Online Appendix to "Portfolio Liquidity and Security Design with Private Information"[†]

Peter M. DeMarzo (*Stanford University*)

David M. Frankel (*Melbourne Business School and Iowa State University*)

Yu Jin (*Shanghai University of Finance and Economics*)

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1. Introduction

This document is the online appendix to "Portfolio Liquidity and Security Design with Private Information" by Peter M. DeMarzo, David M. Frankel, and Yu Jin (2018), henceforth "DFJ". It is organized as follows. Section 2 formally states the technical results that are previewed in section 3.5.2 of DFJ. Section 3 proves these results as well as the results that are stated without proof in section 3 of DFJ. Section 4 relaxes ASSUMPTION A (monotonicity) in the case of 2 assets and 2 issuer types.

2. Technical Results

We first formally state the technical results that we discuss in section 3.5.2 of DFJ. We begin by specifying the continuous and discrete models. The continuous model, denoted "model ∞", is as follows. The issuer's type $t \sim G$ has full support [0,1] and the final asset value Y has conditional and unconditional support $[0, \bar{y}]$ and conditional distribution function H that satisfies the Hazard Rate Ordering property and Lipschitz-H (defined in DFJ, section 3.5.2).

We now define a sequence of discrete models i = 1,2,... that converge to model ∞. Let $(N_i)_{i=1}^{\infty}$ and $(N_i')_{i=1}^{\infty}$ be any two increasing sequences of positive integers. In model i, let

[†] DeMarzo: Stanford, CA 94305-5015; pdemarzo@stanford.edu. Frankel: 200 Leicester St., Carlton, VIC 3053, Australia; d.frankel@mbs.edu. Jin: Shanghai 200433, China; jinyu@shufe.edu.cn.

the gaps between adjacent types t and shocks Z be $\Delta_i = 1/N_i$ and $\Delta_i' = \overline{y}/N_i'$, respectively. That is, t lies in $S_i = \{0, \Delta_i, \ldots, 1-\Delta_i, 1\}$ and Z lies in $S_i' = \{0, \Delta_i', \ldots, \overline{y}-\Delta_i', \overline{y}\}$. By construction, both gaps Δ_i and Δ_i' converge to zero as i goes to infinity. Let the conditional distribution of Y in model i be the restriction of the continuous distribution function H to types in S_i and shocks in S_i'. Similarly, the distribution of t in model i is the restriction of G to types t in S_i.[72]

Let E^i and E^∞ denote the expectations operators in models i and ∞, respectively. Let $v^i(D,t) = E^i[\min(D,Y) \mid t]$ denote the expected payout of simple debt with face value D in model i given a type $t \in S_i$. Let $v^\infty(D,t) = E^\infty[\min(D,Y) \mid t]$ denote the expected payout of the same security in model ∞ given a type $t \in [0,1]$.

Fix equilibria in models i and ∞ in which the issuer's security is simple debt. Let D_t^i and D_t^∞ be the equilibrium face values of these securities for a given type t. The equilibrium price of this security in model i, denoted $p^i(t)$, is simply the security's expected payout $v^i(D_t^i, t)$. And the issuer's expected issuance profit, denoted $u^i(t)$, is simply the gains from trade $(1-\delta)v^i(D_t^i, t)$ as competition drives investors' payoffs to zero. Similarly, in the continuous model the price $p^\infty(t)$ of the security equals the expected payout $v^\infty(D_t^\infty, t)$ and the issuer's profit $u^\infty(t)$ equals the expected gains from trade $(1-\delta)v^\infty(D_t^\infty, t)$.

In the present notation, equation (11) is written as:[73]

[72] That is, in model i, the probability that the type does not exceed some t in S_i is $G(t)$, while the probability, conditional on a type t, that the cash flow Y does not exceed some y in S_i' is $H(y \mid t)$.

[73] Integrating by parts, $v(D,t) = \int_{y=0}^{\overline{y}} \min\{y, D\}\, dH(y \mid t)$ equals $D - \int_{y=0}^{D} H(y \mid t)\, dy$ whence $v_2(D,t) = -\int_{y=0}^{D} H_2(y \mid t)\, dy$. As $H(D \mid t) = \pi(D,t)$ by Lipschitz-H, CIVP is equivalent to (11).

THE CONTINUOUS INITIAL VALUE PROBLEM (CIVP).

$$D_0 = \overline{y} \text{ and, for each } t, \quad \frac{dD_t}{dt} = \frac{1}{1-\delta} \frac{\int_{y=0}^{D_t} H_2(y\,|\,t)\,dy}{1 - H(D_t|t)} < 0.$$

Our first result shows that CIVP and its discrete analog have unique solutions:

PROPOSITION 11. Assume Hazard Rate Ordering and Lipschitz-H.

1. There exists a unique function D^∞ that satisfies CIVP for $v = v^\infty$. This function is decreasing and differentiable in the type t, and takes values in $(0, \overline{y}]$. The associated price and profit functions, p^∞ and u^∞, are decreasing and continuous in the type t.
2. For each discrete model i=1,2,..., there exists a unique, decreasing function D^i with $D_0^i = \overline{y}$ and satisfying (10) with $v = v^i$ for all $t \in S_i$ and $\Delta = \Delta_i$.

The proof for CIVP runs roughly as follows. The Picard-Lindelöf theorem is the usual tool for proving the existence and uniqueness of the solution to a differential equation. Unfortunately, we cannot apply this theorem directly because the differential equation in CIVP is not Lipschitz continuous in D^∞: it approaches negative infinity as D^∞ approaches $\overline{y}$.

We sidestep this difficulty in the following way. We define upper and lower bounds on D^∞ using a modification of CIVP that is Lipschitz continuous with constant k. We then show that as k grows, these upper and lower bounds approach the same limit, which satisfies CIVP and thus must be its unique solution D^∞.

Having shown the existence of unique face value functions in both models, we now show that the face value function in the discrete model converges to that of the continuous model uniformly in the issuer's type t. As the equilibrium of the discrete model uniquely satisfies the intuitive criterion, this supports the use of the continuous equilibrium when convenient.

PROPOSITION 12. Assume HRO and Lipschitz-H. For any $\varepsilon > 0$ there is an $i*$ such that for all models $i > i*$ and all types t in [0,1], $\left|D_t^i - D_t^\infty\right| < \varepsilon$.[74]

The idea of the proof is as follows. For any model i = 1,2,... and constant $k > 0$, we first show that any solution D^i must lie between fixed upper and lower bounds $\bar{D}_k^i$ and $\underline{D}_k^i$, where these bounds are Lipschitz continuous with constant k. Moreover, these bounds converge to the aforementioned upper and lower bounds on D^∞ as i grows. By the prior intuition, these bounds on D^∞ converge in turn to D^∞ as k grows. Thus, by taking i and k to infinity simultaneously, $\bar{D}_k^i$ and $\underline{D}_k^i$ - and thus D^i which lies between them – must converge to the unique solution D^∞ of model ∞.

Our next result gives conditions under which the convergence of the discrete model to the continuous model is uniform in various parameters. This property can be useful in applications in which the security design game is preceded by some interaction in which the issuer also chooses an optimal action, as in Frankel and Jin (2015). In such settings, the result can help establish that the issuer's optimal choices in the discrete model are well-approximated by her optimal choice in the continuous model.

Henceforth, we assume the distribution of types G is continuous with a strictly positive density g that satisfies the following technical condition:

LIPSCHITZ-*G* (L-*G*). There are constants $k_3, k_4 \in (0,\infty)$ such that for all types t and t' in [0,1], $g(t) \le k_3$ and $\left|g(t) - g(t')\right| \le k_4 \left|t - t'\right|$.

We now show that the key functions of the model converge uniformly in the distributions G and H, the type $t \in [0,1]$, and the cash flow parameter $\bar{y}$.[75] These key functions are the face value function D^i, price function p^i, and the conditional profit function u^i. We also show uniform convergence of the issuer's unconditional expected issuance profits $Eu^i = E^i\left[u^i(t)\right]$ to its continuous analogue, $Eu^\infty = E^\infty[u^\infty(t)]$. Finally, let $\Pi^i(t)$ denote

[74] Technically, the function D^i is defined only for types t in the discrete set S_i. We extend it to all types t in [0,1] by evaluating it at the highest type in S_i that does not exceed t.
[75] Uniformity in t does not apply to the expected profit functions Eu and EΠ, as they do not depend on t.

the issuer's conditional *total* profits in model i: the sum of issuance profits $u^i(t)$ and the conditional expected portfolio return $E^i[Y \mid t]$. Let $E\Pi^i = E^i\left[\Pi^i(t)\right]$ denote unconditional expected total profits.[76] We show that these converge uniformly to their continuous counterparts, $\Pi^\infty(t) = u^\infty(t) + E^\infty[Y \mid t]$ and $E\Pi^\infty = E^\infty\left[\Pi^\infty(t)\right]$.

