

## ADAPTIVE EXPECTATIONS AND STOCK MARKET CRASHES\*

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A theory is developed that explains how stocks can crash without fundamental news and why crashes are more common than frenzies. A crash occurs via the interaction of rational and naive investors. Naive traders believe that prices follow a random walk with serially correlated volatility. Their expectations of future volatility are formed adaptively. When the market crashes, naive traders sell stock in response to the apparent increase in volatility. Since rational traders are risk averse as well, a lower price is needed to clear the market: The crash is a self-fulfilling prophecy. Frenzies cannot occur in this model.

### 1. INTRODUCTION

On October 19, 1987, the S&P 500 index fell by 20.5%. Evidence from option prices suggests that investors expect more crashes to occur (e.g., Ait-Sahalia et al., 2001). What causes such jumps in prices? The explanation should reflect the fact that many traders were responding to the price declines themselves, rather than to news about the economy or firm profitability. According to Shiller's (1989, p. 386) postcrash survey, declining prices on October 14–16 and the morning of October 19 were the news items that most influenced investors' views of the stock market on October 19, 1987 (see also Cutler et al., 1989; Shiller, 1998).

The theory should also explain why crashes happen more often than comparable-sized frenzies, in which prices rise sharply. Nine out of the 10 largest one-day price movements in the postwar period were declines (Hong and Stein, 1999). Since 1945, the Dow Jones Industrial Average has fallen by 5% or more on 11 separate days; the average of these declines was 8.1%. The index has risen by over 5% on only 5 days; the average increase was only 6.6%.

We present a new theory in which a crash results from the interaction between rational and naive traders. The naive traders believe that stock prices follow a random walk with serially correlated volatility. ("Volatility" refers to the variance of the change in stock prices.) They predict future volatility adaptively, as a weighted average of recent squared price changes. This contrasts with the rational traders, who predict future volatility correctly using knowledge of other players' strategies.

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The naive traders' model for predicting future volatility lies in the family of Autoregressive Conditional Heteroskedasticity (ARCH) models proposed by Engle (1982) and Bollerslev (1986). These models have become the dominant approach to modelling changing volatility in econometric analysis of asset markets. Our model appears to be the first to explore how equilibrium prices are affected if some agents use this common type of model to predict future return volatility.

There is some historical justification for the idea that the presence of naive traders makes crashes more likely. The largest crashes, in 1929 and 1987, occurred after extended bull markets that attracted many inexperienced investors into the stock market. The investor Bernard Baruch wrote, in reference to the crash of 1929,

Never before had there been such gambling as there was in those last turbulent years of the twenties; but few people realized they were gambling—they thought they had a sure thing. . . . Taxi drivers told you what to buy. The shoeshine boy could give you a summary of the day's financial news as he worked with rag and polish. (Baruch, 1960)

In our model, a crash occurs in the following way. The rational traders observe a common signal that acts as a coordinating device. For certain values of this signal, they lower the price they bid for stocks, causing the stock price to fall. The sharp price change raises the naive traders' assessment of the risk in the market. Since naive traders are risk averse, they become less willing to own stocks. This lowers the market's risk-bearing capacity, so that a lower price clears the market. The crash is thus a self-fulfilling prophecy for the rational traders.

Importantly, this mechanism does not give rise to frenzies. Suppose rational traders were suddenly to *raise* their bid for stocks. The sharp price change would, once again, raise the naive traders' estimate of future volatility, lowering the market's risk-bearing capacity. Accordingly, there would be a surplus of stock at the higher price: The market would not clear. This is consistent with the empirical rarity of frenzies.<sup>2</sup>

This model captures other stylized facts surrounding crashes. Prices jump discontinuously. Some traders—the naive ones—sell in response to the falling price. In addition, crashes are unexpected: Until the crash signal is observed, no one knows a crash is about to happen. This mirrors findings of Bates (1991) that option prices indicated no crash fears in the 2 months leading up to the 1987 crash.

The presence of naive traders is necessary for crashes to occur in our model. If there were only rational agents, they would know that the crash was a transitory event and would thus prevent the crash by bidding prices up on the crash day. But while some naivete is needed for a crash to occur, it takes a very mild form: Naive traders believe that stock prices follow a random walk with serially correlated

<sup>2</sup> Naive traders in the model believe that prices follow a random walk. Thus, they do not believe that price increases will be followed by more increases. If they did believe this (à la the feedback traders of De Long et al., 1990), frenzies might occur. However, the mechanism we describe would still reduce the sizes of frenzies relative to crashes.

volatility. Until recent years, this was a widespread belief among economists (e.g., Bachelier, 1900; Mandelbrot, 1963; Fama, 1965; Malkiel, 1985).

In our theory, naive traders fare worse than rational traders. However, the usual criticism is not valid that naive traders cannot play a role in price dynamics since they will eventually be driven from the market. This is because crashes are rare. Most of the investors in the market during the 1987 crash had not been born in 1929.

In the model, the rational traders sell in response to a common signal that acts as a coordinating device. What played this role in 1987? On the morning of the crash on October 19, 1987, the *Wall Street Journal* published a chart suggesting a similarity between recent market action and stock prices in 1929. This chart, which is discussed by Shiller (1998), is reproduced in Figure 1.

This similarity is more than just casual. On the eve of the 1987 crash, the recent behavior of the Dow Jones Industrial Average (DJIA) was more similar to its behavior on the eve of the 1929 crash than at any Friday between the two dates. More precisely, we computed charts of the closing Dow Jones Industrial Average, in logs, over the 100 trading days ending on each Friday from 1930 through 1987. (We restrict to Fridays since both crashes occurred on a Monday.) We superimposed each of these charts on the corresponding chart for the Friday that preceded the 1929 crash. We then computed the area between the two curves. This area was smaller on the Friday preceding the 1987 crash than on any prior Friday in the 1930–1987 period.<sup>3</sup> These two curves are depicted in Figure 2.

This parallel was noticed independently by other investors. Stanley Druckenmiller, then manager of George Soros's Quantum Fund, states:

That Friday [October 16, 1987] after the close, I happened to speak to Soros. He said that he had a study done by Paul Tudor Jones that he wanted to show me. . . . *The analysis . . . illustrated the extremely close correlation in price action between the 1987 stock market and the 1929 stock market, with the implicit conclusion that we were now at the brink of a collapse* [emphasis added]. I was sick to my stomach when I went home that evening. I realized that I had blown it and that the market was about to crash. (Schwager, 1992)

Shiller argues that such reliance on historical parallels is an example of the *representativeness heuristic*:

a tendency for people to categorize events as typical or representative of a well-known class, and then, in making probability estimates, to overstress the importance of such a categorization, disregarding evidence about the underlying probabilities. (Shiller, 1998)

<sup>3</sup> Let  $D_t$  be the log closing DJIA on day  $t$ . Let  $T$  be 10/25/1929, the Friday preceding the 1929 crash. The minimum area between the 100-day charts ending on days  $t$  and  $T$  is given by  $A_t = \min_x \sum_{i=0}^{99} |D_{t-i} - D_{T-i} - x|$ .  $A_t$  is minimized by setting  $x$  to the difference between the medians of the two series. We measure  $A_t$  for  $t$  equal to each Friday from 1930 to 1987. The smallest  $A_t$  is 2.29 and occurred on the Friday that preceded the 1987 crash (10/16/1987). The next smallest, 2.53, was reached the prior Friday (10/9/1987). The average value of  $A_t$  from 1930 to 1987 (Fridays only) is 5.76 and the maximum is 23.38. (The New York Stock Exchange was open for two hours in the morning on Saturdays until 1952. In order to ensure a consistent relation between trading days and calendar time, we omit these Saturday index levels from our analysis.)

