{\Omega}_{n-1}(k,p,z',r,b)\right) - H\left(\overline{\Omega}(k,p,z',r,b)\right)\right| < \varepsilon/\left(2\overline{S}\right)$, whence

$$\left|\overline{W}_n(s) - w_{\overline{F}}(s)\right| \leq \varepsilon;$$

accordingly, $\lim_{n\to\infty} \overline{W}_n(s) = w_{\overline{F}}(s)$ as claimed.

5. For any function W defined on (k',p',r',z,b), let $f_W(s,k',p',r')$ denote the function $b + e(k,k',p,p',\ell,z,r) + \beta W(k',p',r',z,b)$. We claim that $v^c_{\overline{F}}(s) = \overline{V}^c(s)$ or, equivalently, that

$$\overline{V}^c_n(s) = \max_{(k',p',r')\in\overline{\Gamma}_n(z,b)} f_{\overline{W}_n}(s,k',p',r')$$

converges to $v^c_{\overline{F}}(s) = \max\limits_{(k',p',r')\in\overline{\Gamma}(z,b)} f_{\overline{W}}(s,k',p',r')$ (where we have used parts 3 and 4 to substitute $\overline{\Gamma}$ and $\overline{W}$ for $w_{\overline{F}}$ and $\overline{\gamma}_{\overline{F}}$, respectively, in $v^c_{\overline{F}}(s)$). Henceforth let (k',p',r') denote an element of $\overline{\Gamma}(z,b)$ that satisfies $f_{\overline{W}}(s,k',p',r') = v^c_{\overline{F}}(s)$. Then for all n, $f_{\overline{W}_n}(s,k',p',r') \geq f_{\overline{W}}(s,k',p',r')$ by part 2 of Theorem 1. Moreover, (k',p',r') is in $\overline{\Gamma}_n(z,b)$ by (54). Accordingly, $\overline{V}^c_n(s) \geq v^c_{\overline{F}}(s)$; since weak inequalities are preserved in the limit, this implies $\overline{V}^c(s) \geq v^c_{\overline{F}}(s)$. Now suppose $\overline{V}^c(s) > v^c_{\overline{F}}(s)$ and let $\varepsilon = \overline{V}^c(s) - v^c_{\overline{F}}(s) > 0$. For each n, let $(k'_n,p'_n,r'_n) \in \overline{\Gamma}_n(s)$ satisfy $f_{\overline{W}_n}(s,k'_n,p'_n,r'_n) = \overline{V}^c_n(s)$. Since $\lim_{n\to\infty}\overline{V}^c_n(s) = \overline{V}^c(s)$, there is an $n^* < \infty$ such that for each $n > n^*$,

$$\overline{V}^c_n(s) - v^c_{\overline{F}}(s) = f_{\overline{W}_n}(s,k'_n,p'_n,r'_n) - f_{\overline{W}}(s,k',p',r') > \varepsilon/2. \tag{74}$$

By uniform continuity of $f_{\overline{W}}$, there is a $\delta > 0$ such that for any $(k''_n,p''_n,r''_n) \in B$ satisfying $|(k''_n,p''_n,r''_n) - (k'_n,p'_n,r'_n)| < \delta$, we have

$$C_n \stackrel{d}{=} |f_{\overline{W}}(s,k'_n,p'_n,r'_n) - f_{\overline{W}}(s,k''_n,p''_n,r''_n)| < \varepsilon/4. \tag{75}$$

Further, by part 1 of Lemma 1 and Remark 1, for this particular δ there is a n^{**} such that for each $n > n^{**}$, there is a $(k''_n,p''_n,r''_n) \in \overline{\Gamma}(s)$ such that

$|(k''_n, p''_n, r''_n) - (k'_n, p'_n, r'_n)| < \delta$ and thus (75) holds. Moreover, since $(k''_n, p''_n, r''_n) \in \overline{\Gamma}(s)$, we have $f_{\overline{W}}(s, k''_n, p''_n, r''_n) \leq f_{\overline{W}}(s, k', p', r')$ by definition of (k', p', r'), whence by (74) and the triangle inequality,

$$\varepsilon/2 \quad < \quad f_{\overline{W}_n}(s, k'_n, p'_n, r'_n) \quad - \quad f_{\overline{W}}(s, k''_n, p''_n, r''_n) \quad \leq \quad B_n \quad + \quad C_n.$$

where $B_n = \left|f_{\overline{W}_n}(s, k'_n, p'_n, r'_n) - f_{\overline{W}}(s, k'_n, p'_n, r'_n)\right|$ and C_n is defined in (75). Finally, by part 1 of Lemma 1, there is a n^{***} such that for each $n > n^{***}$, $B_n < \varepsilon/4$. Accordingly, for all $n > \max\{n^*, n^{**}, n^{***}\}$ we have $\varepsilon/2 < B_n + C_n < \varepsilon/2$, a contradiction.