> **PROPOSITION 13.** Fix constants k_0, k_1, k_2, k_3, k_4, and $\mathbf{y}$, all in $(0,\infty)$. Let $\mathbf{H}$ be the set of conditional distribution functions $H(z \mid t)$ that satisfy Hazard Rate Ordering and Lipschitz-H with constants k_0, k_1, and k_2. Let $\mathbf{G}$ be the set of distribution functions G that satisfy Lipschitz-G with constants k_3 and k_4. For all $\varepsilon > 0$ there is an i^* such that for all models $i > i^*$, G in $\mathbf{G}$, H in $\mathbf{H}$, $\overline{y}$ in $[0, \mathbf{y}]$, and t in $[0,1]$, $|\omega^i(t)-\omega^\infty(t)|$ is less than ε for ω equal to D, p, u, and Π; and $|E\omega^i - E\omega^\infty|$ is less than ε for ω equal to u and Π.[77]

We next show a useful comparative statics property: the issuer chooses a higher face value when the rate of change function in (11) (which is negative) is smaller in absolute value. PROPOSITION 10 is essentially a special case of this result (and its proof relies on this result).

> **PROPOSITION 14**. Let the discount factors $\hat{\delta}$ and $\tilde{\delta}$ lie in $(0,1)$. Assume the conditional distribution functions $\hat{H}$ and $\tilde{H}$ satisfy HRO and Lipschitz-H. suppose that for any given D and t, the rate of change function $\frac{1}{1-\delta}\frac{\int_{y=0}^{D} H_2(y \mid t)\,dy}{1-H(D|t)}$ in (11) is no larger in absolute value when (δ, H) equals $(\hat{\delta}, \hat{H})$ than when it equals $(\tilde{\delta}, \tilde{H})$. Let the face value functions $\hat{D}_t$ and $\tilde{D}_t$ solve equation (11) for (δ, H) equal to $(\hat{\delta}, \hat{H})$ and $(\tilde{\delta}, \tilde{H})$, respectively. Then for all t,

[76] This last quantity is especially important in applications: if there is a pregame period, the issuer will act so as to maximize the sum of $E\Pi^i$ (perhaps multiplied by a discount factor) and any pregame payoff.

[77] As in Proposition 14, we extend these functions to all types t in [0,1] by evaluating them at the highest type in S_i that does not exceed t.

$\hat{D}_t \geq \tilde{D}_t$: the issuer does not choose a lower face value under $(\hat{\delta}, \hat{H})$ than under $(\tilde{\delta}, \tilde{H})$.

Finally, we show a homogeneity property that can simplify the analysis of models in which security design is embedded (e.g., Frankel and Jin (2015)).

COROLLARY 15. The face value functions D^i and D^∞, the price functions p^i and p^∞, the profit functions u^i, Π^i, u^∞, and Π^∞, and the issuer's expected profits Eu^i, $E\Pi^i$, Eu^∞, and $E\Pi^\infty$, defined above, are all homogeneous of degree one in the cash flow parameter $\overline{y}$.

3. Proofs

We now give the omitted proofs from section 3 of DFJ, as well as the proofs of the results of section 2 of this online appendix.

PROOF OF PROPOSITION 6. We first define the Intuitive Criterion in the GSD game. Consider an equilibrium and any interim asset vector *I*. If type *t* sticks to her continuation strategy, she will get $u(t \mid I) \overset{d}{=} U(I, P_t(I), \rho_t(I) \mid t, p)$. If instead she chooses some other *ex-post* action (P, ρ), for her to lose from this deviation it suffices that her equilibrium payoff $u(t \mid I)$ exceeds her maximum payoff $\rho - \delta E\left[W_P^I(Y) \mid t\right]$ from the deviation – or, equivalently, that

$$\rho < u(t \mid I) + \delta E\left[W_P^I(Y) \mid t\right]. \tag{13}$$

The right hand side of (13) is thus the minimum revenue that type *t* requires to be willing to deviate to (P, ρ).

The Intuitive Criterion for the GSD game states that on seeing the deviation (P, ρ) following *I*, investors must put zero probability on type *t* if she is never willing to choose (P, ρ) following *I* but some other type might be: if condition (13) holds for *t* but fails for some other type *s*. That is:

THE INTUITIVE CRITERION (GSD GAME). An equilibrium of the GSD game with posterior belief function μ and outcome u is *intuitive* if, on seeing any action (I, P, ρ), investors' posterior probability $\mu(t \mid I, P, \rho)$ is zero for any type t that satisfies (13) as long as there is some type s for which the inequality is reversed: for which $\rho \geq u(s \mid I) + \delta E\left[W_P^I(Y) \mid s\right]$.

Let $E = \left(\hat{I}, P_{\cdot}(\bullet), \rho_{\cdot}(\bullet), p, \mu\right)$ be any equilibrium of the GSD game. We will show that the following equilibrium $\tilde{E} = \left(\tilde{I}, \tilde{P}_{\cdot}(\bullet), \tilde{\rho}_{\cdot}(\bullet), \tilde{p}, \tilde{\mu}\right)$ is also an equilibrium, and is intuitive if E is. Let $\#(V)$ denote the length of a vector V.

1. Securitization. In $\tilde{E}$, the issuer's interim asset vector $\tilde{I} = \left(\tilde{I}^1\right)$ consists of a single security whose payout equals her cash flow: $\tilde{I}^1(Y) = Y$. She then chooses an *ex-post* action as follows for any given t and interim asset vector I. First, if she deviated in the prior stage (so that $I \neq \tilde{I}$), she chooses the *ex-post* action $\left(P_t(I), \rho_t(I)\right)$ that she would choose in E after choosing I. Else she issues an *ex-post* action that consists of her equilibrium revenue cap $\rho_t\left(\hat{I}\right)$ in E together with an *ex-post* asset vector consisting of a single security whose payout equals the aggregate payout $\sum_{j=1}^{\#\left(P_t(\hat{I})\right)} P_t^j\left(\hat{I}\right)\left(\hat{I}(Y)\right)$ of her equilibrium *ex-post* security vector $P_t\left(\hat{I}\right)$ in E.

2. Beliefs. Let $\tilde{T}(P, \rho \mid I)$ denote the set of types whose strategies in $\tilde{E}$ instruct them to choose the *ex-post* action (P, ρ) conditional on having chosen some interim asset vector I. Upon seeing an action $\left(\tilde{I}, P, \rho\right)$ that is expected in $\tilde{E}$, the investors' posterior probability $\mu(t \mid \tilde{I}, P, \rho)$ equals the probability that the issuer's type is t conditional on her type being in $\tilde{T}\left(P, \rho \mid \tilde{I}\right)$.[78] That is, it equals

[78] An action $\left(\tilde{I}, P, \rho\right)$ is *expected* in an equilibrium if it is selected with positive probability in that equilibrium.

$g(t)\left[\sum_{s\in\tilde{T}(P,\rho|\tilde{I})} g(s)\right]^{-1}$ if $t\in\tilde{T}\left(P,\rho\mid\tilde{I}\right)$ and zero otherwise. Upon seeing an action (I,P,ρ) that is unexpected in $\tilde{E}$, $\tilde{\mu}(t\mid I,P,\rho)$ equals $\mu(t\mid I,P,\rho)$: its value in E.

3. Pricing. Prices are given by equation (8) with $\tilde{\mu}$ substituted for μ: for any action (I,P,ρ), the resulting price vector is $\tilde{p}(I,P,\rho)=\sum_t E[P(I(Y))\mid t]\,\tilde{\mu}(t\mid I,P,\rho)$.

Pricing and Beliefs ensure that $\tilde{E}$ satisfies Competitive Pricing and Rational Updating. As for Payoff Maximization, the payoff of an issuer of type t from taking an action (I,P,ρ) that is unexpected in $\tilde{E}$ equals her payoff from taking this action in E, since, by Pricing and Beliefs, investors respond with the same price vector. As for expected actions in $\tilde{E}$, we rely on the following claim, whose proof appears below.

Claim. Let $\left(\tilde{I},P,\rho\right)$ be expected in $\tilde{E}$, and let Σ denote the set of actions $\left(\hat{I},P',\rho\right)$ that are taken in E by any type whose prescribed action in $\tilde{E}$ is $\left(\tilde{I},P,\rho\right)$. Then an issuer of any given type gets the same payoff in $\tilde{E}$ from taking the action $\left(\tilde{I},P,\rho\right)$ as she gets in E from taking any action in Σ.

By this Claim, no action in $\tilde{E}$ affords an issuer of a given type more than her equilibrium payoff in E; and taking her prescribed action in $\tilde{E}$ gives her the same payoff that she gets, in equilibrium, in E. Thus she will take her prescribed action in $\tilde{E}$: Payoff Maximization holds, whence $\tilde{E}$ is an equilibrium as claimed.

Proof of Claim. Let $\left(\hat{I},P',\rho\right)$ be any action in Σ. Note first that $\left(\hat{I},P',\rho\right)$ has the same aggregate payout function $W(Y)=P^1\left(\tilde{I}^1(Y)\right)$ as $\left(\tilde{I},P,\rho\right)$ and thus the same aggregate expected payout conditional on the issuer's type, $E\left[W(Y)\mid t\right]$. Hence the Claim holds if $\left(\hat{I},P',\rho\right)$ raises the same revenue in E as $\left(\tilde{I},P,\rho\right)$ raises in $\tilde{E}$. We first show that all actions in Σ raise the same revenue in E:

LEMMA. In *E*, following the equilibrium interim asset choice $\hat{I}$, if there are two *ex-post* actions that are expected given $\hat{I}$ and that have the same revenue cap and aggregate payout function, they must raise the same revenue.