# Repeating the 1920s? Some Parallels but Some Contrasts

## Tracking the DJIA

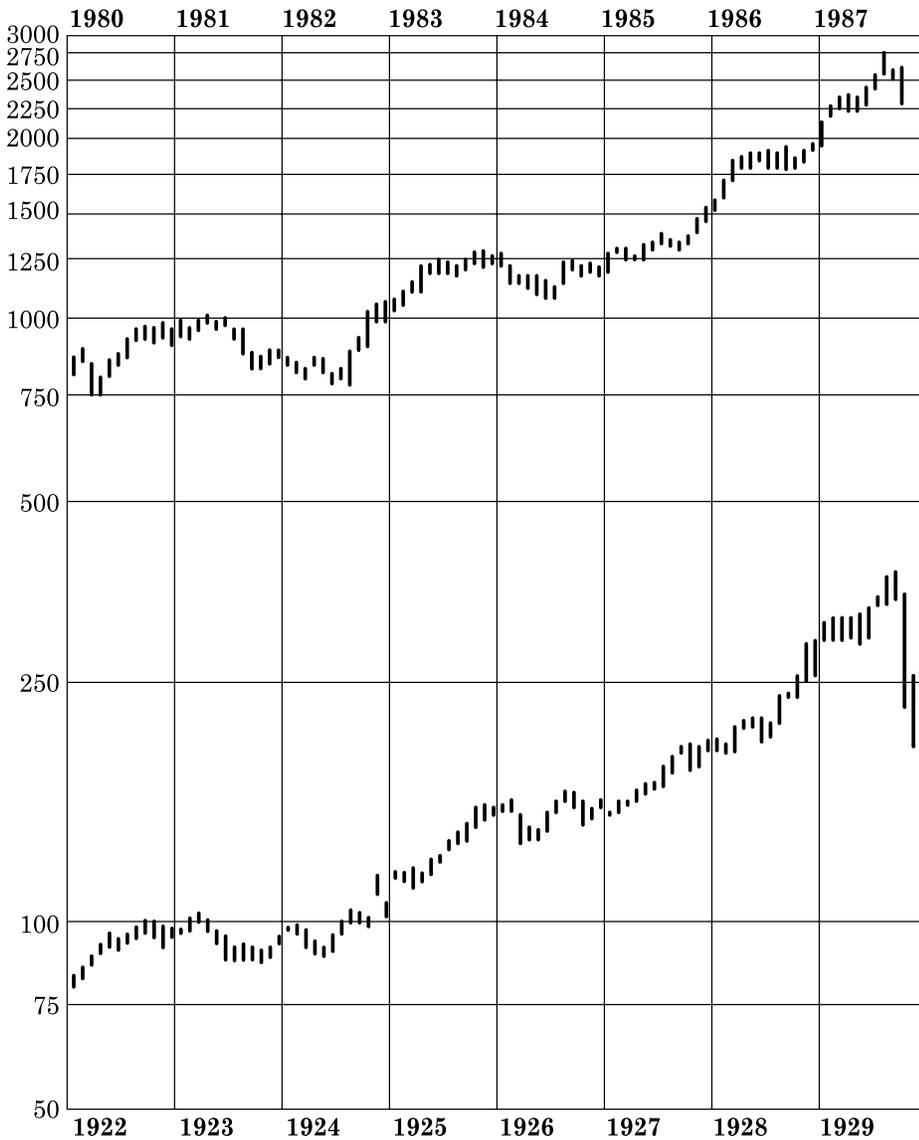


FIGURE 1

A CHART THAT APPEARED IN THE *WALL STREET JOURNAL* ON THE MORNING OF THE 1987 CRASH (OCTOBER 19, 1987)

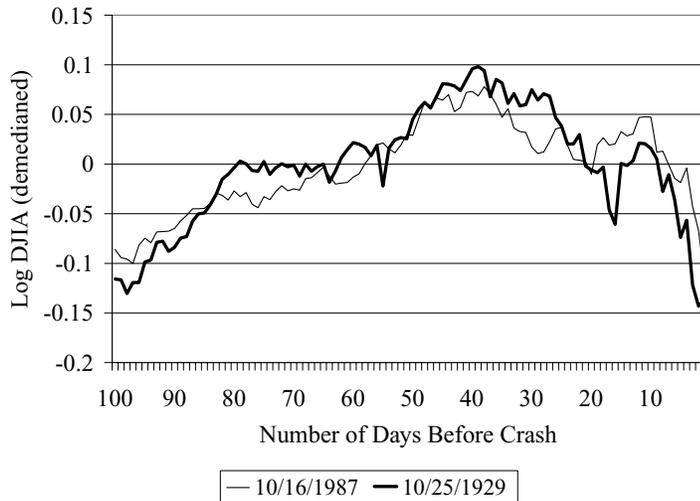


FIGURE 2

BEHAVIOR OF DJIA IN 100 DAYS PRECEDING 1929 AND 1987 CRASHES. THIS CHART DEPICTS THE CLOSING VALUE OF THE DOW JONES INDUSTRIAL INDEX, IN LOGS, FOR THE 100 TRADING DAYS ENDING THE FRIDAYS BEFORE THE 1929 AND 1987 CRASHES. THE CRASH OCCURRED ON THE FOLLOWING MONDAY IN BOTH CASES. THE MEDIAN OF EACH SERIES IS SUBTRACTED IN ORDER TO MINIMIZE THE AREA BETWEEN THE CURVES

This heuristic was first identified by Kahneman and Tversky (1972). It has also been cited as an explanation for stock market overreaction to news (De Bondt and Thaler, 1984; Barberis et al., 1998). Early experimental evidence that decision makers use this heuristic appears in Kahneman and Tversky (1973), Grether (1980), and Johnson (1983).

Camerer (1987) documents the use of the representativeness heuristic in asset markets. In his experiment, subjects are shown an urn that contains three balls. With some prior probability, one ball is black and two are red. With complementary probability, two balls are black and one is red. Subjects know these prior probabilities. Three balls are drawn with replacement from the urn and are shown to the subjects. Subjects then trade in assets whose payoffs depend on the numbers of black and red balls in the urn. Camerer's subjects appear to overestimate the probability that the actual distribution equals the sample distribution. For instance, if two sampled balls are black, then subjects tend to overvalue assets that pay off when the urn contains exactly two black balls.

Why is the representativeness heuristic used? One possible answer comes from Gilboa and Schmeidler (1995, 2001). They study "case-based decision making": the practice of choosing among possible actions by considering how they have performed in similar situations in the past. Gilboa and Schmeidler suggest that this practice may be a reasonable way to copy with complex situations.<sup>4</sup> Our

<sup>4</sup> Although both theories predict that agents will choose actions that have performed well in similar situations, case-based decision theory is not the same as the representativeness heuristic. For instance, whereas the representativeness heuristic involves underweighting of priors, in case-based decision theory there are no priors (Gilboa and Schmeidler, 1995).

theory gives another context in which the heuristic may be useful: It may serve as a coordinating device. This would provide an additional reason for individual agents to use it in some strategic settings.

The rest of the article is organized as follows. Relevant literature is reviewed in Section 2. Section 3 explains the naive traders' beliefs. The model is presented in Section 4 and solved in Section 5.

## 2. LITERATURE REVIEW

Explaining crashes is a central problem in economics. Occasional crashes are an essential feature of the aggregate stock market in modern times and appear to be crucial for understanding the empirical patterns of option prices (e.g., Ait-Sahalia et al., 2001). A satisfactory model of crashes should generate asymmetric price jumps from little or no fundamental news. By and large, prior models of crashes do not yield this phenomenon.

One group of models studies how crashes can occur if small changes in the environment lead substantial information to be revealed to partially informed investors. This class of models includes Abreu and Brunnermeier (2003), Caplin and Leahy (1994), Hong and Stein (1999), Kraus and Smith (1989), Lee (1998), Romer (1993), and Zeira (1999). Although these models yield price jumps with little or no fundamental news, they generally do not yield the prediction that crashes are more common than frenzies. However, there are two exceptions. In Abreu and Brunnermeier (2003), negative skew is generated by the assumption that investors overestimate the dividend growth rate. If investors were to underestimate this rate, the skew would be positive. In Hong and Stein (1999), negative skew comes from short-sale constraints. If these were replaced with margin constraints on leveraged buying, crashes would be replaced by frenzies. In our model, negative skew is generated by risk aversion. If naive investors were risk-loving, there would be frenzies instead of crashes.

The models of Gennotte and Leland (1990), Grossman (1988), and Jacklin et al. (1992) explore how rational investors can mistake the informational content of the trades of nonrational investors. These models assume the existence of portfolio insurers, who mechanically sell stocks when prices fall and buy when they rise. If rational traders underestimate the extent of this behavior, they will mistake it for informed trading. This can magnify the price effects of minor news. The first two papers interpret the crash as coming from such a mistake. Jacklin et al. (1992) interpret the price increase before the crash as coming from underestimation of portfolio insurance, and the crash itself occurred when informed traders realized their mistake. These models do not generate skewed returns: Crashes and frenzies are equally likely. In addition, in most of these models, the crash is caused by a misinterpretation by rational investors. One would expect prices to recover quickly as this confusion is cleared up. In practice, prices returned to precrash levels only a year after the 1987 crash. According to our theory, prices can remain low if naive investors remain "crashophobic" after the crash. Evidence from option prices indicates that crash fears have been present in the years since 1987 but were not present before the crash (Jackwerth and Rubinstein, 1996).