6. We claim that $v_{\overline{F}}(s) = \overline{V}(s)$ or, equivalently, that

$$\overline{V}_n(s) = \max\left\{0, \alpha k - \phi(p, r), \overline{V}^c_n(s)\right\}$$

converges to $v_{\overline{F}}(s) = \max\left\{0, \alpha k - \phi(p, r), \overline{V}^c(s)\right\}$ (where we have used part 5 to substitute $\overline{V}^c$ for $v^c_{\overline{F}}$ in $v_{\overline{F}}(s)$). The result holds by part 5 and since max is a continuous function.[32]

7. We claim that $\underline{d}_{\overline{F}}(s) = \underline{D}(s)$ or, equivalently, that

$$\underline{D}_n(s) = \min\left\{\{1 + \varepsilon\} \cup \left\{\ell' \in [0, 1] : \phi(p, r) \leq \overline{V}_n(k, 0, \ell', z, 0, b)\right\}\right\}$$

converges to $\underline{d}_{\overline{F}}(s) = \min\left\{\{1 + \varepsilon\} \cup \left\{\ell' \in [0, 1] : \phi(p, r) \leq \overline{V}(k, 0, \ell', z, 0, b)\right\}\right\}$ (where we have used part 6 to substitute $\overline{V}$ for $v_{\overline{F}}$ in $\underline{d}_{\overline{F}}(s)$). Let $\ell_n = \underline{D}_n(s)$, $\ell_\infty = \lim_{n\to\infty} \ell_n = \underline{D}(s)$, and $\widetilde{\ell} = \underline{d}_{\overline{F}}(s)$. We will show that $\ell_\infty = \widetilde{\ell}$ in the following three (exhaustive) cases.

[32] The result can also be easily shown from first principles . If $\overline{V}^c(s) \geq \max\{0, \alpha k - \phi(p, r)\}$ then $v_{\overline{F}}(s)$ equals $\overline{V}^c(s)$ and, moreover, $\overline{V}^c_n(s) \geq \max\{0, \alpha k - \phi(p, r)\}$ by part 2 of Theorem 1 whence $\overline{V}_n(s)$ equals $\overline{V}^c_n(s)$; the result then holds from part 5. If instead $\overline{V}^c(s) < \max\{0, \alpha k - \phi(p, r)\}$ then $v_{\overline{F}}(s)$ equals $\max\{0, \alpha k - \phi(p, r)\}$ and, moreover, there is an $n^* < \infty$ such that for all $n > n^*$, $\overline{V}^c_n(s) < \max\{0, \alpha k - \phi(p, r)\}$ and thus $\overline{V}_n(s) = \max\{0, \alpha k - \phi(p, r)\} = v_{\overline{F}}(s)$ as claimed.

(a) Suppose $\ell_\infty = 1+\varepsilon$. In this case, there is an $n^* < \infty$ such that for all $n > n^*$, $\ell_n = 1+\varepsilon$ whence $\phi(p,r) > \overline{V}_n(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$. But then by part 2 of Theorem 1, $\phi(p,r) > \overline{V}(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$, whence $\widetilde{\ell} = 1+\varepsilon = \ell_\infty$.

(b) Suppose $\widetilde{\ell} = 1+\varepsilon$. Then $\phi(p,r) > \overline{V}(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$. Hence, by part 1 of Lemma 1, there is an n^* such that for all $n > n^*$, $\phi(p,r) > \overline{V}_n(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$ and thus $\ell_n = 1+\varepsilon$. Accordingly, $\ell_\infty = \lim_{n\to\infty} \ell_n = 1+\varepsilon = \widetilde{\ell}$.

(c) Suppose $\ell_\infty, \widetilde{\ell} \in [0,1]$. This implies $\phi(p,r) \le \overline{V}_n(k,0,\ell_n,z,0,b)$ and $\phi(p,r) \le \overline{V}\left(k,0,\widetilde{\ell},z,0,b\right)$. Since $\overline{V}_n$ is nonincreasing in n by part 2 of Theorem 1, we have $\phi(p,r) \le \overline{V}\left(k,0,\widetilde{\ell},z,0,b\right) \le \overline{V}_n\left(k,0,\widetilde{\ell},z,0,b\right)$ whence $\widetilde{\ell} \ge \ell_n$ and hence, taking limits, $\widetilde{\ell} \ge \ell_\infty$. Now suppose $\widetilde{\ell} > \ell_\infty$, whence $\phi(p,r) - \overline{V}(k,0,\ell_\infty,z,0,b) = \varepsilon$ for some $\varepsilon > 0$; we will derive a contradiction. By the triangle inequality

$$\left|\overline{V}_n(k,0,\ell_n,z,0,b) - \overline{V}(k,0,\ell_\infty,z,0,b)\right| \le A_n + B_n$$

where $A_n = \left|\overline{V}_n(k,0,\ell_n,z,0,b) - \overline{V}(k,0,\ell_n,z,0,b)\right|$ and

$$B_n = \left|\overline{V}(k,0,\ell_n,z,0,b) - \overline{V}(k,0,\ell_\infty,z,0,b)\right|.$$

By part 1 of Lemma 1, there must be an $n^* < \infty$ such that for all $n > n^*$, $A_n < \varepsilon/2$. By part 2 of Lemma 1 and since $\ell_\infty = \lim_{n\to\infty} \ell_n$, there is an $n^{**} < \infty$ such that for all $n > n^{**}$, $B_n < \varepsilon/2$. Thus, for all $n > \max\{n^*, n^{**}\}$, $A_n + B_n < \varepsilon$ and thus $\phi(p,r) - \overline{V}_n(k,0,\ell_n,z,0,b) \ge \varepsilon - (A_n + B_n) > 0$, which contradicts the definition of ℓ_n. We conclude that $\widetilde{\ell} = \ell_\infty$ in this case as well.