PROOF OF LEMMA. Suppose not: there are types *t* and t' (possibly equal) who, after choosing $\hat{I}$ and seeing their types, choose *ex-post* actions (P,ρ) and (P',ρ), respectively, that have the same revenue cap and aggregate payout function, such that (P,ρ) yields less revenue than (P',ρ): $\rho \wedge \sum_{j=1}^{\#(P)} p^j(I,P,\rho) < \rho \wedge \sum_{j=1}^{\#(P')} p^j(I,P',\rho)$. Since the aggregate payout functions are identical, their conditional expectations are the same conditional on the issuer's type being *t*: $E\left[W_P^{\hat{I}}(Y)\,|\,t\right] = E\left[W_{P'}^{\hat{I}}(Y)\,|\,t\right]$. But then by (7), type *t*'s payoff $U\left(\hat{I},P',\rho\,|\,t,p\right)$ from (P',ρ) exceeds her payoff $U\left(\hat{I},P,\rho\,|\,t,p\right)$ from (P,ρ) which she is therefore unwilling to choose - a contradiction. ◆

By this Lemma, each action in Σ raises the same issuance revenue R_Σ in *E*. And if, in *E*, the issuer chooses action $\left(\hat{I},P',\rho\right)$ in Σ then, by (6) and (8), investors' willingness to pay $WTP_E\left(\hat{I},P',\rho\right) \overset{d}{=} \sum_{j=1}^{\#(P')} p^j\left(\hat{I},P',\rho\right)$ for her *ex-post* assets equals their estimate $\sum_t E\left[W(Y)\,|\,t\right]\mu(t\,|\,\hat{I},P',\rho)$ of the aggregate payout given what the action $\left(\hat{I},P',\rho\right)$ reveals about the issuer's type. The issuance revenue R_Σ in *E* is then the minimum of $WTP_E\left(\hat{I},P',\rho\right)$ and the cap ρ. Since, by the above Lemma, this issuance revenue is the same for any action $\left(\hat{I},P',\rho\right)$ in Σ, it follows that either (i) $WTP_E\left(\hat{I},P',\rho\right) > \rho$ for every such action or (ii) $WTP_E\left(\hat{I},P',\rho\right) \le \rho$ for every such action. Moreover, in case (ii), $WTP_E\left(\hat{I},P',\rho\right)$ must take a common value $WTP_E(\Sigma)$ for any action $\left(\hat{I},P',\rho\right)$ in Σ.

Likewise, if the issuer chooses action $(\tilde{I}, P, \rho)$ in $\tilde{E}$ then the amount $\tilde{p}^1(\tilde{I}, P, \rho)$ that investors are willing to pay for her *ex-post* security is just their estimate $\sum_t E[W(Y) \mid t]\, \tilde{\mu}(t \mid \tilde{I}, P, \rho)$ of this security's payout conditional on the action $(\tilde{I}, P, \rho)$; the resulting issuance revenue is the minimum of $\tilde{p}^1(\tilde{I}, P, \rho)$ and the cap ρ.

But for each *t*, by Beliefs, $\tilde{\mu}(t \mid \tilde{I}, P, \rho)$ can be written as $\dfrac{g(t) 1_{t \in \tilde{T}(P, \rho \mid \tilde{I})}}{\sum_s g(s) 1_{s \in \tilde{T}(P, \rho \mid \tilde{I})}}$. And the sets $T(P', \rho \mid \hat{I})$ of types who, in *E*, take actions $(\hat{I}, P', \rho) \in \Sigma$, is a partition of the set $\tilde{T}(P, \rho \mid \tilde{I})$ of types who take the action $(\tilde{I}, P, \rho)$ in $\tilde{E}$. Thus we can rewrite $\tilde{\mu}(t \mid \tilde{I}, P, \rho)$ as a weighted sum $\sum_{(\hat{I}, P', \rho) \in \Sigma} \mu(t \mid \hat{I}, P', \rho) w_{(\hat{I}, P', \rho)}$ where the weight $w_{(\hat{I}, P', \rho)} = \dfrac{\sum_s g(s) 1_{s \in T(P', \rho \mid \hat{I})}}{\sum_s g(s) 1_{s \in \tilde{T}(P, \rho \mid \tilde{I})}}$ is the probability that the issuer would have chosen $(\hat{I}, P', \rho)$ in *E* conditional on her having chosen $(\tilde{I}, P, \rho)$ in $\tilde{E}$. Hence we can rewrite $\tilde{p}^1(\tilde{I}, P, \rho)$ as $\sum_{(\hat{I}, P', \rho) \in \Sigma} w_{(\hat{I}, P', \rho)} \sum_t E[W(Y) \mid t] \mu(t \mid \hat{I}, P', \rho)$ or, equivalently, as $\sum_{(\hat{I}, P', \rho) \in \Sigma} w_{(\hat{I}, P', \rho)} R_E(\hat{I}, P', \rho)$. Since, the weights $w_{(\hat{I}, P', \rho)}$ sum to one, the revenue $\rho \wedge R_{\tilde{E}}(\tilde{I}, P, \rho)$ raised by the action $(\tilde{I}, P, \rho)$ in $\tilde{E}$ equals ρ in case (i) above (where the revenue raised in *E* by each $(\hat{I}, P', \rho)$ in Σ is also ρ) and the common value $WTP_E(\Sigma)$ in case (ii) (when the revenue raised in *E* by each $(\hat{I}, P', \rho)$ in Σ is also $WTP_E(\Sigma)$). This proves the Claim. ◆

It remains to show that if *E* is intuitive, so is $\tilde{E}$. Suppose *E* is intuitive. Let $u(\cdot \mid I)$ and $\tilde{u}(\cdot \mid I)$ denote the equilibrium payoffs in *E* and $\tilde{E}$, respectively, of an issuer of type *t* who chooses the interim asset vector I and then follows her continuation strategy in the given equilibrium. Consider any action (I,P,ρ) for which there are types t and s satisfying:

$$\rho < \tilde{u}(t \mid I) + \delta E\left[W_P^I(Y) \mid t\right] \text{ and } \rho \geq \tilde{u}(s \mid I) + \delta E\left[W_P^I(Y) \mid s\right]. \qquad (14)$$

First assume (I,P,ρ) is unexpected in $\tilde{E}$. As previously noted, the payoff of an issuer of type t from taking the action (I,P,ρ) in $\tilde{E}$ equals her payoff from taking this action in E. It follows that $\tilde{u}(v \mid I) = u(v \mid I)$ for $v = s,t$. Hence $\rho < u(t \mid I) + \delta E\left[W_P^I(Y) \mid t\right]$ and $\rho \geq u(s \mid I) + \delta E\left[W_P^I(Y) \mid s\right]$ whence, since E is intuitive, $\mu(t \mid I,P,\rho)$ is zero. But since (I,P,ρ) is unexpected in $\tilde{E}$, Beliefs implies that $\mu(t \mid I,P,\rho)$ equals $\tilde{\mu}(t \mid I,P,\rho)$ which thus also is zero as required.

Now suppose instead that (I,P,ρ) is expected in $\tilde{E}$: $\tilde{T}(P,\rho \mid I)$ is nonempty. By (14), $\tilde{u}(t \mid I)$ exceeds $\rho - \delta E\left[W_P^I(Y) \mid t\right]$ which is the highest payoff type t can expect from (I,P,ρ). Hence t is not in $\tilde{T}(P,\rho \mid I)$ whence, by Beliefs, $\tilde{\mu}(t \mid I,P,\rho)$ is zero. We conclude that $\tilde{E}$ is intuitive as claimed. ◆

PROOF OF PROPOSITION 7. We first define the notions of equilibrium, intuitive beliefs, and fair pricing for an EPSD game. Each definition is the natural restriction of the analogous definition for a GSD game.

> **EPSD EQUILIBRIUM.** A perfect Bayesian equilibrium of the EPSD game is a security design $S_t \in M$ and price cap $\overline{\rho}_t \in \Re_+$ for each type t, as well as a price function $\hat{p}(S,\overline{\rho})$ and belief function $\hat{\mu}(t \mid S,\overline{\rho})$, with the following properties:
>
> 1. Payoff Maximization: for all t, the issuer's choice $(S_t,\overline{\rho}_t)$ solves $\max_{S,\overline{\rho}} \{\hat{p}(S,\overline{\rho}) - \delta E[S(Y) \mid t]\}$ subject to $S \in$ M.
>
> 2. Competitive Pricing: for any monotone security $S \in$ M and price cap $\overline{\rho}$, the price function $\hat{p}(S,\overline{\rho})$ equals $\min\{\overline{\rho}, \sum_t E[S(Y) \mid t]\, \hat{\mu}(t \mid S,\overline{\rho})\}$.

3. Rational Updating: the investors' belief function $\hat{\mu}(t \mid S, \overline{\rho})$ follows Bayes's rule when applicable.

FAIR PRICING (EPSD GAME). An equilibrium $\left(\left(S_t, \overline{\rho}_t\right)_{t=0}^{T}, \hat{p}, \hat{\mu}\right)$ of the GSD game is *fairly priced* if, for each type *t*, the price of *ex-post* security S_t equals its expected value conditional on the issuer's type: $\hat{p}(S_t, \overline{\rho}_t) = E\left[S_t(Y) \middle| t\right]$.

The *outcome* of the EPSD game is the function $\hat{u}(t) = \hat{p}(S_t, \overline{\rho}_t) - \delta E\left[S_t(Y) \mid t\right]$ giving the securitization payoff of each type *t*.