Two other models of crashes are Barlevy and Veronesi (2003) and Yuan (2005), in which all investors are rational and some are uninformed. A price decline signals negative information to the uninformed investors, which lowers the price further, which signals the possibility of even worse information, and so on. These models share the property that crashes and frenzies can occur without assuming irrational or mistaken investors. In Barlevy and Veronesi (2003), this is due to nonstandard assumptions about the signal distribution. In Yuan (2005), it is due to borrowing constraints. Both papers find that the stock price can depend discontinuously on fundamentals. However, like Gennotte and Leland (1990), both of these are static, one period models and thus do not bear directly on the asymmetry of changes in price.

The model of Grossman and Zhou (1996), although not aimed at explaining crashes, does yield some of their properties. They study a model with symmetric information and two types of risk-averse investors who each maximize expected consumption utility. One type, the “portfolio insurers,” have an additional constraint that their wealth must not fall below a certain level. As fundamentals worsen, the portfolio insurers sell stock at an accelerating rate, leading to an increase in volatility. This model does not yield news-free price jumps.

Our theory is related to the “volatility feedback” effect first studied by French et al. (1987), Malkiel (1979), and Pindyck (1984). They point out that greater stock market volatility can lead to a higher risk premium and thus to lower stock prices.<sup>5</sup> Campbell and Hentschel (1992) show that this effect can also give rise to negative skew: Price declines are larger, on average, than price advances. They assume a fully rational representative agent who sees dividends that follow a process with serially correlated volatility. Large dividend shocks lead to lower prices since they indicate an increase in volatility and the agent is risk averse. This “volatility feedback effect” dampens the price effects of positive dividend news and exaggerates the price effects of negative news.

Although the model of Campbell and Hentschel (1992) generates negative skew, it does not give news-free jumps: Prices are a continuous function of fundamentals. In their calibrated model, the crash of 1987 results from a substantial negative dividend shock. This is inconsistent with the evidence, cited above, that this crash was not caused by any fundamental news that was revealed around the crash date (Cutler et al., 1989; Shiller, 1989). One point of our model is that if risk-averse traders believe that *prices* display serially correlated volatility, then their presence in the market can yield news-free crashes without frenzies.

### 3. NAIVE TRADERS' BELIEFS

The naive traders in our model believe that large price changes tend to be followed by more large changes: that price volatility is serially correlated. This has

<sup>5</sup> In response to Pindyck (1984), Poterba and Summers (1986) produced evidence that volatility changes are not persistent enough to effect stock prices much. They model volatility as an AR(1) process. However, Chou (1988) subsequently found much stronger persistence using GARCH, a more flexible specification.

been the dominant view among academic researchers since Mandelbrot (1963) and Fama (1965). It has reached a broader audience through popular textbooks such as Brealey and Myers (1988) and Sharpe (1981).

This view also underlies many econometric studies of asset prices. In 1982, Engle first proposed the ARCH (Autoregressive Conditional Heteroskedasticity) model, in which next period's volatility is a weighted sum of past realized volatilities. In 1986, Bollerslev generalized this to GARCH (Generalized ARCH) by letting next period's volatility depend also on past predicted volatilities. In the past two decades, over 200 journal articles have used ARCH or GARCH to model the changing volatility of asset returns.<sup>6</sup> Our naive traders use a restricted GARCH model to predict future volatility.

We also assume naive traders believe that prices follow a random walk or, more precisely, a Martingale. This view has been widely promulgated in textbooks and the popular literature. In his best-selling textbook, Sharpe (1981) writes:

Stock returns exhibit almost no serial correlation: the particular value of return in the last period provides little if any help in predicting the likelihood of various possible returns in the next period.

Malkiel makes the same point forcefully in his well-known book *A Random Walk Down Wall Street* (1985). The view that price changes are unpredictable dates from Bachelier (1900).

To first order, the market behaves in accordance with naive traders' beliefs. For the S&P Composite Index from 1929 to 1999, the serial correlation of daily return volatility was 0.23; in comparison, the serial correlation of daily returns was only 0.055.<sup>7</sup> For the Dow Jones Industrial Average over the same period, the analogous figures were 0.22 and 0.052, respectively. Daily returns on S&P futures are essentially uncorrelated.<sup>8</sup> These statistics support the view that stock prices follow a random walk with serially correlated volatility.

Naive traders' beliefs are as follows. Let  $p_t$  be the stock price in period  $t$ . Naive traders believe that  $p_{t+1}$  will be normally distributed with mean  $p_t$  and variance or "volatility"  $V_t$ . They predict future volatility adaptively:

$$(1) \quad V_t = \alpha(p_t - p_{t-1})^2 + \beta V_{t-1}$$

for some fixed, positive constants  $\alpha$  and  $\beta$ . By substituting repeatedly for  $V$  on the right hand side, one can express  $V_t$  as a geometric weighted sum of past squared returns:

<sup>6</sup> Author's tabulation from Econlit.

<sup>7</sup> These statistics are based on the usual definition of the ex-dividend return,  $r_t = \frac{p_t - p_{t-1}}{p_{t-1}}$ . The serial correlation of returns is the sample correlation of  $r_t$  with  $r_{t-1}$ ; the serial correlation of volatility is the sample correlation of  $r_t^2$  with  $r_{t-1}^2$ .

<sup>8</sup> MacKinlay and Ramaswamy (1988) compute daily autocorrelations in log returns for the S&P 500 index and for futures contracts on this index during the 1983–1987 period. They find an average autocorrelation of 6.04% for daily index returns versus -0.24% for daily futures returns.

$$(2) \quad V_t = \alpha \sum_{i=1}^{\infty} \beta^{i-1} r_{t-i}^2$$

where  $r_t = p_t - p_{t-1}$ .

Equation (1) is a restricted GARCH(1,1) model. To see this, note that the unrestricted GARCH(1,1) model is

$$r_t = \gamma + \varepsilon_t$$

$$E(\varepsilon_t^2) = V_t = \omega + \alpha r_{t-1}^2 + \beta V_{t-1},$$

where  $r_t$  is the return (Bollerslev, 1986). The belief that prices follow a Martingale corresponds to setting  $\gamma$  to zero. We also set  $\omega$  and  $V_0$  to zero: As long as naive traders have not seen any price changes, they do not expect them. Indeed, we will show that there is an equilibrium in which the price is constant. Finally, for analytical convenience we define  $r_t$  to be the absolute price change  $p_t - p_{t-1}$  rather than the log return.

#### 4. THE MODEL

The game takes place in three periods:  $t = 0, 1, 2$ . There is a measure  $\mu$  of fully rational traders and  $1 - \mu$  of naive traders. Agents consume only in period 2; they maximize expected utility  $EU(W_2) = E[-e^{-\lambda W_2}]$ , where  $W_t$  is wealth in period  $t$  and  $\lambda$  is the coefficient of absolute risk aversion. Note that agents are not myopic; both types are forward looking. There are two assets: One ("stocks") pays an i.i.d. dividend  $\delta_t \sim N(\bar{\delta}, \sigma_\delta^2)$  per share<sup>9</sup> after the market closes in each period  $t = 0, 1$ , and a fixed liquidating dividend of  $D$  in period 2. The other asset, bonds, is in infinitely elastic supply and pays interest at a fixed net rate of  $r$  after the market closes in periods 0 and 1.

Let  $p_t$  be the price of a share of stock in period  $t$ . If an agent buys  $x_t$  shares of stock in period  $t = 0, 1$ , costing her  $p_t x_t$ , her wealth in period  $t + 1$  is

$$(3) \quad W_{t+1} = x_t(p_{t+1} + \delta_t) + (W_t - p_t x_t)(1 + r).$$

The sequence of events is as follows.

**Period 0.** No signals are observed and all traders trade. The role of this period is to permit optimal risk-sharing and to establish a base price  $p_0$  for the stock. At the end of the period, the dividend  $\delta_0$  per share is announced and distributed.

<sup>9</sup>The assumption of negative exponential utility and normally distributed dividends is common in the theoretical finance literature. This is an advantage: It shows that crashes can be obtained in a standard framework by adding a certain type of naive trader. The assumption that dividends are i.i.d. implies that there is no fundamental news that is relevant to the stock price. This stylized assumption is made to show that crashes can occur without any fundamental news. Serially correlated dividend shocks, although perhaps making the model more realistic, would obscure this point without essentially changing the results.

The naive traders' prediction  $V_0$  of the variance of the initial price  $p_0$  is taken to be zero.

**Period 1.** With probability  $\varepsilon$ , the rational traders all observe a crash signal; with probability  $1 - \varepsilon$ , no signal is seen. The crash signal is a pure coordinating device. All traders then trade at some price  $p_1$ . Finally, the dividend  $\delta_1$  is announced and paid.