8. We claim that $\underline{l}_{\overline{F}}(s) = \underline{L}(s)$ or, equivalently, that

$$\underline{L}_n(s) = \min\left\{\{1+\varepsilon\} \cup \left\{\ell' \in [0,1] : \alpha k \le \overline{V}_n^c(k,0,\ell',z,0,b)\right\}\right\}$$

converges to

$$\underline{l}_{\overline{F}}(s) = \min\left\{\{1+\varepsilon\} \cup \left\{\ell' \in [0,1] : \alpha k \leq \overline{V}^c(k,0,\ell',z,0,b)\right\}\right\}$$

where we have used part 5 to substitute $\overline{V}^c$ for $v^c_{\overline{F}}$ in $\underline{l}_{\overline{F}}(s)$. Let $\ell_n = \underline{L}_n(s)$, $\ell_\infty = \lim_{n\to\infty} \ell_n = \underline{L}(s)$, and $\widetilde{\ell} = \underline{l}_{\overline{F}}(s)$. We will show that $\ell_\infty = \widetilde{\ell}$ in the following three (exhaustive) cases.

(a) Suppose $\ell_\infty = 1+\varepsilon$. In this case, there is an $n^* < \infty$ such that for all $n > n^*$, $\ell_n = 1+\varepsilon$ whence $\alpha k > \overline{V}^c_n(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$. But then by part 2 of Theorem 1, $\alpha k > \overline{V}^c(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$, whence $\widetilde{\ell} = 1+\varepsilon = \ell_\infty$.

(b) Suppose $\widetilde{\ell} = 1+\varepsilon$. Then $\alpha k > \overline{V}^c(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$. Hence, by part 1 of Lemma 1, there is an n^* such that for all $n > n^*$, $\alpha k > \overline{V}^c_n(k,0,\ell,z,0,b)$ for all $\ell \in [0,1]$ and thus $\ell_n = 1+\varepsilon$. Accordingly, $\ell_\infty = \lim_{n\to\infty} \ell_n = 1+\varepsilon = \widetilde{\ell}$.

(c) Suppose $\ell_\infty, \widetilde{\ell} \in [0,1]$. This implies $\alpha k \leq \overline{V}^c_n(k,0,\ell_n,z,0,b)$ and $\alpha k \leq \overline{V}^c\left(k,0,\widetilde{\ell},z,0,b\right)$. Since $\overline{V}^c_n$ is nonincreasing in n by part 2 of Theorem 1, we have $\alpha k \leq \overline{V}^c\left(k,0,\widetilde{\ell},z,0,b\right) \leq \overline{V}^c_n\left(k,0,\widetilde{\ell},z,0,b\right)$ whence $\widetilde{\ell} \geq \ell_n$ and hence, taking limits, $\widetilde{\ell} \geq \ell_\infty$. Now suppose $\widetilde{\ell} > \ell_\infty$, whence $\alpha k - \overline{V}^c(k,0,\ell_\infty,z,0,b) = \varepsilon$ for some $\varepsilon > 0$; we will derive a contradiction. By the triangle inequality

$$\left|\overline{V}^c_n(k,0,\ell_n,z,0,b) - \overline{V}^c(k,0,\ell_\infty,z,0,b)\right| \leq A_n + B_n$$

where $A_n = \left|\overline{V}^c_n(k,0,\ell_n,z,0,b) - \overline{V}^c(k,0,\ell_n,z,0,b)\right|$ and

$$B_n = \left|\overline{V}^c(k,0,\ell_n,z,0,b) - \overline{V}^c(k,0,\ell_\infty,z,0,b)\right|.$$

By part 1 of Lemma 1, there must be an $n^* < \infty$ such that for all $n > n^*$, $A_n < \varepsilon/2$. By part 2 of Lemma 1 and since $\ell_\infty = \lim_{n\to\infty} \ell_n$, there is an $n^{**} < \infty$ such that for all $n > n^{**}$, $B_n < \varepsilon/2$. Thus, for all $n > \max\{n^*, n^{**}\}$, $A_n + B_n < \varepsilon$ and thus $\alpha k - \overline{V}^c_n(k, 0, \ell_n, z, 0, b) \geq \varepsilon - (A_n + B_n) > 0$, which contradicts the definition of ℓ_n. We conclude that $\widetilde{\ell} = \ell_\infty$ in this case as well.

9. We claim that $\underline{\lambda_{\overline{F}}}(s) = \underline{\Lambda}(s)$ or, equivalently, that

$$\underline{\Lambda}_n(s) = \min\{\underline{L}_n(k, z, 0), \max\{\underline{D}_n(s), \underline{L}_n(k, z, b)\}\}$$

converges to $\underline{\lambda_{\overline{F}}}(s) = \min\{\underline{L}(k, z, 0), \max\{\underline{D}(s), \underline{L}(k, z, b)\}\}$ (where we have used parts 7 and 8 to substitute $\underline{D}$ and $\underline{L}$ for $\underline{d_{\overline{F}}}$ and $\underline{l_{\overline{F}}}$, respectively, in $\underline{\lambda_{\overline{F}}}(s)$). The result holds by parts 7 and 8 and since min and max are continuous functions.