THE INTUITIVE CRITERION (EPSD GAME). A perfect Bayesian equilibrium $\left(\left(S_t, \overline{\rho}_t\right)_{t=0}^{T}, \hat{p}(\cdot,\cdot), \hat{\mu}(\cdot \mid \cdot,\cdot)\right)$ of the EPSD game, with outcome $\hat{u}(\cdot)$, is *intuitive* if, for any security $S \in M$ and revenue cap $\overline{\rho} \in \Re_+$ for which $\overline{\rho} \geq \hat{u}(t) + \delta E\left[S(Y) \mid t\right]$ for some type *t*, $\overline{\rho} < \hat{u}(s) + \delta E\left[S(Y) \mid s\right]$ implies $\hat{\mu}(s \mid S, \overline{\rho}) = 0$.

Define the following two sets.

1. I_{AS} is the set of intuitive Asset Sale Equilibria $\left(q(\cdot), \overline{p}(\cdot), p(\cdot,\cdot), \mu(\cdot \mid \cdot,\cdot)\right)$ of the asset sale game G_{AS} with portfolio (F^*, *a*) in which investor beliefs following any issuance choice $\left(q, \overline{p}\right)$ depend only on the quantity choices *q* and the issuer's maximum issuance revenue $\overline{\rho} = \overline{p}q$. Changing notation slightly, we now denote these beliefs as $\mu\left(\cdot \mid q, \overline{p}q\right)$ rather than $\mu\left(\cdot \mid q, \overline{p}\right)$.

2. I_{SD} is the set of intuitive equilibria $\left(\left(S_t, \overline{\rho}_t\right)_{t=0}^{T}, \hat{p}(\cdot,\cdot), \hat{\mu}(\cdot \mid \cdot,\cdot)\right)$ of the Ex Post Security Design game G_{SD}.

These two sets are equivalent in the following sense

LEMMA.

1. For any equilibrium *e* in I_{AS}, there is an equilibrium $\hat{e}$ in I_{SD} with the same outcome. If type *t* sells the quantities *q*(*t*) in *e*, then this type issues the security $S_t(Y) = q(t) F^*(Y)$ in $\hat{e}$.

2. For any equilibrium $\hat{e}$ in I_{SD}, there is a set of equilibria e in I_{AS} with the same outcome. If type t issues the security S_t in $\hat{e}$, then this type sells the quantities q^{S_t} in any such equilibrium e.

Proof of Lemma. Consider any equilibrium e in I_{AS}. Its outcome is $u(t) = q(t)\left[p(q(t), \overline{p}(t)) - \delta f^*(t)\right]$ and investors' price function is $p(q, \overline{p}) = \overline{p} \wedge \sum_t f^*(t)\mu(t \mid q, \overline{p}q)$. We build an equivalent Security Design Equilibrium $\hat{e}$ in I_{SD} as follows. Type t issues the monotone security $S_t(y) = q(t)F^*(y)$ with maximum revenue $\overline{\rho}_t$ equal to $\overline{p}(t)q(t)$. This implies, in particular, that $q_i^{S_t} = q(t)\left[F^*(y_i) - F^*(y_{i-1})\right] / (y_i - y_{i-1}) = q_i(t)$; collecting terms, $q^{S_t} = q(t)$. Given any security design choice $(S, \overline{\rho})$, investors respond with beliefs $\hat{\mu}(\cdot \mid S, \overline{\rho}) = \mu(\cdot \mid q^S, \overline{\rho})$ and associated price

$$\hat{p}(S, \overline{\rho}) = \max\left\{\overline{\rho}, \sum_t E\left[S(Y) \mid t\right]\hat{\mu}(t \mid S, \overline{\rho})\right\},$$

whence Competitive Pricing holds in $\hat{e}$. A type t issuer's payoff from the choice $(S, \overline{\rho})$ is $\hat{p}(S, \overline{\rho}) - \delta E\left[S(Y) \mid t\right] = \max\left\{\overline{\rho}, q^S \sum_t f^*(t)\mu(t \mid q^S, \overline{\rho})\right\} - \delta q^S f^*(t)$, which is identical to her payoff $q^S\left[p(q^S, \overline{p}) - \delta f^*(t)\right]$ from the issuance choice $(q^S, \overline{p})$ in e where $\overline{p}$ is any n-vector of asset price caps that gives the same maximum revenue: $\overline{\rho} = \overline{p}q^S$. Moreover, all such asset price cap vectors give the same payoff in e because – by assumption - investor beliefs and thus issuance revenue depend only on the quantity vector q and maximum revenue $q\overline{p}$. Hence, since $(q(t), \overline{p}(t))$ maximizes the payoff of type t in e, it follows that $(S_t, \overline{\rho}_t)$ maximizes the payoff of type t in $\hat{e}$: Payoff Maximization holds in $\hat{e}$. By restricting to choices made in equilibrium, the preceding also implies that the outcome $\hat{u}(t) = \hat{p}(S_t, \rho_t) - \delta E\left[S_t(Y) \mid t\right]$ of $\hat{e}$ equals the outcome $u(t) = q(t)\left[p(q(t), \overline{p}(t)) - \delta f^*(t)\right]$ of e. By construction, $\hat{\mu}(\cdot \mid S_t, \overline{\rho}_t) = \mu(\cdot \mid q^{S_t}, \overline{\rho}_t) = \mu(\cdot \mid q(t), \overline{p}(t))$ and the latter is given by Bayes's Rule whenever possible. Since the issuer's behavior is also the same in the two cases, Rational Updating holds: $\hat{e}$ is a Security Design Equilibrium. Finally, consider any security design choice $(S,\bar{\rho})$ in $\hat{e}$ such that, for some type *t*,

$$\bar{\rho} \geq \hat{u}(t) + \delta E[S(Y) \mid t] = u(t) + \delta q^S f^*(t).$$

Then following the issuance choice $(q^S, \bar{p})$ in *e*, investors put zero weight on any type *s* for which $\bar{\rho}$ is less than $u(s) + \delta q^S f^*(s)$ which equals $\hat{u}(s) + \delta E[S(Y) \mid s]$. Hence $\hat{\mu}(s \mid S, \bar{\rho}) = \mu(s \mid q^S, \bar{\rho}) = 0$: $\hat{e}$ is intuitive.

Consider any equilibrium $\hat{e}$ in I_{SD}. Its outcome is $\hat{u}(t) = \hat{p}(S_t, \bar{\rho}_t) - \delta E[S_t(Y) \mid t]$ and the investors' price function is $\hat{p}(S,\bar{\rho}) = \max\left\{\bar{\rho}, \sum_t E[S(Y) \mid t]\hat{\mu}(t \mid S,\bar{\rho})\right\}$. We build an equivalent Asset Sale Equilibrium *e* in I_{AS} as follows. (As the price cap vector will not be unique, this defines a set of equilibria *e*, as indicated in the statement of this proposition.) Type *t* chooses the quantity vector $q(t) = q^{S_t}$ and *any* price cap vector $\bar{p}(t)$ that satisfies $\bar{p}(t)q(t) = \bar{\rho}_t$. In response to any asset sale choice $(q,\bar{p})$, investors' posterior beliefs are $\mu(\cdot \mid q,\bar{p}) = \hat{\mu}(\cdot \mid qF^*, \bar{p}q)$.[79] The associated price vector is $p(q,\bar{p}) = \bar{p} \wedge \sum_t f^*(t)\mu(t \mid q,\bar{p})$, which implies Competitive Pricing in *e*. The price functions in *e* and $\hat{e}$ are thus related by $p(q,\bar{p})q = \hat{p}(qF^*, \bar{p}q)$. Hence, a type *t* issuer's payoff $q[p(q,\bar{p}) - \delta f^*(t)]$ in *e* from the issuance choice $(q,\bar{p})$ equals her payoff $\hat{p}(qF^*, \bar{p}q) - \delta E[qF^*(Y) \mid t]$ in $\hat{e}$ from the security design $(qF^*, \bar{p}q)$. Thus, since $(S_t, \bar{\rho}_t)$ maximizes the payoff of type *t* in $\hat{e}$, it follows that $(q^{S_t}, \bar{p})$ maximizes the payoff of type *t* in *e*, where $\bar{p}$ is any price cap vector that satisfies $\bar{p}q^{S_t} = \bar{\rho}_t$: Payoff Maximization holds in *e*. By restricting to choices made in equilibrium, the preceding also implies that the outcome $u(t) = q(t)[p(q(t),\bar{p}(t)) - \delta f^*(t)]$ of *e* equals the outcome $\hat{u}(t) = \hat{p}(S_t, \rho_t) - \delta E[S_t(Y) \mid t]$ of $\hat{e}$. By construction,

[79] The security $qF^*(Y)$ in G_{SD} promises the same realized payout to investors as the quantity vector q in G_{AS}.

$$\mu(\cdot \mid q(t), \overline{p}(t)) = \hat{\mu}(\cdot \mid q(t) F^*, \overline{p}(t) q(t)) = \hat{\mu}(\cdot \mid S_t, \overline{\rho}_t)$$

and the last is given by Bayes's Rule whenever possible. Since the issuer's behavior is also the same in the two settings, Rational Updating holds: e is an Asset Sale Equilibrium. Finally, consider any issuance choice $(q, \overline{p})$ in *e* such that, for some type *t*, $\overline{p}q \geq u(t) + \delta q f^*(t) = \hat{u}(t) + \delta E[S(Y) \mid t]$ where *S* denotes the security qF^*. Then, following the corresponding issuance choice $(S, \overline{p}q)$ in $\hat{e}$, investors put zero weight on any type s for which $\overline{p}q < \hat{u}(s) + \delta E[S(Y) \mid s] = u(s) + \delta q f^*(s)$. Hence, $\mu(s \mid q, \overline{p}) = \hat{\mu}(s \mid qF^*, \overline{p}q) = 0$: e is intuitive. ◆

Now let e^*_{AS} denote the intuitive equilibrium e^* of the Asset Sale game, specialized to the case in which the endowment (a, f) equals the endowment (a^*, f^*) of G_{AS}. We claim that e^*_{AS} lies in I_{AS}. Why? In e^*_{AS}, each type *t* can choose any price cap vector that exceeds $f^*(t)$ and investors ignore these caps. Thus, beliefs on the equilibrium path do not depend on the price cap vector. Following any out-of-equilibrium choice $(q, \overline{p})$, investors' beliefs are given by (3) and (4), which are also independent of the price cap vector. So investor beliefs in e^*_{AS} do not depend on the price cap vector at all. Thus, as e^*_{AS} is intuitive by PROPOSITION 1, it must lie in I_{AS}.