**Period 2.** The liquidating dividend  $D$  is paid. There is no trade in this period.

When trade takes place, naive agents simultaneously submit demand functions: the quantity of shares they wish to buy at each price. Each rational trader submits a single limit order of the form "I will buy  $x$  shares if the price per share is no greater than  $p$ ." Since there is a continuum of agents, they will act as price takers.

By making limit orders, the rational traders collectively determine whether a crash will occur by picking a particular point on their demand curves. This permits decentralized crashes: many different investors suddenly deciding to pay less for stock since they *predict* that the price will be lower. If rational agents were instead to submit entire demand curves, crashes would be centralized. They would take the form of a Walrasian auctioneer's occasionally choosing a low market-clearing price. The reason is that the risky asset is liquidated in the next period with a fixed payoff distribution, so a rational agent's demand curve in period 1 is the same regardless of whether or not a crash signal is seen. This is not the case with more trading periods. In Frankel (2006) we analyze an infinite-horizon extension in which the crash signal leads rational agents to lower their entire demand curves, which causes the crash to occur.

The model's environment is nonstationary: There is a finite horizon and agents consume only at the end of the game.<sup>10</sup> To ensure the existence of a baseline equilibrium with a constant stock price, we make two normalizations. The first is to fix the liquidating dividend  $D$  at  $\frac{1}{r}[\bar{\delta} - \lambda\sigma_\delta^2]$ . The second is to fix the number of shares of stock at  $\frac{1}{1+r}$  in period zero and one in period 1. Why? Agents in period 1 face a one-period problem with objective function  $e^{-\lambda W_2}$ . But agents in period 0 invest as if they are more risk averse. Their objective function is proportional to  $e^{-\lambda(1+r)W_1}$ : By (3),  $W_2$  equals  $W_1(1+r)$  plus a term that is independent of  $W_1$ .<sup>11</sup> As a result, stock demand in any constant-price equilibrium must be lower in period 0 than in period 1. For the price to be constant, the initial supply must also be lower.

We allow for the possibility of margin constraints: In every period, a trader can buy no more than  $\kappa$  shares of stock, where  $\kappa \geq 1$ . The case of no constraints can be captured by setting  $\kappa = \infty$ . The main barrier to a crash is that rational traders demand more stock when the price falls. The margin constraint tempers this effect. Indeed, we will show that a sufficient condition for crashes to occur is that the

<sup>10</sup> The main advantage of the finite horizon formulation is that analytical results are possible. In Frankel (2006), we show by simulation that crashes can also occur in a stationary infinite-horizon version of the model.

<sup>11</sup> This is because the amount the agent invests in stock in period 1 does not depend on her wealth  $W_1$ .

margin constraint be tight enough to prevent rational traders from buying all the stock in the market in period 1. Note that the margin constraint cannot bind for both groups of traders in a given period. Otherwise, total stock demand would be  $\mu\kappa + (1 - \mu)\kappa = \kappa$ , which exceeds the supply.

The market-clearing price is determined by the condition that the demand for stocks equal the supply:

$$(4) \quad \text{Period 0: } \mu x_0^R + (1 - \mu)x_0^N = \frac{1}{1 + r}$$

$$(5) \quad \text{Period 1: } \mu x_1^R + (1 - \mu)x_1^N = 1,$$

where  $x_t^R$  and  $x_t^N$  are the time- $t$  stock demands of rational and naive traders, respectively.

### 5. RESULTS

We first solve for investors' stock demand functions. We will make use of the following well-known property. Proofs of all results are in the Appendix.

LEMMA 1. *Suppose an agent has wealth  $W$  and the share price is  $p$ . The agent buys  $x$  shares of a risky asset that pays a gross return  $R$  in the next period and invests the rest of her wealth in a riskless asset that pays the gross return  $1 + r$ . There is a margin constraint:  $x$  cannot exceed some constant  $\kappa$ . Let the agent's wealth in the next period be  $W' = W(1 + r) + x(R - (1 + r)p)$ . The agent seeks to maximize the expectation of  $-e^{-\lambda W'}$ . Assume the agent believes that  $R \sim N(\mu, \sigma^2)$ . Then the agent's unconstrained stock demand is  $x^* = \frac{\mu - (1+r)p}{\lambda\sigma^2}$ . She will buy  $x = \min\{\kappa, x^*\}$  shares and she believes that her expected utility is*

$$EU = \begin{cases} -\exp\left[-\frac{(\mu - (1+r)p)^2}{2\sigma^2} - \lambda W(1 + r)\right] & \text{if } x = x^* \\ -\exp\left[\frac{\lambda^2\sigma^2}{2}\kappa^2 - \lambda(\mu - (1+r)p)\kappa - \lambda W(1 + r)\right] & \text{if } x = \kappa. \end{cases}$$

In period 1, each share yields the gross return  $D + \delta_1$ , which is normally distributed with mean  $D + \bar{\delta}$  and variance  $\sigma_\delta^2$ . By Lemma 1, the unconstrained demand of rational agents is thus

$$(6) \quad x_1^{R*} = \frac{D + \bar{\delta} - (1 + r)p_1}{\lambda\sigma_\delta^2}.$$

Note that rational traders demand more stock when the price falls.

Naive traders expect each share to yield the gross return  $p_2 + \delta_1$ , which they believe is normally distributed with mean  $p_1 + \bar{\delta}$  and variance  $V_1 + \sigma_\delta^2$  where  $V_1 = \alpha(p_1 - p_0)^2$ . (Recall our assumption that naive traders' initial variance estimate  $V_0$  is zero.) By Lemma 1, naive traders' unconstrained demand must equal

$$(7) \quad x_1^{N*} = \frac{\bar{\delta} - rp_1}{\lambda(\alpha(p_1 - p_0)^2 + \sigma_\delta^2)}.$$

This equation is the key for understanding how crashes can occur. A falling price in period 1 has two effects. It raises the numerator: The higher dividend yield has a positive effect on naive demand, as in the case of rational traders. But it also raises the denominator: Declining prices raise naive traders' estimate of future volatility, which lowers their demand for stock. If this second, "volatility feedback" effect dominates, then naive traders' demand for stock will be upwards sloping.

We now solve for demand in period 0. Naive traders expect each share to yield the gross return  $p_1 + \delta_0$ , which they believe is normally distributed with mean  $p_0 + \bar{\delta}$  and variance  $\sigma_\delta^2$ . By Lemma 1, their expected utility in period 1 is proportional to  $\exp(-\lambda(1+r)W_1)$ , where  $W_1$  is period 1 wealth. Applying Lemma 1 with a risk aversion coefficient of  $\lambda(1+r)$ , naive traders' unconstrained demand in period 0 equals

$$(8) \quad x_0^{N*} = \frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2}.$$

Rational traders' demand in period 0 cannot be computed explicitly if  $p_1$  is not normally distributed. However, we can compute it in a constant-price equilibrium, in which  $p_1 = p_0$  for sure. In this case, rational traders have the same beliefs as naive traders, so their unconstrained demand is also given by

$$(9) \quad x_0^{R*} = \frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2}.$$

We first show that there is only one constant-price equilibrium:

**PROPOSITION 1.** *There is only one equilibrium in which the same price occurs in periods 0 and 1. In this equilibrium, this price is  $\bar{p} = D = \frac{1}{r}[\bar{\delta} - \lambda\sigma_\delta^2]$ .*

We call  $\bar{p}$  the variance-free price. Proposition 2 shows that even in equilibria with changing prices, the stock price can never exceed this fundamental level of  $\bar{p}$ . Thus, frenzies cannot occur in this model.

**PROPOSITION 2.** *In any equilibrium, the stock price can never exceed  $\bar{p}$ .*

The intuition for Proposition 2 is as follows. In a given equilibrium, let  $p^{\max}$  be the maximum price that can be attained in any period. When the price is  $p^{\max}$ , rational traders must expect the next period's return to be zero or negative. Naive traders, by assumption, expect it to be zero. Agents' expected returns in the constant price equilibrium are at least as optimistic as this; furthermore, agents in the constant-price equilibrium expect zero volatility. So if the price is  $p^{\max}$ , stocks cannot offer a more attractive return distribution to either type of agent than in

the constant-price equilibrium. But then no agent will ever be willing to pay more than the price in that equilibrium, which is  $\bar{p}$ ; hence,  $p^{\max} \leq \bar{p}$ .

We now consider equilibria in which there are two possible prices in period 1: a high price that is close to the prior period's price and a lower "crash" price that occurs with some small probability. A crash has two effects. It raises rational traders' demand for stock since a lower stock price implies a higher dividend yield. However, it also lowers naive trader stock demand if their prediction of future volatility puts enough weight on past volatility (i.e., if  $\alpha$  is high enough). If this volatility feedback effect is sufficiently strong, stock demand can be upward sloping. In this case, there can be multiple market-clearing prices.