10. We claim that $\overline{\omega}_{\overline{F}}(s) = \underline{\Omega}(s)$ or, equivalently, that

$$\overline{\Omega}_n = u - (\zeta - \xi)\min\{1, \underline{\Lambda}_n(s)\} - \xi \min\{1, \max\{\underline{D}_n(s), \underline{\Lambda}_n(s)\}\}$$

converges to $\overline{\omega}_{\overline{F}}(s) = u - (\zeta - \xi)\min\{1, \underline{\Lambda}(s)\} - \xi\min\{1, \max\{\underline{D}(s), \underline{\Lambda}(s)\}\}$ (where we have used parts 7 and 9 to substitute $\underline{D}$ and $\underline{\Lambda}$ for $\underline{d_{\overline{F}}}$ and $\underline{\lambda_{\overline{F}}}$, respectively, in $\overline{\omega}_{\overline{F}}(s)$). The result hold by parts 7 and 9 and since min and max are continuous functions.

We next show that at each state s in S, $\underline{F}_n(s)$ converges to $b_{\underline{F}}(s)$ whence $\underline{F}(s) = b_{\underline{F}}(s)$. As above, we consider each component in turn.

1. We claim that $\underline{\upsilon_F}(s) = \underline{\Upsilon}(s)$ or, equivalently, that

$$\underline{\Upsilon}_n(s) = 1_{\ell > \overline{D}_{n-1}(s)} p(1 + r) + 1_{\ell \leq \overline{D}_{n-1}(s)} \underline{V}_{n-1}(k, 0, \ell, z, 0, 0)$$

converges to $\underline{\upsilon_F}(s) = 1_{\ell > \overline{D}(s)} p(1 + r) + 1_{\ell \leq \overline{D}(s)} \underline{V}(k, 0, \ell, z, 0, 0)$. There are two cases.

   (a) $\ell > \overline{D}(s)$, whence $\underline{\upsilon_F}(s)$ equals $p(1 + r)$. Since $\overline{D}(s) = \lim_{n\to\infty} \overline{D}_{n-1}(s)$, there is an n^* such that for all $n > n^*$, $\ell > \overline{D}_{n-1}(s)$ and thus $\underline{\Upsilon}_n(s)$ equals $p(1 + r)$ as well, proving the result.

(b) $\ell \leq \overline{D}(s)$, whence $\upsilon_{\underline{F}}(s)$ equals $\underline{V}(k,0,\ell,z,0,0)$. As $\overline{D}_{n-1}(s)$ is nonincreasing in n, for all n we have $\ell \leq \overline{D}_{n-1}(s)$ whence $\underline{\Upsilon}_n(s)$ equals $\underline{V}_{n-1}(k,0,\ell,z,0,0)$ which converges to $\underline{V}(k,0,\ell,z,0,0)$ as $n \to \infty$.

2. We claim that $\underline{\psi}_{\underline{F}}(s) = \underline{\Psi}(s)$ or, equivalently, that

$$\underline{\Psi}_n(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \underline{\Upsilon}_n(k,p,1,z',r,b)\, H\left(\underline{\Omega}_{n-1}(k,p,z',r,b)\right) \\ +\underline{\Upsilon}_n(k,p,0,z',r,b)\left[1 - H\left(\underline{\Omega}_{n-1}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1})$$

converges to

$$\underline{\psi}_{\underline{F}}(s) = \int_{z'=\underline{z}}^{\overline{z}} \left\{ \begin{array}{c} \underline{\Upsilon}(k,p,1,z',r,b)\, H(\underline{\Omega}(k,p,z',r,b)) \\ +\underline{\Upsilon}(k,p,0,z',r,b)\left[1 - H(\underline{\Omega}(k,p,z',r,b))\right] \end{array} \right\} dG(z'|z_{-1})$$

where we have used part 1 to substitute $\underline{\Upsilon}$ for $\upsilon_{\underline{F}}$ in $\underline{\psi}_{\underline{F}}(s)$. By Lemma 1 and (3), for any $\varepsilon > 0$ there is an $n^* < \infty$ such that for $n > n^*$ and every z', $|\underline{\Upsilon}_n(k,p,\ell,z',r,b) - \underline{\Upsilon}(k,p,\ell,z',r,b)| < \varepsilon/2$ for $\ell \in \{0,1\}$ and

$$\left|H\left(\underline{\Omega}_{n-1}(k,p,z',r,b)\right) - H(\underline{\Omega}(k,p,z',r,b))\right| < \varepsilon/\left(2\overline{S}\right),$$

whence $\left|\underline{\Psi}_n(s) - \underline{\psi}_{\underline{F}}(s)\right| \leq \varepsilon$; accordingly, $\lim_{n\to\infty} \underline{\Psi}_n(s) = \underline{\psi}_{\underline{F}}(s)$.