By the above Lemma, then, there exists an equilibrium e^*_{SD} in I_{SD} with the same outcome as e^*_{AS}, in which each type *t* issues the security $S^*_t(Y) = q^*(t) F^*(Y)$ where $q^*(t)$ is chosen by type *t* in e^*_{AS} and solves RLP. Finally, FOSD together with the fact that F^* is monotone implies that ASSUMPTION A holds. Hence, by PROPOSITION 1, the quantity choice function in any equilibrium in I_{AS} must be a solution $q^*(\cdot)$ to RLP. Thus, by the above Lemma, in *any* equilibrium in I_{SD}, the issuer's optimal security $S^*_t(Y)$ must equal $q^*(t)F^*(Y)$ where $q^*(t)$ solves RLP. ◆

PROOF OF PROPOSITION 8. From PROPOSITION 7, $S_t^*(Y) = q^*(t)F^*(Y)$. By PROPOSITION 5, IIS holds so that from PROPOSITION 3, $q_i^*(t) = 1$ for $i < h(t)$ and $q_i^*(t) = 0$ for $i > h(t)$. Therefore $S_t^*(y_i) = q^*(t)F^*(y_i) = y_i$ for $i < h(t)$ and

$$S_t^*(y_i) = q^*(t)F^*(y_i) = y_{h(t)-1} + q_{h(t)}^*(t)\left(y_{h(t)} - y_{h(t)-1}\right)$$

for $i \geq h(t)$. Thus, $S_t^*(Y) = \min\left(D_t^*, Y\right)$ where $D_t^* = y_{h(t-1)} + q_{h(t)}^*(t)\left(y_{h(t)} - y_{h(t)-1}\right)$. Finally, since q^* and h are decreasing in t by PROPOSITION 3, D_t^* is decreasing in t. That D_t^* satisfies (9) follows from PROPOSITION 4. Finally, as the face value function is strictly decreasing, the equilibrium is fully revealing; hence, $u^*(t) = (1-\delta)v(D_t, t)$. Off-equilibrium, suppose debt $D \in (D_{t+1}^*, D_t^*)$ is issued. The price p that would make type t willing to deviate satisfies $p = u^*(t) + \delta v(D,t) = (1-\delta)v(D_t^*, t) + \delta v(D,t)$. For any type $s < t$, the incentive constraint for s implies $v(D_t^*, t) - \delta v(D_t^*, s) \leq u^*(s)$. Combining these, we have $p + \delta \underbrace{\left[v(D_t^*, t) - v(D,t) - \left(v(D_t^*, s) - v(D,s)\right)\right]}_{>0} \leq u^*(s) + \delta v(D,s)$, where the term in square brackets is positive: the value of junior debt with face value $D_t^* - D$ is higher for type t than for type $s < t$. Therefore, types $s < t$ would not find it profitable to deviate given price p. The beliefs specified in (3) must therefore concentrate on t, and so the market price following a deviation to D is $v(D,t)$. ◆

PROOF OF PROPOSITION 9: Let the range of D^∞ be $(\underline{D}, \bar{y}]$. Define $U\left(t, \widehat{t}, D\right)$ to be the payoff $E\left[\min(D,Y)\,|\widehat{t}\right] - \delta E\left[\min(D,Y)\,|t\right]$ of an issuer with type t when she issues debt with face value D and the investors believe her type is $\widehat{t}$. Then

$$\frac{U_D}{U_{\widehat{t}}} = \frac{\Pr\left(Y > D|\widehat{t}\right) - \delta \Pr\left(Y > D|t\right)}{\frac{\partial}{\partial \widehat{t}} E\left[\min(D,Y)\,|\widehat{t}\right]}$$

is nonincreasing in t by First-Order Stochastic Dominance (FOSD). Since by assumption D_t is decreasing in t and since $U_2\left(t, \widehat{t}, D\right) \geq 0$ by FOSD, part 1 of Theorem 6 in Mailath and von Thadden (2013) implies that an issuer of type t will not imitate any other type: she will not sell debt with face value $D^\infty_{\widehat{t}}$ for any $\widehat{t} \neq t$.

It remains to consider whether she will sell debt with a face value not in the range of D^∞, as well as other types of monotone securities. We may assume that investors respond to any such deviation with the the most pessimistic beliefs: that her type is zero.

First consider a deviation of a type t to debt with a face value D that is not in the range $[\underline{D}, \bar{y}]$ of D^∞. Assume this deviation makes type t strictly better off than sticking to the equilibrium. First, suppose $D > \bar{y}$. This security has the same payout for any Y as debt with face value $\bar{y}$, and both lead investors to believe that $t = 0$. By the preceding result, no type strictly prefers such a deviation. Now consider $D < \underline{D}$. Investor beliefs cannot be more optimistic than those that result from debt with face value $\underline{D}$, since these are the beliefs that the type is one. Moreover, there are gains from trade, so a higher face value is more profitable for the issuer (holding investor beliefs constant). Hence, any $D < \underline{D}$ is worse for the issuer than $D = \underline{D}$.

Finally, we consider deviations of type t to a general monotone security $\widehat{S}$. The issuer's securitization profit from $\widehat{S}$ equals the price $E\left[\widehat{S}(Y)\,|0\right]$ assigned by investors less the discounted expected security payout $\delta E\left[\widehat{S}(Y)\,|t\right]$. This profit can be written as

$$\int_{Y=0}^{\bar{y}} \widehat{S}(Y)\, d\left[H(Y|0) - \delta H(Y|t)\right]$$

where H denotes the conditional distribution of Y given t. The security $\widehat{S}$ is monotone if and only if $\widehat{S}(0) = 0$ and, for all Y, the control variable $c(Y) = \widehat{S}'(Y)$ lies in $[0,1]$. For all

$Y \in [0,\bar{y}]$ define the Hamiltonian

$$\mathscr{H} = \lambda(Y)c(Y) + \widehat{S}(Y)\left[H'(Y|0) - \delta H'(Y|t)\right] + \mu_0(Y)c(Y) + \mu_1(Y)[1-c(Y)]$$

where $\mu_0, \mu_1 \geq 0$ are Lagrange multipliers that capture the constraints on c and $\lambda(Y)$ is a costate variable. By Pontryagin's maximization principle (Pontryagin *et al* 1962), the optimal control c must maximize the Hamiltonian $\mathscr{H}$ while the multipliers $\mu_0, \mu_1 \geq 0$ must minimize it. As $\mathscr{H}$ is linear in c, this implies the first order condition

$$0 = \frac{\partial}{\partial c(Y)}\mathscr{H} = \lambda(Y) + \mu_0(Y) - \mu_1(Y) \tag{16}$$

as well as the Kuhn-Tucker (complementary slackness) conditions:

$$\mu_0(Y)c(Y) = 0 \text{ and } \mu_1(Y)[1-c(Y)] = 0. \tag{17}$$

Additionally, for all Y, the costate equation

$$\lambda'(Y) = -\mathscr{H}_{\widehat{S}} = -H'(Y|0) + \delta H'(Y|t) \tag{18}$$

must be satisfied. Finally, as the final state $\widehat{S}(\bar{y})$ is not fixed, the terminal costate must be zero:

$$\lambda(\bar{y}) = 0. \tag{19}$$

Solving (18) subject to (19), we obtain $\lambda(Y) = [1-H(Y|0)]\left[1-\delta\frac{1-H(Y|t)}{1-H(Y|0)}\right]$. By HRO, $\frac{1-H(Y|t)}{1-H(Y|0)}$ is increasing in Y since $t > 0$. Hence, by (16), (17), and the nonnegativity of μ_0 and μ_1, there exists an $D \in \Re$ such that $c(Y) = 1$ for all $Y < D$ and $c(Y) = 0$ for all $Y > D$: the optimal security is debt with face value D and thus, as shown above, $D = D_t^{\infty}$. Q.E.D.PROPOSITION 9

PROOF OF PROPOSITION 10: First, $1-H(y|t)$ can be written as $1-H(y|1)+\int_{s=t}^{1} H_2(y|s)\,ds$, which is smaller for $H = \widetilde{H}$ than for $H = \widehat{H}$. Hence, the rate of change function in (1) is no larger in absolute value when H equals $\widehat{H}$ than when it equals $\widetilde{H}$. The result then follows from PROPOSITION 14. Q.E.D.PROPOSITION 10