The following proposition gives two alternative conditions that guarantee this. The first is that margin constraints be strict enough that rational traders alone cannot purchase all the stock and that naive traders' beliefs be sufficiently sensitive to realized volatility. The second, alternative condition is that there exist any equilibrium other than the constant-price equilibrium. If either condition holds, then crashes can occur, in the following sense. For any sufficiently small  $\varepsilon > 0$  there is an equilibrium in which (a) the period-0 price is close to the variance-free price; (b) with probability  $1 - \varepsilon$  the period-1 price is also close to the variance-free price; (c) with probability  $\varepsilon$  the period-1 price is below and not close to the variance-free price. As the crash probability  $\varepsilon$  shrinks to zero, the period-0 price and the noncrash price in period 1 converge to the variance-free price,  $\bar{p}$ , and the period-1 crash price converges to a level that is strictly below the variance-free price.

**PROPOSITION 3.** *The following properties hold generically. Fix  $\bar{\delta} > 0$ ,  $\lambda > 0$ ,  $\sigma_\delta^2 > 0$ ,  $\mu \in (0, 1)$ , and  $r > 0$ , such that the variance-free price  $\bar{p} = (\bar{\delta} - \lambda\sigma_\delta^2)/r$  is positive. Then if either*

1. *rational traders cannot buy all the stock ( $\mu\kappa < 1$ ) and naive traders are sufficiently sensitive to past price volatility ( $\alpha > \alpha^*$  for some  $\alpha^* < \infty$ ), or*
2. *there exists any equilibrium other than the constant-price equilibrium,*

*then*

1. *there is an  $\bar{\varepsilon} > 0$  such that for any  $\varepsilon \in (0, \bar{\varepsilon})$  there are prices  $p_0, p_1^L$ , and  $p_1^H$ , such that the following is an equilibrium:*
  - (a) *the period 0 price equals  $p_0$ , and*
  - (b) *the period 1 price equals  $p_1^L$  with probability  $\varepsilon$  and  $p_1^H$  with probability  $1 - \varepsilon$ ;*
2. *as  $\varepsilon \rightarrow 0$ ,  $p_0$  and  $p_1^H$  both converge to  $\bar{p}$ , but  $p_1^L$  converges to a price that is strictly lower than  $\bar{p}$ .*

By Proposition 3, infrequent crashes are the easiest type of price uncertainty to sustain in this model. An intuition is that if the crash is unlikely, then it has little effect on rational traders' stock demand in period 0. Hence, the period-0 price will be very close to its maximum possible value of  $\bar{p}$ . This maximizes naive traders'

prediction of future volatility,  $\alpha(p_0 - p_1)^2$ , at any given crash price. Hence, the volatility feedback effect is greatest when crashes are rare, which maximizes the chance that there exists a market-clearing crash price.

We now study how the exogenous parameters of the model affect whether or not crashes can occur in equilibrium.<sup>12</sup> The following proposition shows that equilibria with crashes are easier to sustain when naive traders are either more numerous or more sensitive to past price volatility, or the margin constraint is tighter. Each of these has the effect of magnifying the importance of the volatility-feedback effect. The margin constraint does this since it limits the increase in the demand of rational traders in a crash.

**PROPOSITION 4.** *Each of the following expands the set of other parameters for which there exist equilibria with rare crashes.*

1. *An increase in  $\alpha$ , the sensitivity of naive traders to past price volatility.*
2. *A decrease (tightening) of  $\kappa$ , the margin constraint.*
3. *A decrease in  $\mu$ , the proportion of rational traders.*

The third result is consistent with the fact, cited above, that the 1929 and 1987 crashes occurred after sustained bull markets that drew many novice investors into the stock market.

5.1. *Simulations.* A simulation appears in Figure 3. The chart shows excess demand (demand minus supply) for stocks in period 1. The horizontal axis shows the ratio  $p_1/p_0$  and the vertical axis gives excess demand. Each curve shows not excess demand per investor but rather excess demand for the group as a whole. Hence, the two dashed curves (excess demand of naive and rational traders) add up to the solid curve (total excess demand). Equilibrium in period 1 requires that total excess demand (the solid curve) be zero.<sup>13</sup>

In this simulation, naive traders' update sensitivity is moderate ( $\alpha = 0.1$ ) and a majority (60%) of traders are rational. Despite this, there is a small but positive chance that prices will fall by 23% in period 1. Substantial crashes can occur even though rational traders as a group can buy almost all ( $\mu\kappa = 0.9$  units, or 90%) of the stock.

Without margin constraints ( $\kappa = \infty$ ), Proposition 3 does not guarantee that crashes can occur since the rational traders can purchase all the stock. However, simulations show that crashes can occur if naive traders are sufficiently numerous and put high enough weight on recent volatility in updating their beliefs. An

<sup>12</sup> We do not study the effects of exogenous parameters on crash sizes and probabilities, since these are not uniquely determined: When a crash is possible, there must exist multiple crash equilibria, each with its own crash size and probability (Proposition 3). In addition, the effects of model parameters on, say, the maximum possible crash size or probability are generally not monotonic, making it hard to derive useful empirical implications. We also do not study the effects of changes in risk aversion or dividend risk on the equilibrium set since these effects are likely to depend on the assumptions of exponential utility and normally distributed dividends.

<sup>13</sup> The chart depicts the limiting case in which the crash risk goes to zero. It is approximately correct when the crash risk is small but nonzero.

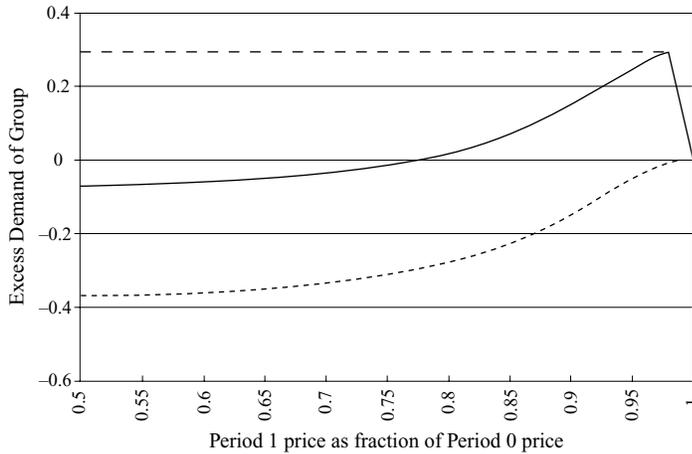


FIGURE 3

EXCESS DEMAND IN PERIOD ONE: EXAMPLE WITH MARGIN CONSTRAINTS. THIS CHART DEPICTS EXCESS DEMAND FOR STOCK AMONG RATIONAL TRADERS (LONG DASHES), NAIVE TRADERS (SHORT DASHES), AND ALL TRADERS (SOLID), IN PERIOD 1, IN THE LIMIT AS THE CRASH RISK GOES TO ZERO, ASSUMING INVESTORS CAN BORROW UP TO 50% OF THEIR WEALTH. THE PRICE RATIO  $p_1/p_0$  APPEARS ON THE HORIZONTAL AXIS; EXCESS DEMAND APPEARS ON THE VERTICAL AXIS. PARAMETERS ARE  $\alpha = 0.1$ ,  $\mu = 0.6$ ,  $\bar{\delta} = 1.25$ ,  $\lambda = \sigma_{\delta}^2 = 1$ ,  $r = 1\%$ , AND  $\kappa = 1.5$

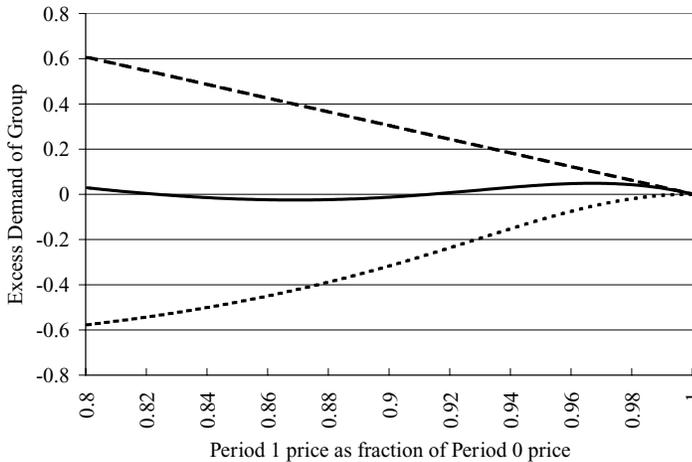


FIGURE 4

EXCESS DEMAND IN PERIOD ONE: EXAMPLE WITHOUT MARGIN CONSTRAINTS. THIS CHART DEPICTS EXCESS DEMAND FOR STOCK AMONG RATIONAL TRADERS (LONG DASHES), NAIVE TRADERS (SHORT DASHES), AND ALL TRADERS (SOLID), IN PERIOD 1, IN THE LIMIT AS THE CRASH RISK GOES TO ZERO, ASSUMING INVESTORS CAN BORROW ANY AMOUNT. THE PRICE RATIO  $p_1/p_0$  APPEARS ON THE HORIZONTAL AXIS; EXCESS DEMAND APPEARS ON THE VERTICAL AXIS. PARAMETERS ARE  $\alpha = 0.3$ ,  $\mu = 0.2$ ,  $\bar{\delta} = 1.15$ ,  $\lambda = \sigma_{\delta}^2 = 1$ ,  $r = 1\%$ , AND  $\kappa = \infty$

example in which this weight,  $\alpha$ , equals 0.3 and 80% of traders are naive appears in Figure 4. Since excess demand crosses the zero axis in two places for these parameters, crashes can be either large (about 18% of the precrash price) or small (about 9%).