3. We claim that $\underline{\gamma}_{\underline{F}}(s) = \underline{\Gamma}(s)$ or, equivalently, that

$$\underline{\Gamma}_n(s) = \{(k',p',r') \in B : \underline{\Psi}_n(k',p',r',z,b) > p'(1+\rho)\}$$

converges to $\underline{\gamma}_{\underline{F}}(s) = \{(k',p',r') \in B : \underline{\Psi}(k',p',r',z,b) > p'(1+\rho)\}$ (where we have used part 2 to substitute $\underline{\Psi}$ for $\underline{\psi}_{\underline{F}}$ in $\underline{\gamma}_{\underline{F}}(s)$). First suppose (k',p',r') is in $\underline{\Gamma}(s)$. Then by (54) there is an $n^* < \infty$ such that (k',p',r') is in $\underline{\Gamma}_{n^*}(s)$ whence, by part 2 of Theorem 1, (k',p',r') is in $\underline{\Gamma}_n(s)$ for all $n > n^*$ and thus $\underline{\Psi}_n(k',p',r',z,b) > p'(1+\rho)$ for all such n; but then, by part 2 of Theorem 1, $\underline{\Psi}(k',p',r',z,b) > p'(1+\rho)$ as well whence (k',p',r') is in $\underline{\gamma}_{\underline{F}}(s)$. Conversely, suppose that (k',p',r') is in $\underline{\gamma}_{\underline{F}}(s)$. Then by (53) and part 2 of Theorem 1, there is an $n^* < \infty$ such that, for all $n > n^*$, $\underline{\Psi}_n(k',p',r',z,b) > p'(1+\rho)$ and thus (k',p',r') is in $\underline{\Gamma}_n(s)$; but then (k',p',r') is in $\underline{\Gamma}(s)$ by (54).

4. We claim that $w_{\underline{F}}(s) = \underline{W}(s)$ or, equivalently, that

$$\underline{W}_n = \int_{z'=\underline{z}}^{\bar{z}} \left\{ \begin{array}{c} \underline{V}_{n-1}(k,p,1,z',r,b) H\left(\underline{\Omega}_{n-1}(k,p,z',r,b)\right) \\ +\underline{V}_{n-1}(k,p,0,z',r,b)\left[1 - H\left(\underline{\Omega}_{n-1}(k,p,z',r,b)\right)\right] \end{array} \right\} dG(z'|z_{-1})$$

converges to

$$w_{\underline{F}}(s) = \int_{z'=\underline{z}}^{\bar{z}} \left\{ \begin{array}{c} \underline{V}(k,p,1,z',r,b) H(\underline{\Omega}(k,p,z',r,b)) \\ +\underline{V}(k,p,0,z',r,b)\left[1 - H(\underline{\Omega}(k,p,z',r,b))\right] \end{array} \right\} dG(z'|z_{-1}).$$

By Lemma 1 and (3), for any $\varepsilon > 0$ there is an $n^* < \infty$ such that for $n > n^*$ and every z', $|\underline{V}_n(k,p,\ell,z',r,b) - \underline{V}(k,p,\ell,z',r,b)| < \varepsilon/2$ for $\ell \in \{0,1\}$ and $\left|H\left(\underline{\Omega}_{n-1}(k,p,z',r,b)\right) - H(\underline{\Omega}(k,p,z',r,b))\right| < \varepsilon/\left(2\overline{S}\right)$, whence

$$|\underline{W}_n(s) - w_{\underline{F}}(s)| \le \varepsilon;$$

accordingly, $\lim_{n\to\infty} \underline{W}_n(s) = w_{\underline{F}}(s)$ as claimed.

5. For any function W defined on (k',p',r',z,b), let $f_W(s,k',p',r')$ denote the function $b + e(k,k',p,p',\ell,z,r) + \beta W(k',p',r',z,b)$. We claim that $v^c_{\underline{F}}(s) = \underline{V}^c(s)$ or, equivalently, that[33]

$$\underline{V}^c_n(s) = \max_{(k',p',r') \in \underline{\Gamma}_n(z,b)} f_{\underline{W}_n}(s,k',p',r')$$

converges to $v^c_{\underline{F}}(s) = \max_{(k',p',r') \in \underline{\Gamma}(z,b)} f_{\underline{W}}(s,k',p',r')$ (where we have used parts 3 and 4 above to substitute $\underline{\Gamma}$ and $\underline{W}$ for $w_{\underline{F}}$ and $\gamma_{\underline{F}}$, respectively, in $v^c_{\underline{F}}(s)$). Henceforth let (k',p',r') denote an element of $\underline{\Gamma}(z,b)$ that satisfies

$$f_{\underline{W}}(s,k',p',r') = v^c_{\underline{F}}(s).$$

Then for all n, $f_{\underline{W}_n}(s,k',p',r') \ge f_{\underline{W}}(s,k',p',r')$ by part 2 of Theorem 1. Moreover, (k',p',r') is in $\underline{\Gamma}_n(z,b)$ by (54). Accordingly, $\underline{V}^c_n(s) \ge v^c_{\underline{F}}(s)$; since weak inequalities are preserved in the limit, this implies $\underline{V}^c(s) \ge v^c_{\underline{F}}(s)$. Now

[33] The max always exists since $\underline{\Gamma}_n(z,b)$ is finite.