PROOF OF PROPOSITIONS 11-14, AND COROLLARY 15: In this proof, we will write $D^{\infty}(t)$ and $D^{i}(t)$ in place of D_t^{∞} and D_t^{i}. We will also treat $\bar{y}$ as a parameter and the relative cash flow $Z = Y/\bar{y}$ as an exogenous random variable with support $[0,1]$ and conditional distribution function $\widehat{H}$ defined by $\widehat{H}(z|t) = H(z\bar{y}|t)$ for z in $[0,1]$. Lipschitz-H then entails the following property.[1]

Lipschitz-$\widehat{H}$. There are constants $\widehat{k}_0, \widehat{k}_1, \widehat{k}_2 \in (0,\infty)$ such that for all $z,t,t',t'' \in [0,1]$, $\frac{\partial \widehat{H}(z|t)}{\partial z} \in \left(\widehat{k}_0, \widehat{k}_1\right)$, $-\frac{\partial \widehat{H}(z|t)}{\partial t} \in \left[\widehat{k}_0 z(1-z), \widehat{k}_1\right)$, $\left|\frac{\partial \widehat{H}(z|t')}{\partial z} - \frac{\partial \widehat{H}(z|t'')}{\partial z}\right| < \widehat{k}_2 |t'-t''|$, and

$$\left| \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t''} \right| < \widehat{k}_2 \left|t'-t''\right|.$$

Henceforth we rename $\widehat{H}$ to H and each $\widehat{k}_n$ to k_n (for $n = 0,1,2$), and refer to Lipschitz-$\widehat{H}$ as Lipschitz-H or L-H.

Recall that E^i and E^{∞} denote the expectation operators in models i and ∞. For any $D \in \Re$, let

$$v^{H\bar{y}i}(D,t) = E^i\left[\min\{D, \bar{y}Z\}|t\right] = \sum_{c=1}^{1/\Delta_i'} \min\left\{D, \bar{y}c\Delta_i'\right\}\left[H\left(c\Delta_i'|t\right) - H\left((c-1)\Delta_i'|t\right)\right] \quad (20)$$

[1] Lipschitz-H implies that there are constants $k_0, k_1, k_2 \in (0,\infty)$ such that for all $z,t,t',t'' \in [0,1]$,

$$\frac{\partial \widehat{H}(z|t)}{\partial z} = \frac{\partial H(z\bar{y}|t)}{\partial z} = \frac{\partial H(z\bar{y}|t)}{\partial (z\bar{y})}\frac{d(z\bar{y})}{dz} \in (k_0\bar{y}, k_1\bar{y}),$$

$$-\frac{\partial \widehat{H}(z|t)}{\partial t} = -\frac{\partial H(z\bar{y}|t)}{\partial t} \in [k_0 y(\bar{y}-y), k_1) = \left[\bar{y}^2 k_0 z(1-z), k_1\right),$$

$$\begin{aligned}\left|\frac{\partial \widehat{H}(z|t')}{\partial z} - \frac{\partial \widehat{H}(z|t'')}{\partial z}\right| &= \left|\frac{\partial H(z\bar{y}|t')}{\partial (z\bar{y})} - \frac{\partial H(z\bar{y}|t'')}{\partial (z\bar{y})}\right| \frac{d(z\bar{y})}{dz} \\ &= \left|\frac{\partial H(y|t')}{\partial y} - \frac{\partial H(y|t'')}{\partial y}\right| \bar{y} < k_2\bar{y}\left|t'-t''\right|,\end{aligned}$$

and

$$\begin{aligned}\left| \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t''} \right| &= \left| \left.\frac{\partial H(z\bar{y}|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial H(z\bar{y}|t)}{\partial t}\right|_{t=t''} \right| \\ &= \left| \left.\frac{\partial H(y|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial H(y|t)}{\partial t}\right|_{t=t''} \right| < k_2\left|t'-t''\right|.\end{aligned}$$

This implies Lipschitz-$\widehat{H}$ with constants $\widehat{k}_0 = k_0 \min\left\{\bar{y}, \bar{y}^2\right\}$, $\widehat{k}_1 = k_1 \max\{\bar{y}, 1\}$, and $\widehat{k}_2 = k_2 \max\{\bar{y}, 1\}$.

and

$$v^{H\bar{y}}(D,t) = E^{\infty}[\min\{D,\bar{y}Z\}|t] = \int_{z=0}^{1} \min\{D,\bar{y}z\}\, dH(z|t) \tag{21}$$

denote the expected payout of debt with face value D in models i and ∞, respectively, conditional on the issuer's type t. Integrating by parts yields

$$v^{H\bar{y}}(D,t) = D - \bar{y}\int_{z=0}^{D/\bar{y}} H(z|t)\, dz. \tag{22}$$

Given the above notation, the incentive compatibility condition in model i becomes:

Discrete Initial Value Problem with Parameters H, δ, $\bar{y}$ (DP$^{H\delta\bar{y}i}$). The condition

$$\begin{aligned} & v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t+\Delta_i), t+\Delta_i\right) - \delta v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t+\Delta_i), t\right) \\ = \; & (1-\delta)\, v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), t\right) \end{aligned} \tag{23}$$

with $D^{H\delta\bar{y}i} : S_i \to \Re$, together with the initial value $D^{H\delta\bar{y}i}(0) = \bar{y}$.

Define the function

$$f^{H\delta\bar{y}}(D,t) = -\frac{1}{1-\delta}\frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} = \frac{\bar{y}}{1-\delta}\frac{\int_{z=0}^{\frac{D}{\bar{y}}} \frac{\partial H(z|t)}{\partial t} dz}{1-H\left(\frac{D}{\bar{y}}|t\right)} \leq 0, \tag{24}$$

where the second equality follows from (22). A restatement of CP in model ∞ is as follows.

Continuous Initial Value Problem with Parameters H, δ, $\bar{y}$ (CP$^{H\delta\bar{y}}$). The differential equation

$$\frac{dD^{H\delta\bar{y}}}{dt} = f^{H\delta\bar{y}}\left(D^{H\delta\bar{y}}, t\right), \tag{25}$$

with $D^{H\delta\bar{y}} : [0,1] \to \Re$, together with the initial value $D^{H\delta\bar{y}}(0) = \bar{y}$.

In model i, let $p^{H\delta\bar{y}i}(t)$ denote $v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), t\right)$: the price and equilibrium expected payout of a standard debt contract, conditional on the type t. Let $u^{H\delta\bar{y}i}(t)$ denote $(1-\delta)\, p^{H\delta\bar{y}i}(t)$: the issuer's profit and expected gains from trade from such a contract conditional on the type t. Let $Eu^{GH\delta\bar{y}i} = E^{i}\left[u^{H\delta\bar{y}i}(t)\right]$ denote the unconditional expected gains from trade and issuer's profit. Analogously, in the continuous model let

$p^{H\delta\bar{y}}(t)$ denote $v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t),t\right)$, let $u^{H\delta\bar{y}}(t)$ denote $(1-\delta)\,p^{H\delta\bar{y}}(t)$, and let $Eu^{GH\delta\bar{y}}$ denote $E^{\infty}\left[u^{H\delta\bar{y}}(t)\right]$. Let $\Pi^{GH\delta\bar{y}i}(t)=u^{GH\delta\bar{y}i}(t)+E^{i}[\bar{y}Z|t]$ denote the issuer's conditional (on t) expected total profits in model i and let $E\Pi^{GH\delta\bar{y}i}=E^{i}\left[\Pi^{GH\delta\bar{y}i}(t)\right]$ denote her unconditional expected total profits. In the continuous model, define the analogous quantities $\Pi^{GH\delta\bar{y}}(t)=u^{GH\delta\bar{y}}(t)+E^{\infty}[\bar{y}Z|t]$ and $E\Pi^{GH\delta\bar{y}}=E^{\infty}\left[\Pi^{GH\delta\bar{y}}(t)\right]$. As noted, we extend the functions in model i to any type $t\in[0,1]$ by evaluating them them at the highest type in S_i that does not exceed t.

With this notation, Propositions 11-14, together with Corollary 15, are combined as follows.