Without margin constraints, the proportion of naive traders must be fairly high to sustain crashes—in these simulations, 80%. The reason is that rational traders' demand rises in a crash. For there to be multiple market-clearing prices in period 1, the reduced demand of naive traders in a crash must have the potential to offset the greater demand of rational traders. This holds only if there are relatively many naive traders in the market.

## 6. DISCUSSION

In the model presented here, rational traders collectively cause a crash by making limit orders that correspond to a lower point on their (fixed) demand curves: They bid a lower price, but for a larger quantity of stock. This makes them net buyers of stock on the day of the crash. In Frankel (2006), we study an infinite horizon, overlapping generations version of the model presented here. In that model, rational traders actually lower their entire demand curves for stock and become net sellers of stock on the crash day. This occurs because we now permit trading after the crash, unlike in the original model. On the day after the crash, naive traders sell their shares en masse. This causes prices to continue to fall, though by far less than on the crash day. Thus, the market actually reaches its lowest point the day *after* the crash. Anticipating this, rational traders have an incentive to sell on the crash day.<sup>14</sup>

The analysis in Frankel (2006) indicates that naive traders make two mistakes that cause them to sell after the crash. First, they do not take into account that, due to risk aversion, expected returns are higher when volatility is high. Instead, they continue to believe after the crash that prices follow a random walk—neither rising nor falling on average. In Frankel (2006) we show that this mistake is not essential: Crashes can occur even if naive traders take into account the empirical relation between volatility and expected returns. The reason is that, empirically, this relation is not very strong.

Naive traders also overestimate postcrash volatility. This error is essential: If naive traders correctly predict the variance of future returns, crashes cannot occur. However, this error is plausible. First, we show that traders who used adaptive expectations to predict future volatility following the 1987 crash would have made this mistake. More generally, the GARCH model overpredicts the persistence of large shocks to volatility (see, e.g., Longin, 1997). In addition, the 1987 crash did lead to a large and enduring increase in the market's assessment of the likelihood of future crashes (Jackwerth and Rubinstein, 1996)

## 7. CONCLUSION

From time to time, stock indices have jumped by several percentage points in a single day. These jumps tend to be negative and they do not appear to be driven by public news about fundamentals. Although these jumps are infrequent, evidence

<sup>14</sup> In the extension, naive traders react to increased volatility with a lag. This allows them to be net buyers of stock on the crash day and net sellers the next day.

from derivatives markets suggests that their anticipation has a significant effect on asset prices. By extension, these “crash fears” may raise the cost of corporate capital and depress economic growth.

For these reasons, it is important to understand the mechanisms that underlie crashes. The existing literature has revealed many mechanisms that help us understand markets for risky assets. However, most models lack one or more of the essential features of crashes. Some models give skewed returns but require large fundamental shocks to generate crashes. Others yield news-free jumps but symmetric returns. In the few models that do exhibit both phenomena, the negative skew is due to special assumptions of the model that can be plausibly reversed, yielding positive skew. This article presents a theory that does not appear to have this weakness. Positive skew can be generated by assuming that investors are risk-loving, but this is not plausible.

The basic idea of our theory is that some investors are naive: Rather than taking into account the strategic behavior of other agents, they believe that stock prices follow a random walk with serially correlated volatility. This belief is approximately true in an empirical sense and was the mainstream view in the finance community for decades. In this simple model, we show that prices cannot exceed fundamentals but they can suddenly fall significantly below fundamentals. Simulations show that this can occur with reasonable updating by naive traders. Moreover, with mild margin constraints, naive traders can also constitute a minority of investors in the market.

#### APPENDIX: PROOFS

PROOF OF LEMMA 1. Expected utility is

$$\begin{aligned} & -\frac{1}{2\pi\sigma} \int_{R=-\infty}^{\infty} \exp[-(R-\mu)^2/2\sigma^2] \exp(-\lambda[W(1+r) + x(R-(1+r)p)]) dR \\ & = -\frac{1}{2\pi\sigma} \int_{R=-\infty}^{\infty} \exp[-(R-\mu)^2/2\sigma^2 - \lambda x R - \lambda[W(1+r) - x(1+r)p]] dR. \end{aligned}$$

But

$$\begin{aligned} & -(R-\mu)^2/2\sigma^2 - \lambda x R \\ & = -\frac{1}{2\sigma^2}(R^2 - 2\mu R + \mu^2 + 2\sigma^2\lambda x R) \\ & = -\frac{1}{2\sigma^2}((R + \sigma^2\lambda x - \mu)^2 - (\sigma^2\lambda x - \mu)^2 + \mu^2) \\ & = -\frac{1}{2\sigma^2}((R + \sigma^2\lambda x - \mu)^2 - [(\sigma^2\lambda x)^2 + \mu^2 - 2\mu\sigma^2\lambda x] + \mu^2) \\ & = -\frac{1}{2\sigma^2}((R + \sigma^2\lambda x - \mu)^2 - (\sigma^2\lambda x)^2 + 2\mu\sigma^2\lambda x), \end{aligned}$$

so expected utility is

(A.1)

$$\begin{aligned}
& -\frac{1}{2\pi\sigma} \int_{R=-\infty}^{\infty} \exp \left[ +\frac{1}{2\sigma^2} ((\sigma^2\lambda x)^2 - 2\mu\sigma^2\lambda x) - \frac{1}{2\sigma^2} (R + \sigma^2\lambda x - \mu)^2 - \lambda(W(1+r) - x(1+r)p) \right] dR \\
& = -\exp \left[ \frac{1}{2\sigma^2} ((\sigma^2\lambda x)^2 - 2\mu\sigma^2\lambda x) - \lambda(W(1+r) - x(1+r)p) \right] \\
& = -\exp \left[ \frac{\lambda^2\sigma^2}{2} x^2 - \lambda(\mu - (1+r)p)x - \lambda W(1+r) \right].
\end{aligned}$$

The expression in (A.1) is globally concave. To see why, note that (A.1) is of the form  $-\exp(ax^2 - bx - c)$  and that

$$\frac{\partial^2}{\partial x^2} (-\exp(ax^2 - bx - c)) = e^{ax^2 - bx - c} f(x),$$

where  $f(x) = 4abx - 2a - b^2 - 4a^2x^2$ .  $f(x)$  is concave in  $x$  so its first derivative is zero at a maximum of  $f$ . One can verify that  $f' = 0$  only at  $x = b/2a$ . At this value of  $x$ ,  $f(x) = 4ab\frac{b}{2a} - 2a - b^2 - 4a^2(\frac{b}{2a})^2 = -2a$ . This is negative since  $a = \frac{\lambda^2\sigma^2}{2} > 0$ . So  $f(x)$  is always negative: (A.1) is globally concave.