suppose $\underline{V}^c(s) > v^c_{\underline{F}}(s)$ and let $\varepsilon = \underline{V}^c(s) - v^c_{\underline{F}}(s) > 0$. For each n, let $(k'_n, p'_n, r'_n) \in \underline{\Gamma}_n(s)$ satisfy $f_{\underline{W}_n}(s, k'_n, p'_n, r'_n) = \underline{V}^c_n(s)$. Since $\lim_{n\to\infty} \underline{V}^c_n(s) = \underline{V}^c(s)$, there is an $n^* < \infty$ such that for each $n > n^*$,

$$\underline{V}^c_n(s) - v^c_{\underline{F}}(s) = f_{\underline{W}_n}(s, k'_n, p'_n, r'_n) - f_{\underline{W}}(s, k', p', r') > \varepsilon/2. \tag{76}$$

By uniform continuity of $f_{\underline{W}}$, there is a $\delta > 0$ such that for any $(k''_n, p''_n, r''_n) \in B$ satisfying $|(k''_n, p''_n, r''_n) - (k'_n, p'_n, r'_n)| < \delta$, we have

$$C_n \overset{d}{=} |f_{\underline{W}}(s, k'_n, p'_n, r'_n) - f_{\underline{W}}(s, k''_n, p''_n, r''_n)| < \varepsilon/4. \tag{77}$$

Further, by part 1 of Lemma 1 and Remark 1, for this particular δ there is a n^{**} such that for each $n > n^{**}$, there is a $(k''_n, p''_n, r''_n) \in \underline{\Gamma}(s)$ such that $|(k''_n, p''_n, r''_n) - (k'_n, p'_n, r'_n)| < \delta$ and thus (77) holds. Moreover, since $(k''_n, p''_n, r''_n) \in \underline{\Gamma}(s)$, we have $f_{\underline{W}}(s, k''_n, p''_n, r''_n) \leq f_{\underline{W}}(s, k', p', r')$ by definition of (k', p', r'), whence by (76) and the triangle inequality,

$$\varepsilon/2 \quad < \quad f_{\underline{W}_n}(s, k'_n, p'_n, r'_n) \quad - \quad f_{\underline{W}}(s, k''_n, p''_n, r''_n) \quad \leq \quad B_n \quad + \quad C_n.$$

where $B_n = \left|f_{\underline{W}_n}(s, k'_n, p'_n, r'_n) - f_{\underline{W}}(s, k'_n, p'_n, r'_n)\right|$ and C_n is defined in (77). Finally, by part 1 of Lemma 1, there is a n^{***} such that for each $n > n^{***}$, $B_n < \varepsilon/4$. Accordingly, for all $n > \max\{n^*, n^{**}, n^{***}\}$ we have $\varepsilon/2 < B_n + C_n < \varepsilon/2$, a contradiction.

6. We claim that $v_{\underline{F}}(s) = \underline{V}(s)$ or, equivalently, that

$$\underline{V}_n(s) = \max\{0, \alpha k - \phi(p, r), \underline{V}^c_n(s)\}$$

converges to $v_{\underline{F}}(s) = \max\{0, \alpha k - \phi(p, r), \underline{V}^c(s)\}$ (where we have used part 5 to substitute $\underline{V}^c$ for $v^c_{\underline{F}}$ in $v_{\underline{F}}(s)$). The result holds by part 5 and since max is a continuous function.[34]

[34]The result can also be easily shown from first principles . If $\underline{V}^c(s) \geq \max\{0, \alpha k - \phi(p, r)\}$ then

7. We claim that $\overline{d}_{\underline{F}}(s) = \overline{D}(s)$ or, equivalently, that[35]

$$\overline{D}_n(s) = \min\{\{1+\varepsilon\} \cup \{\ell' \in S_\ell : \phi(p,r) < \underline{V}_n(k,0,\ell',z,0,b)\}\}$$

converges to

$$\overline{d}_{\underline{F}}(s) = \min\{\{1+\varepsilon\} \cup \{\ell' \in S_\ell : \phi(p,r) < \underline{V}(k,0,\ell',z,0,b)\}\}$$

where we have used part 6 to substitute $\underline{V}$ for $v_{\underline{F}}$ in $\overline{d}_{\underline{F}}(s)$. Let $\ell_n = \overline{D}_n(s)$, $\ell_\infty = \lim_{n\to\infty} \ell_n = \overline{D}(s)$, and $\widetilde{\ell} = \overline{d}_{\underline{F}}(s)$. We will show that $\ell_\infty = \widetilde{\ell}$ in the following three (exhaustive) cases.

(a) Suppose $\ell_\infty = 1+\varepsilon$. In this case, there is an $n^* < \infty$ such that for all $n > n^*$, $\ell_n = 1+\varepsilon$ whence $\phi(p,r) \geq \underline{V}_n(k,0,\ell,z,0,b)$ for all $\ell \in S_\ell$. But then by part 2 of Theorem 1, $\phi(p,r) \geq \underline{V}(k,0,\ell,z,0,b)$ for all $\ell \in S_\ell$, whence $\widetilde{\ell} = 1+\varepsilon = \ell_\infty$.