THEOREM 1 *Fix constants k_0, k_1, k_2, k_3, k_4, and $\mathbf{y}$, all in $(0,\infty)$. Let $\mathscr{G}$ be the set of distribution functions G that satisfy Lipschitz-G with constants k_3 and k_4. Let $\mathscr{H}$ be the set of conditional distribution functions H that satisfy Lipschitz-H with constants k_0, k_1, and k_2, and Hazard Rate Ordering. For any distribution function G in $\mathscr{G}$, conditional distribution function H in $\mathscr{H}$, parameter $\bar{y}$ in $(0,\mathbf{y}]$, and discount factor δ in $(0,1)$:*

1. *There exists a unique function $D^{H\delta\bar{y}}$ that satisfies $CP^{H\delta\bar{y}}$. This function is decreasing and differentiable, and takes values in $(0,\bar{y}]$. The associated price and profit functions, $p^{H\delta\bar{y}}$ and $u^{H\delta\bar{y}}$, are continuous and decreasing in the type t as well.*

2. *For each discrete model $i=1,2,\ldots$, there exists a unique, decreasing function $D^{H\delta\bar{y}i}$ that satisfies $DP^{H\delta\bar{y}i}$.*

3. *The sequences of face value functions, price functions, conditional and unconditional expected securitization profit functions, and conditional and unconditional expected total profit functions in the discrete models converge to their continuous counterparts as i grows, uniformly in the distributions G and H, the parameter y, and (except in the case of the unconditional expected profit functions which do not depend on the type) the type $t\in[0,1]$. More precisely:*

   - *For all $\varepsilon>0$ there is an i^* such that for all models $i>i^*$, G in $\mathscr{G}$, H in $\mathscr{H}$, $\bar{y}$*

in $(0,\mathbf{y}]$, *and* $t\in[0,1]$, $\left|\omega^{H\delta\bar{y}i}(t)-\omega^{H\delta\bar{y}}(t)\right|<\varepsilon$ *for each* $\omega=D,p,u,\Pi$, *and* $\left|E\omega^{H\delta\bar{y}i}-E\omega^{H\delta\bar{y}}\right|<\varepsilon$ *for each* $\omega=u,\Pi$.

4. *All of the functions defined above are homogeneous of degree one in the parameter* $\bar{y}$*:* $\omega^{H\delta\bar{y}}=\bar{y}\omega^{H\delta 1}$ *and* $\omega^{H\delta\bar{y}i}=\bar{y}\omega^{H\delta 1i}$ *for each* $\omega=D,p,u,Eu,\Pi,E\Pi$.

5. *Let* $\widetilde{H},\widehat{H}\in\mathscr{H}$ *and* $\widetilde{\delta},\widehat{\delta}\in(0,1)$ *satisfy* $f^{\widetilde{H}\widetilde{\delta}\bar{y}}\leq f^{\widehat{H}\widehat{\delta}\bar{y}}\leq 0$. *Then* $D^{\widetilde{H}\widetilde{\delta}\bar{y}}(t)\leq D^{\widehat{H}\widehat{\delta}\bar{y}}(t)$ *for all* $t\in[0,1]$.

We now prove Theorem 1. Without loss of generality, we restrict to face values D that do not exceed the maximum final asset value $\bar{y}$.[2] First, we define an integration by parts formula for the function $v^{H\bar{y}i}$ defined in (20).

CLAIM 2 *For any face value* $D\in[0,\bar{y}]$ *and type* t*,*

$$v^{H\bar{y}i}(D,t)=D-\bar{y}\int_{z=0}^{\frac{D}{\bar{y}}}H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right)dz. \tag{26}$$

PROOF OF CLAIM 2**.** One can easily verify (using $H(1|t)=1$ and $H(0|t)=0$) that for $D\in[0,\bar{y}]$, $v^{H\bar{y}i}(D,t)=\min\{D,\bar{y}\}-\sum_{c=1}^{\frac{1}{\Delta_i'}-1}H(c\Delta_i'|t)\left[\min\{D,\bar{y}(c+1)\Delta_i'\}-\min\{D,\bar{y}c\Delta_i'\}\right]$ by (20). As c is an integer, $c\leq x$ iff $c\leq\lfloor x\rfloor$. Thus,

$$\min\left\{D,\bar{y}(c+1)\Delta_i'\right\}-\min\left\{D,\bar{y}c\Delta_i'\right\}=\begin{cases}\bar{y}\Delta_i' & \text{if } c\leq\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor-1\\ D-\bar{y}c\Delta_i' & \text{if } c=\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\\ 0 & \text{if } c>\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\end{cases},$$

whence, as $D\leq\bar{y}$,

$$v^{H\bar{y}i}(D,t)=D-\bar{y}\Delta_i'\left[\begin{array}{c}\sum_{c=1}^{\lfloor D/(\bar{y}\Delta_i')\rfloor-1}H(c\Delta_i'|t)\\ +H\left(\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\Delta_i'|t\right)\left(\frac{D}{\bar{y}\Delta_i'}-\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\right)1\left(\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\leq\frac{1}{\Delta_i'}-1\right)\end{array}\right]. \tag{27}$$

[2] Choosing a higher face value is equivalent to choosing $\bar{y}$ since the underlying assets cannot be worth more than $\bar{y}$.

If $z \in [c\Delta_i', (c+1)\Delta_i')$, then $\lfloor z/\Delta_i' \rfloor = c$. So

$$\begin{aligned}
\sum_{c=1}^{\lfloor D/(\bar{y}\Delta_i')\rfloor -1} H\left(c\Delta_i'|t\right) &= \sum_{c=1}^{\lfloor D/(\bar{y}\Delta_i')\rfloor -1} \frac{1}{\Delta_i'} \int_{z=c\Delta_i'}^{(c+1)\Delta_i'} H\left(\Delta_i' \left\lfloor \frac{z}{\Delta_i'} \right\rfloor |t\right) dz \\
&= \frac{1}{\Delta_i'} \int_{z=\Delta_i'}^{\Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor} H\left(\Delta_i' \left\lfloor \frac{z}{\Delta_i'} \right\rfloor |t\right) 1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right) dz \qquad (28)
\end{aligned}$$

since $1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right)$ equals one for all $z < \Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor$. Moreover,

$$\begin{aligned}
& H\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \Delta_i'|t\right) \left(\frac{D}{\bar{y}\Delta_i'} - \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor\right) 1\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \le \frac{1}{\Delta_i'} - 1\right) \\
= & \frac{1}{\Delta_i'} \int_{z=\Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor}^{\frac{D}{\bar{y}}} H\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \Delta_i'|t\right) 1\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \le \frac{1}{\Delta_i'} - 1\right) dz \qquad (29) \\
= & \frac{1}{\Delta_i'} \int_{z=\Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor}^{\frac{D}{\bar{y}}} H\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor \Delta_i'|t\right) 1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right) dz \qquad (30)
\end{aligned}$$

where the last equality holds for two reasons. First, $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor = \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor$ in the interval of integration in line (29). Second, since $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor$ and $\frac{1}{\Delta_i'}$ are both integers, $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}$ if and only if $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor \le \frac{1}{\Delta_i'} - 1$. Combining (27), (28), and (30), we obtain

$$v^{H\bar{y}i}(D,t) = D - \bar{y} \int_{z=\Delta_i'}^{\frac{D}{\bar{y}}} H\left(\Delta_i' \left\lfloor \frac{z}{\Delta_i'} \right\rfloor |t\right) 1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right) dz.$$

As $1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right)$ equals one except possibly at the upper endpoint of the integral, it can be omitted. Finally, the lower limit of integration can be reduced to zero since $H(0|t) = 0$. Q.E.D.$_{\text{Claim 2}}$

We next show that the first derivatives with respect to D and t of the expected payout $v^{H\bar{y}i}(D,t)$ in model i are bounded above and below (parts 1 and 2) and converge uniformly to the respective first derivatives of the expected payout $v^{H\bar{y}}(D,t)$ in the continuous case (parts 3 and 4):

CLAIM 3 *Assume L-H. Define*

$$\Omega^{H\bar{y}i}(t',t'',D',D'') = v^{H\bar{y}i}(D',t') - v^{H\bar{y}i}(D'',t'') - \left[v^{H\bar{y}}(D',t') - v^{H\bar{y}}(D'',t'')\right].$$

1. *For all* $D \in (\bar{y}\Delta_i', \bar{y}]$*, and all* t', t'' *in* S_i *such that* $t' > t''$*,*

$$\max\left\{0, k_0 \frac{(D')^2 (3\bar{y} - 2D')}{6\bar{y}^2}\right\} < \frac{v^{H\bar{y}i}(D,t') - v^{H\bar{y}i}(D,t'')}{t' - t''} < \bar{y}k_1 \min\left\{\frac{D}{\bar{y}}, 1 - \Delta_i'\right\},$$

where $D' = D - 2\bar{y}\Delta_i'$*.*

2. *For all* D', D'' *in* $[0, \bar{y}]$ *such that* $D' > D''$ *and for all* $t \in S_i$*,*

$$\begin{aligned} k_1\left(1 - \frac{D''}{\bar{y}} + \Delta_i\right) &> \frac{v^{H\bar{y}i}(D',t) - v^{H\bar{y}i}(D'',t)}{D' - D''} \\ &> k_0\left(1 - \Delta_i'\left\lceil \frac{D'}{\bar{y}\Delta_i'}\right\rceil + \Delta_i'\right) \geq k_0\left(1 - \frac{D'}{\bar{y}}\right). \end{aligned}$$

3. *For all* $\varepsilon > 0$ *there exists an* i^* *such that if* $i > i^*$*, then for all* $G \in \mathscr{G}$*,* $H \in \mathscr{H}$*,* $\bar{y} \in (0, \mathbf{y}]$*,* D *in* $[0, \bar{y}]$ *and* t', t'' *in* S_i *such that* $t' > t''$*,* $\left|\Omega^{H\bar{y}i}(t', t'', D, D)\right| < \varepsilon(t' - t'')$*.*

4. *For all* $\varepsilon > 0$ *there exists an* i^* *such that if* $i > i^*$*, then for all* $G \in \mathscr{G}$*,* $H \in \mathscr{H}$*,* $\bar{y} \in (0, \mathbf{y}]$*,* $t \in S_i$*, and* D', D'' *in* $[0, \bar{y}]$ *such that* $D' > D''$*,* $\left|\Omega^{H\bar{y}i}(t, t, D', D'')\right| < \varepsilon(D' - D'')$*.*