Consequently, (A.1) has a unique maximum given by  $x = b/2a = \frac{\mu - (1+r)p}{\lambda\sigma^2} = x^*$ . If  $x^* \leq \kappa$ , then the agent sets  $x = x^*$  and receives the utility

$$\begin{aligned}
& -\exp \left[ \frac{\lambda^2\sigma^2}{2} \left( \frac{\mu - (1+r)p}{\lambda\sigma^2} \right)^2 - \lambda(\mu - (1+r)p) \frac{\mu - (1+r)p}{\lambda\sigma^2} - \lambda W(1+r) \right] \\
& = -\exp \left[ \frac{(\mu - (1+r)p)^2}{2\sigma^2} - \frac{(\mu - (1+r)p)^2}{\lambda\sigma^2} - \lambda W(1+r) \right] \\
& = -\exp \left[ -\frac{(\mu - (1+r)p)^2}{2\sigma^2} - \lambda W(1+r) \right].
\end{aligned}$$

If  $x^* > \kappa$ , then the agent chooses  $x = \kappa$ ; substituting into (A.1), her utility is

$$-\exp \left[ \frac{\lambda^2\sigma^2}{2} \kappa^2 - \lambda(\mu - (1+r)p)\kappa - \lambda W(1+r) \right]. \quad \blacksquare$$

**PROOF OF PROPOSITION 1.** By definition,  $p_0 = p_1$  in a constant-price equilibrium. By (5), (6), and (7), market clearing in period 1 implies

$$\begin{aligned}
(A.2) \quad 1 & = \mu x_1^R + (1 - \mu)x_1^N = \mu \min \left\{ \kappa, \frac{D + \bar{\delta} - (1+r)p_1}{\lambda\sigma_\delta^2} \right\} \\
& \quad + (1 - \mu) \min \left\{ \kappa, \frac{\bar{\delta} - rp_1}{\lambda\sigma_\delta^2} \right\}.
\end{aligned}$$

If the rational traders' margin constraint binds in period 1, then the naive traders' margin constraint cannot bind, so by (A.2),

$$1 = \mu\kappa + (1 - \mu) \frac{\bar{\delta} - rp_1}{\lambda\sigma_\delta^2} \implies p_1 = \frac{1}{r} \left( \bar{\delta} - \lambda\sigma_\delta^2 \frac{1 - \mu\kappa}{1 - \mu} \right) > D,$$

but this is a contradiction: If  $p_1 > D$ , then by (6) and (7), the rational traders' unconstrained demand is less than the naive traders' unconstrained demand. An analogous argument implies that naive traders' margin constraint cannot bind. Since neither group's margin constraint can bind in period 1, (A.2) implies that  $p_1 = D = \frac{1}{r}[\bar{\delta} - \lambda\sigma_\delta^2]$ .

We now consider period 0. Since  $p_1$  is known to equal  $D$ , rational traders' demand equals  $x_0^R = \min\{\kappa, \frac{D + \bar{\delta} - (1+r)p_0}{\lambda(1+r)\sigma_\delta^2}\}$  by Lemma 1. Hence, by (4) and (8),

$$(A.3) \quad \frac{1}{1+r} = \mu x_0^R + (1 - \mu)x_0^N \\ = \mu \min \left\{ \kappa, \frac{D + \bar{\delta} - (1+r)p_0}{\lambda(1+r)\sigma_\delta^2} \right\} + (1 - \mu) \min \left\{ \kappa, \frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2} \right\}.$$

By an analogous argument to the case of period 1, one can show that neither margin constraint binds; given this,  $p_0 = D$  is the only solution to (A.3). ■

PROOF OF PROPOSITION 2. First suppose  $p_1 > \bar{p}$ . By (7),  $x_1^N \leq \frac{\bar{\delta} - rp_1}{\lambda\sigma_\delta^2} < \frac{\bar{\delta} - r\bar{p}}{\lambda\sigma_\delta^2} = 1$ . So by (5),  $x_1^R > 1$ . However, by (6),

$$x_1^R = \min \left\{ \kappa, \frac{D + \bar{\delta} - (1+r)p_1}{\lambda\sigma_\delta^2} \right\} > 1 \\ \implies \frac{D + \bar{\delta} - (1+r)p_1}{\lambda\sigma_\delta^2} > 1 \implies p_1 < \frac{1}{r}[\bar{\delta} - \lambda\sigma_\delta^2] = \bar{p},$$

a contradiction. Now suppose  $p_0 > \bar{p}$ . Since  $p_1 \leq \bar{p}$ , rational trader demand is maximized by the belief that  $p_1 = \bar{p}$ . Hence, by Lemma 1, rational traders' unconstrained demand is at most  $\frac{\bar{p} + \bar{\delta} - (1+r)p_0}{\lambda(1+r)\sigma_\delta^2}$ . Since  $p_0 > \bar{p}$ , this is less than  $\frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2}$ , which is an upper bound on naive traders' demand. Hence, total demand is less than  $\frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2}$ . Since demand must equal supply,  $1/(1+r)$ , it must be that

$$\frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2} > \frac{1}{1+r} \implies p_0 < D,$$

a contradiction. ■

PROOF OF PROPOSITION 3. We cannot explicitly compute rational agents' demands in period 0 in this case, since returns are not normally distributed. We will overcome this by using instead their period-0 demand for the case in which they expect the price in period 1 to equal the variance-free price for sure. Under this assumption, there can be two market clearing prices in period 1: the variance-free price and a lower price, which is unanticipated in period 0. We will then show that there is continuity: For any small probability  $\varepsilon$ , there is an equilibrium in which the price is close to the lower price in period 1 with probability  $\varepsilon$  and close to the variance-free price with probability  $1 - \varepsilon$ .

If rational traders expect the price in period 1 to be the variance-free price, their demand in period zero is  $\min\{\kappa, \frac{\bar{p} + \bar{\delta} - (1+r)p_0}{\lambda(1+r)\sigma_\delta^2}\}$  by Lemma 1. Hence, market-clearing in period 0 implies

$$(A.4) \quad \mu \min \left\{ \kappa, \frac{\bar{p} + \bar{\delta} - (1+r)p_0}{\lambda(1+r)\sigma_\delta^2} \right\} + (1 - \mu) \min \left\{ \kappa, \frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2} \right\} = \frac{1}{1+r}.$$

But this is just Equation (A.3), whose only solution was shown to be  $p_0 = \bar{p}$ : The variance-free price clears the market in period 0 as well.

Now suppose that traders reach period 1 after observing  $p_0 = \bar{p}$ . By (5), (6), and (7), market clearing in period 1 requires that

$$(A.5) \quad \mu \min \left\{ \kappa, \frac{D - p_1 + \bar{\delta} - rp_1}{\lambda\sigma_\delta^2} \right\} + (1 - \mu) \min \left\{ \kappa, \frac{\bar{\delta} - rp_1}{\lambda(\alpha(p_1 - \bar{p})^2 + \sigma_\delta^2)} \right\} = 1.$$

One solution to this equation is  $p_1 = \bar{p}$ . When is there a lower market clearing price? As  $p_1$  falls below  $\bar{p}$ , both numerators in (A.5) rise to first order. The denominator of the naive traders' demand is unchanged to first order but rises to second order. Hence, for  $p_1$  less than but arbitrarily close enough to  $\bar{p}$ , the left hand side exceeds 1. However, if  $\alpha$  is large enough, we can guarantee that as  $p_1$  continues to fall, naive traders' demand will begin to shrink. Indeed, by taking  $\alpha$  arbitrarily large, we can ensure that naive traders' unconstrained demand shrinks to zero arbitrarily quickly as  $p_1 < \bar{p}$  falls. Since by assumption the rational traders cannot buy all the stock, there must exist an  $\alpha^*$  such that if  $\alpha > \alpha^*$ , then there is another market clearing price  $p_1$  that is strictly less than  $\bar{p}$ .

Thus, for any sufficiently high  $\alpha$  there is an equilibrium in which  $p_0 = p_1 = \bar{p}$  for sure but there also exists another price  $p'_1$ , strictly lower than  $\bar{p}$ , that would also clear the market in period 1.

We now prove that for any small enough crash risk  $\varepsilon > 0$ , there are equilibria that are close to this equilibrium. There are five variables: the crash probability,  $\varepsilon$ ; rational traders' unconstrained demand in period 0,  $x_0^{R*}$ ; the period 0 price  $p_0$ ; the low period 1 price,  $p_1^L$ ; the high period 1 price  $p_1^H$ . The equations for an

equilibrium are as follows. The equation for optimality of unconstrained rational trader demand:

$$0 = f^1(\varepsilon, x_0^{R*}, p_0, p_1^L, p_1^H)$$

$$= \frac{\partial}{\partial x_0^{R*}} \left\{ \begin{array}{l} \varepsilon E_{\delta_1} \left[ -\exp \left( -\lambda \left( +x_0^{R*} (p_1^L + \delta_1 - (1+r)p_0) \right) (1+r) \right) \right. \\ \left. + \frac{(D+\bar{\delta}-(1+r)p_1^L)^2}{\sigma_\delta^2} \right] \\ + (1-\varepsilon) E_{\delta_1} \left[ -\exp \left( -\lambda \left( +x_0^{R*} (p_1^H + \delta_1 - (1+r)p_0) \right) (1+r) \right) \right. \\ \left. + \frac{(D+\bar{\delta}-(1+r)p_1^H)^2}{\sigma_\delta^2} \right] \end{array} \right\}$$