(b) Suppose $\widetilde{\ell} = 1+\varepsilon$. Then $\phi(p,r) \geq \underline{V}(k,0,\ell,z,0,b)$ for all $\ell \in S_\ell$. Hence, for all n, $\phi(p,r) \geq \underline{V}_n(k,0,\ell,z,0,b)$ for all $\ell \in S_\ell$ by part 2 of Lemma 1 and thus $\ell_n = 1+\varepsilon$. Accordingly, $\ell_\infty = \lim_{n\to\infty} \ell_n = 1+\varepsilon = \widetilde{\ell}$.

(c) Suppose $\ell_\infty, \widetilde{\ell} \in S_\ell$. This implies $\phi(p,r) < \underline{V}_n(k,0,\ell_n,z,0,b)$ and $\phi(p,r) < \underline{V}\left(k,0,\widetilde{\ell},z,0,b\right)$. Since $\underline{V}_n$ is nondecreasing in n by part 2 of Theorem 1, we have $\phi(p,r) < \underline{V}_n(k,0,\ell_n,z,0,b) \leq \underline{V}(k,0,\ell_n,z,0,b)$ whence $\widetilde{\ell} \leq \ell_n$ and hence, taking limits, $\widetilde{\ell} \leq \ell_\infty$. Now suppose $\widetilde{\ell} < \ell_\infty$, whence

$$\varepsilon = \underline{V}\left(k,0,\ell_\infty - \frac{1}{N},z,0,b\right) - \phi(p,r)$$

$v_{\underline{F}}(s)$ equals $\underline{V}^c(s)$ and, moreover, $\underline{V}^c_n(s) \geq \max\{0, \alpha k - \phi(p,r)\}$ by part 2 of Theorem 1 whence $\underline{V}_n(s)$ equals $\underline{V}^c_n(s)$; the result then holds from part 5. If instead $\underline{V}^c(s) < \max\{0, \alpha k - \phi(p,r)\}$ then $v_{\underline{F}}(s)$ equals $\max\{0, \alpha k - \phi(p,r)\}$ and, moreover, there is an $n^* < \infty$ such that for all $n > n^*$, $\underline{V}^c_n(s) < \max\{0, \alpha k - \phi(p,r)\}$ and thus $\underline{V}_n(s) = \max\{0, \alpha k - \phi(p,r)\} = v_{\underline{F}}(s)$ as claimed.

[35]The min exists as S_ℓ is finite.

is positive where N is the grid size of S_ℓ. By the triangle inequality

$$\left|\underline{V}_n\left(k, 0, \ell_n - \frac{1}{N}, z, 0, b\right) - \underline{V}\left(k, 0, \ell_\infty - \frac{1}{N}, z, 0, b\right)\right| \leq A_n + B_n$$

where $A_n = \left|\underline{V}_n\left(k, 0, \ell_n - 1/N, z, 0, b\right) - \underline{V}\left(k, 0, \ell_n - 1/N, z, 0, b\right)\right|$ and

$$B_n = \left|\underline{V}\left(k, 0, \ell_n - 1/N, z, 0, b\right) - \underline{V}\left(k, 0, \ell_\infty - 1/N, z, 0, b\right)\right|.$$

By part 1 of Lemma 1, there must be an $n^* < \infty$ such that for all $n > n^*$, $A_n < \varepsilon/2$. By part 2 of Lemma 1 and since $\ell_\infty = \lim_{n\to\infty} \ell_n$, there is an $n^{**} < \infty$ such that for all $n > n^{**}$, $B_n < \varepsilon/2$. Thus, for all $n > \max\{n^*, n^{**}\}$, $A_n + B_n < \varepsilon$ and thus $\underline{V}_n\left(k, 0, \ell_n - 1/N, z, 0, b\right) - \phi\left(p, r\right) > \varepsilon - \left(A_n + B_n\right) > 0$, which contradicts the definition of ℓ_n. We conclude that $\widetilde{\ell} = \ell_\infty$ in this case as well.

8. We claim that $\bar{l}_{\underline{F}}\left(s\right) = \overline{L}\left(s\right)$ or, equivalently, that[36]

$$\overline{L}_n\left(s\right) = \min\left\{\{1+\varepsilon\} \cup \left\{\ell' \in S_\ell : \alpha k < \underline{V}_n^c\left(k, 0, \ell', z, 0, b\right)\right\}\right\}$$

converges to $\bar{l}_{\underline{F}}\left(s\right) = \min\left\{\{1+\varepsilon\} \cup \left\{\ell' \in S_\ell : \alpha k \leq \underline{V}^c\left(k, 0, \ell', z, 0, b\right)\right\}\right\}$ (where we have used part 6 to substitute $\underline{V}^c$ for $v_{\underline{F}}^c$ in $\bar{l}_{\underline{F}}\left(s\right)$). Let $\ell_n = \overline{L}_n\left(s\right)$, $\ell_\infty = \lim_{n\to\infty} \ell_n = \overline{L}\left(s\right)$, and $\widetilde{\ell} = \bar{l}_{\underline{F}}\left(s\right)$. We will show that $\ell_\infty = \widetilde{\ell}$ in the following three (exhaustive) cases.

(a) Suppose $\ell_\infty = 1+\varepsilon$. In this case, there is an $n^* < \infty$ such that for all $n > n^*$, $\ell_n = 1+\varepsilon$ whence $\alpha k \geq \underline{V}_n^c\left(k, 0, \ell, z, 0, b\right)$ for all $\ell \in S_\ell$. But then by part 2 of Theorem 1, $\alpha k \geq \underline{V}^c\left(k, 0, \ell, z, 0, b\right)$ for all $\ell \in S_\ell$, whence $\widetilde{\ell} = 1+\varepsilon = \ell_\infty$.

(b) Suppose $\widetilde{\ell} = 1+\varepsilon$. Then $\alpha k \geq \underline{V}^c\left(k, 0, \ell, z, 0, b\right)$ for all $\ell \in S_\ell$. Hence, for all n, $\alpha k \geq \underline{V}_n^c\left(k, 0, \ell, z, 0, b\right)$ for all $\ell \in S_\ell$ by part 2 of Lemma 1 and thus $\ell_n = 1+\varepsilon$. Accordingly, $\ell_\infty = \lim_{n\to\infty} \ell_n = 1+\varepsilon = \widetilde{\ell}$.

[36]The min exists as S_ℓ is finite.

(c) Suppose $\ell_\infty, \widetilde{\ell} \in S_\ell$. This implies $\alpha k < \underline{V}_n^c(k,0,\ell_n,z,0,b)$ and $\alpha k < \underline{V}^c\left(k,0,\widetilde{\ell},z,0,b\right)$. Since $\underline{V}_n^c$ is nondecreasing in n by part 2 of Theorem 1, we have $\alpha k < \underline{V}_n^c(k,0,\ell_n,z,0,b) \leq \underline{V}^c(k,0,\ell_n,z,0,b)$ whence $\widetilde{\ell} \leq \ell_n$ and hence, taking limits, $\widetilde{\ell} \leq \ell_\infty$. Now suppose $\widetilde{\ell} < \ell_\infty$, whence

$$\varepsilon = \underline{V}^c\left(k,0,\ell_\infty - \frac{1}{N},z,0,b\right) - \alpha k$$

is positive where N is the grid size of S_ℓ. By the triangle inequality

$$\left|\underline{V}_n^c\left(k,0,\ell_n - \frac{1}{N},z,0,b\right) - \underline{V}^c\left(k,0,\ell_\infty - \frac{1}{N},z,0,b\right)\right| \leq A_n + B_n$$

where $A_n = \left|\underline{V}_n^c(k,0,\ell_n - 1/N,z,0,b) - \underline{V}^c(k,0,\ell_n - 1/N,z,0,b)\right|$ and

$$B_n = \left|\underline{V}^c(k,0,\ell_n - 1/N,z,0,b) - \underline{V}^c(k,0,\ell_\infty - 1/N,z,0,b)\right|.$$

By part 1 of Lemma 1, there must be an $n^* < \infty$ such that for all $n > n^*$, $A_n < \varepsilon/2$. By part 2 of Lemma 1 and since $\ell_\infty = \lim_{n\to\infty} \ell_n$, there is an $n^{**} < \infty$ such that for all $n > n^{**}$, $B_n < \varepsilon/2$. Thus, for all $n > \max\{n^*, n^{**}\}$, $A_n + B_n < \varepsilon$ and thus $\underline{V}_n^c(k,0,\ell_n - 1/N,z,0,b) - \alpha k > \varepsilon - (A_n + B_n) > 0$, which contradicts the definition of ℓ_n. We conclude that $\widetilde{\ell} = \ell_\infty$ in this case as well.

9. We claim that $\overline{\lambda}_{\underline{F}}(s) = \overline{\Lambda}(s)$ or, equivalently, that

$$\overline{\Lambda}_n(s) = \min\left\{\overline{L}_n(k,z,0), \max\left\{\overline{D}_n(s), \overline{L}_n(k,z,b)\right\}\right\}$$

converges to $\overline{\lambda}_{\underline{F}}(s) = \min\left\{\overline{L}(k,z,0), \max\left\{\overline{D}(s), \overline{L}(k,z,b)\right\}\right\}$ (where we have used parts 7 and 8 to substitute $\overline{D}$ and $\overline{L}$ for $\overline{d}_{\underline{F}}$ and $\overline{l}_{\underline{F}}$, respectively, in $\overline{\lambda}_{\underline{F}}(s)$). The result holds by parts 7 and 8 and since min and max are continuous functions.

10. We claim that $\underline{\omega}_{\underline{F}}(s) = \underline{\Omega}(s)$ or, equivalently, that

$$\underline{\Omega}_n = u - (\zeta - \xi)\min\left\{1, \overline{\Lambda}_n(s)\right\} - \xi \min\left\{1, \max\left\{\overline{D}_n(s), \overline{\Lambda}_n(s)\right\}\right\}$$

converges to $\underline{\omega}_{\underline{F}}(s) = u - (\zeta - \xi)\min\left\{1, \overline{\Lambda}(s)\right\} - \xi\min\left\{1, \max\left\{\overline{D}(s), \overline{\Lambda}(s)\right\}\right\}$ (where we have used parts 7 and 9 to substitute $\overline{D}$ and $\overline{\Lambda}$ for $\overline{d}_{\underline{F}}$ and $\overline{\lambda}_{\underline{F}}$, respectively, in $\underline{\omega}_{\underline{F}}(s)$). The result hold by parts 7 and 9 and since min and max are continuous functions.

Q.E.D.$_{\textbf{Theorem 2}}$

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