(by Lemma 1); the equation for market clearing in period 0:

$$0 = f^2(\varepsilon, x_0^{R*}, p_0, p_1^L, p_1^H) = \mu \min \{ \kappa, x_0^{R*} \} + (1-\mu) \min \left\{ \kappa, \frac{\bar{\delta} - rp_0}{\lambda(1+r)\sigma_\delta^2} \right\} - 1;$$

and the equations for market clearing in period 1:

$$0 = f^3(\varepsilon, x_0^{R*}, p_0, p_1^L, p_1^H)$$

$$= \mu \min \left\{ \kappa, \frac{D - p_1^L + \bar{\delta} - rp_1^L}{\lambda\sigma_\delta^2} \right\} + (1-\mu) \min \left\{ \kappa, \frac{\bar{\delta} - rp_1^L}{\lambda(\alpha(p_1^L - p_0)^2 + \sigma_\delta^2)} \right\} - 1;$$

$$0 = f^4(\varepsilon, x_0^{R*}, p_0, p_1^L, p_1^H)$$

$$= \mu \min \left\{ \kappa, \frac{D - p_1^H + \bar{\delta} - rp_1^H}{\lambda\sigma_\delta^2} \right\} + (1-\mu) \min \left\{ \kappa, \frac{\bar{\delta} - rp_1^H}{\lambda(\alpha(p_1^H - p_0)^2 + \sigma_\delta^2)} \right\} - 1.$$

As shown above, one solution to this system is

$$(\varepsilon, x_0^{R*}, p_0, p_1^L, p_1^H) = \left( 0, \frac{\bar{\delta} - r\bar{p}}{\lambda(1+r)\sigma_\delta^2}, \bar{p}, \bar{p}', \bar{p} \right).$$

For generic parameters, the functions  $f^n$  are continuously differentiable in a neighborhood of this solution, since differentiability fails only when one of the following nongeneric conditions holds:  $x_0^{R*} = \kappa$ , or either  $\frac{D-p+\bar{\delta}-rp}{\lambda\sigma_\delta^2} = \kappa$  or  $\frac{\bar{\delta}-rp}{\lambda(\alpha(p-p_0)^2+\sigma_\delta^2)} = \kappa$  for  $p = p_1^L$  or  $p_1^H$ . In addition,

$$\det \begin{bmatrix} \frac{\partial f^1}{\partial x_0^R} & \frac{\partial f^1}{\partial p_0} & \frac{\partial f^1}{\partial p_1^L} & \frac{\partial f^1}{\partial p_1^H} \\ \frac{\partial f^2}{\partial x_0^R} & \frac{\partial f^2}{\partial p_0} & \frac{\partial f^2}{\partial p_1^L} & \frac{\partial f^2}{\partial p_1^H} \\ \frac{\partial f^3}{\partial x_0^R} & \frac{\partial f^3}{\partial p_0} & \frac{\partial f^3}{\partial p_1^L} & \frac{\partial f^3}{\partial p_1^H} \\ \frac{\partial f^4}{\partial x_0^R} & \frac{\partial f^4}{\partial p_0} & \frac{\partial f^4}{\partial p_1^L} & \frac{\partial f^4}{\partial p_1^H} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial f^1}{\partial x_0^R} & \frac{\partial f^1}{\partial p_0} & \frac{\partial f^1}{\partial p_1^L} & \frac{\partial f^1}{\partial p_1^H} \\ \frac{\partial f^2}{\partial x_0^R} & \frac{\partial f^2}{\partial p_0} & 0 & 0 \\ 0 & \frac{\partial f^3}{\partial p_0} & \frac{\partial f^3}{\partial p_1^L} & 0 \\ 0 & \frac{\partial f^4}{\partial p_0} & 0 & \frac{\partial f^4}{\partial p_1^H} \end{bmatrix}$$

is generically nonzero. By the Implicit Function Theorem, there is a neighborhood  $N$  of 0 such that for  $\varepsilon \in N$  there are unique, continuously differentiable functions  $x_0^R(\varepsilon)$ ,  $p_0(\varepsilon)$ ,  $p_1^L(\varepsilon)$ , and  $p_1^H(\varepsilon)$ , such that  $(\varepsilon, x_0^R(\varepsilon), p_0(\varepsilon), p_1^L(\varepsilon), p_1^H(\varepsilon))$  solve the system of equations (and thus constitute an equilibrium). The claim follows.

We now show that if there is an equilibrium with a nonconstant price, then there are equilibria with occasional crashes. The nonconstant price must occur in period 1, since there is no way for rational traders to randomize in a coordinated way in period 0. We first show that there must be a positive probability that  $p_1$  will take a value  $\underline{p}_1 < p_0$ . Otherwise, rational traders in period 0 would know that the price could not fall in period 1. A lower bound on their demand would thus be their demand under the belief that  $p_1$  would equal  $p_0$  for sure. Under this belief, rational trader demand equals  $\min\{\kappa, \frac{\bar{\delta} - r p_0}{\lambda(1+r)\sigma_\delta^2}\}$  by Lemma 1. Since this is also the expression for naive demand, neither group's margin constraint could bind. Accordingly, rational and naive trader demand in period 0 would both equal  $\frac{\bar{\delta} - r p_0}{\lambda(1+r)\sigma_\delta^2}$ . Since this is a lower bound on actual market demand, it must be no greater than market supply,  $\frac{1}{1+r}$ ; solving, we find  $p_0 \geq \bar{p}$ . But no price can exceed  $\bar{p}$  by Proposition 2. Since by hypothesis the period 1 price is never below the period 0 price, it must be that  $p_0 = p_1 = \bar{p}$ , which contradicts the assumption that  $p_1$  is not constant.

This shows that there must be a price  $\underline{p}_1$  that is strictly below  $p_0$  and that occurs with positive probability. Market clearing in period 1 requires that  $\mu \min\{\kappa, x_1^{R*}\} + (1 - \mu) \min\{\kappa, x_1^{N*}\} = 1$ , where  $x_1^{R*} = \frac{D - \underline{p}_1 + \bar{\delta} - r \underline{p}_1}{\lambda \sigma_\delta^2}$  is unconstrained rational trader demand and  $x_1^{N*} = \frac{\bar{\delta} - r \underline{p}_1}{\lambda(\alpha(\underline{p}_1 - p_0)^2 + \sigma_\delta^2)}$  is unconstrained naive trader demand. Since  $\underline{p}_1 < p_0 \leq \bar{p}$ , one can show easily that  $x_1^{N*} < x_1^{R*}$ , so the margin constraint does not bind for naive traders in period 1.

Now consider what would happen if the period-0 price were increased to the variance-free price  $\bar{p}$ . This would increase the denominator of  $x_1^{N*}$ , lowering market demand in period 1 at the price  $\underline{p}_1$ . Hence, market demand at  $p_1 = \underline{p}_1$  would be strictly less than one. Market demand at  $p_1 = \bar{p}$  would equal one by Proposition 1. Moreover, for  $p_1$  slightly less than  $\bar{p}$ , market demand would strictly exceed one, since (a) the margin constraint cannot bind when both prices equal  $\bar{p}$  and (b) the effect on the denominator of  $x_1^{N*}$  is zero to first order whereas the numerators of  $x_1^{R*}$  and  $x_1^{N*}$  both rise to first order. Hence, if the period 0 price were to equal  $\bar{p}$ , then both  $p_1 = \bar{p}$  and a price strictly below  $\bar{p}$  would clear the market in period 1. One can now show that there exist equilibria with rare crashes using the implicit function theorem as shown above. ■

PROOF OF PROPOSITION 4. Equilibria with rare crashes exist if there are market-clearing prices strictly below  $\bar{p}$  in period 1 following a period 0 price of  $\bar{p}$ . Each such  $p_1$  must satisfy  $\mu \min\{\kappa, x_1^{R*}\} + (1 - \mu) \min\{\kappa, x_1^{N*}\} = 1$ , where  $x_1^{R*} = \frac{D - p_1 + \delta - r p_1}{\lambda \sigma_\delta^2}$  is unconstrained rational trader demand and  $x_1^{N*} = \frac{\delta - r p_1}{\lambda(\alpha(p_1 - \bar{p})^2 + \sigma_\delta^2)}$  is unconstrained naive trader demand. As  $p_1$  begins to fall below  $\bar{p}$ , both unconstrained demands rise to first order; for a lower price to clear the market, eventually the second order term  $\alpha(p_1 - \bar{p})^2$  in the denominator of  $x_1^{N*}$  must dominate the first order effect. By raising this second order term, an increase in  $\alpha$  will expand the range of other parameters where equilibria with rare crashes exist. Since  $x_1^{R*}$  exceeds  $x_1^{N*}$  for all  $p_1 < \bar{p}$ , a decrease in either  $\kappa$  or  $\mu$  will lower market demand and thus also expand this range. ■

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