PROOF OF CLAIM 3. Part 1. By (26),

$$v^{H\bar{y}i}(D,t') - v^{H\bar{y}i}(D,t'') = \bar{y}\int_{z=0}^{\frac{D}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t''\right) - H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t'\right)\right]dz$$

which by L-H is less than $\bar{y}k_1(t' - t'')\left(\frac{D}{\bar{y}} - \Delta_i'\right)$ and at least

$$\bar{y}k_0(t' - t'')(\Delta_i')^2 \int_{z=0}^{\frac{D}{\bar{y}}}\left\lfloor\frac{z}{\Delta_i'}\right\rfloor\left(\frac{1}{\Delta_i'} - \left\lfloor\frac{z}{\Delta_i'}\right\rfloor\right)dz, \tag{31}$$

which is zero if $D \leq \bar{y}\Delta_i'$ and positive otherwise. Let $c = \left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor$ where $D' = D - 2\bar{y}\Delta_i'$. Recall $N_i' = 1/\Delta_i'$. Hence, for $D > \bar{y}\Delta_i'$, the integral in (31) is at least

$$\begin{aligned} \int_{z=0}^{c\Delta_i'}\left\lfloor\frac{z}{\Delta_i'}\right\rfloor\left(N_i' - \left\lfloor\frac{z}{\Delta_i'}\right\rfloor\right)dz &= \Delta_i'\sum_{n=1}^{c-1} n(N_i' - n) = \Delta_i'\frac{c(c-1)(3N_i' - 2c + 1)}{6} \\ &\geq \Delta_i'\frac{\left(\frac{D'}{\bar{y}\Delta_i'} + 1\right)\frac{D'}{\bar{y}\Delta_i'}\left(3N_i' - 2\frac{D'}{\bar{y}\Delta_i'} - 1\right)}{6} \text{ as } c \geq \frac{D'}{\bar{y}\Delta_i'} + 1 \\ &> \frac{(D')^2(3\bar{y} - 2D')}{6\bar{y}^3(\Delta_i')^2} \text{ as } D' \leq \bar{y}(1 - 2\Delta_i'). \end{aligned}$$

This proves the result.

Part 2. By L-H and (20),

$$
\begin{aligned}
& v^{H\bar{y}i}\left(D',t\right)-v^{H\bar{y}i}\left(D'',t\right) \\
= & \sum_{c=1}^{1/\Delta_i'}\left[\min\left\{D',\bar{y}c\Delta_i'\right\}-\min\left\{D'',\bar{y}c\Delta_i'\right\}\right]\left[H\left(c\Delta_i'|t\right)-H\left(\left(c-1\right)\Delta_i'|t\right)\right]
\end{aligned}
$$

but

$$
\begin{aligned}
\min\left\{D',\bar{y}c\Delta_i'\right\}-\min\left\{D'',\bar{y}c\Delta_i'\right\} &= \begin{cases} 0 & \text{if} \quad c\leq\frac{D''}{\bar{y}\Delta_i'} \\ \bar{y}c\Delta_i'-D'' & \text{if} \quad \frac{D''}{\bar{y}\Delta_i'}<c<\frac{D'}{\bar{y}\Delta_i'} \\ D'-D'' & \text{if} \quad c\geq\frac{D'}{\bar{y}\Delta_i'} \end{cases} \\
&= \begin{cases} 0 & \text{if} \quad c\leq\left\lfloor\frac{D''}{\bar{y}\Delta_i'}\right\rfloor \\ \bar{y}c\Delta_i'-D'' & \text{if} \quad \left\lfloor\frac{D''}{\bar{y}\Delta_i'}\right\rfloor<c<\left\lceil\frac{D'}{\bar{y}\Delta_i'}\right\rceil \\ D'-D'' & \text{if} \quad c\geq\left\lceil\frac{D'}{\bar{y}\Delta_i'}\right\rceil \end{cases}
\end{aligned}
$$

so

$$
\begin{aligned}
v^{H\bar{y}i}\left(D',t\right)-v^{H\bar{y}i}\left(D'',t\right) &\geq \sum_{c=\left\lceil\frac{D'}{\bar{y}\Delta_i'}\right\rceil}^{1/\Delta_i'}\left(D'-D''\right)\left[H\left(c\Delta_i'|t\right)-H\left(\left(c-1\right)\Delta_i'|t\right)\right] \\
&> \left(D'-D''\right)k_0\Delta_i'\left(\frac{1}{\Delta_i'}-\left\lceil\frac{D'}{\bar{y}\Delta_i'}\right\rceil+1\right) \\
&\geq \left(D'-D''\right)k_0\left(1-D'/\bar{y}\right).
\end{aligned}
$$

Moreover, $v^{H\bar{y}i}\left(D',t\right)-v^{H\bar{y}i}\left(D'',t\right)$ is at most

$$
\begin{aligned}
& \sum_{c=\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil}^{1/\Delta_i'}\left(D'-D''\right)\left[H\left(c\Delta_i'|t\right)-H\left(\left(c-1\right)\Delta_i'|t\right)\right] \\
< & \left(D'-D''\right)k_1\Delta_i'\left(\frac{1}{\Delta_i'}-\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil+1\right)\leq\left(D'-D''\right)k_1\left(1-D''/\bar{y}+\Delta_i\right).
\end{aligned}
$$

Part 3. Let i be large enough that $\mathbf{y}k_2\Delta_i' < \varepsilon$. Then

$$\begin{aligned}\left|\Delta\left(t',t'',D,D\right)\right| &= \left|\sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\left[\min\left(D,\bar{y}c\Delta_i'\right)-\min\left(D,\bar{y}z\right)\right]d\left[H\left(z|t'\right)-H\left(z|t''\right)\right]\right| \\ &\le \sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\left|\min\left(D,\bar{y}c\Delta_i'\right)-\min\left(D,\bar{y}z\right)\right|d\left[H\left(z|t'\right)-H\left(z|t''\right)\right] \\ &\le \sum_{c=1}^{1/\Delta_i'}\bar{y}k_2\left(\Delta_i'\right)^2\left|t'-t''\right| = \bar{y}k_2\Delta_i'\left|t'-t''\right| < \varepsilon\left|t'-t''\right|\end{aligned}$$

since $\left|\min\left(D,\bar{y}c\Delta_i'\right)-\min\left(D,\bar{y}z\right)\right| < \bar{y}\Delta_i'$ and by L-H, $\left|\int_{z=(c-1)\Delta_i'}^{c\Delta_i'} d\left[H\left(z|t'\right)-H\left(z|t''\right)\right]\right| \le k_2\Delta_i'\left|t'-t''\right|$.

Part 4. As $\bar{y} > 0$, $\left|\Omega^{H\bar{y}i}\left(t,t,D',D''\right)\right| = \left|\sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)dH\left(z|t\right)\right|$ where $\eta\left(\zeta',\zeta'',\zeta_0,\zeta_1\right) = \max\left\{\zeta_0,\min\left\{\zeta_1,\zeta'\right\}\right\} - \max\left\{\zeta_0,\min\left\{\zeta_1,\zeta''\right\}\right\}$.

REMARK 4 *$\eta\left(\zeta',\zeta'',\zeta_0,\zeta_1\right)$ lies in $\left[0,\zeta'-\zeta''\right]$ if $\zeta e' \ge \zeta e''$ and is zero if either*

$$\max\left\{\zeta',\zeta''\right\} \le \zeta e_0$$

or $\min\left\{\zeta',\zeta''\right\} \ge \zeta e_1$.

Since z lies in $\left[(c-1)\Delta_i', c\Delta_i'\right]$, the integrand $\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)$ is zero if either $c \le \frac{D''}{\bar{y}\Delta_i'}$ or $c \ge \frac{D'}{\bar{y}\Delta_i'}+1$. Hence,

$$\begin{aligned}&\sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)dH\left(z|t\right) \\ =&\sum_{c=\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil}^{\min\left\{1,\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\right\}}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)dH\left(z|t\right).\end{aligned} \tag{32}$$

Since $c\Delta_i' \ge z$ in each integral, the integrands $\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)$ are all nonnegative, so we may dispense with the absolute value signs. The first summand, which corresponds to

$c = \lceil D''/\bar{y}\Delta_i' \rceil$, is

$$\begin{aligned}
& \int_{z=\left(\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - 1\right)\Delta_i'}^{\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil \Delta_i'} \eta\left(\bar{y}\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil \Delta_i', \bar{y}z, D'', D'\right) dH(z|t) \\
= & \left[\begin{array}{c} \eta\left(\bar{y}\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil \Delta_i', D'', D'', D'\right)\left[H\left(\frac{D''}{\bar{y}}|t\right) - H\left(\left(\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - 1\right)\Delta_i'|t\right)\right] \\ + \int_{z=D''/\bar{y}}^{\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil \Delta_i'} \eta\left(\bar{y}\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil \Delta_i', \bar{y}z, D'', D'\right) dH(z|t) \end{array}\right] \\
\leq & \left[\begin{array}{c} \bar{y}k_1(\Delta_i')^2\left(\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - \frac{D''}{\bar{y}\Delta_i'}\right)\left[\frac{D''}{\bar{y}\Delta_i'} - \left(\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - 1\right)\right] \\ + \bar{y}k_1(\Delta_i')^2\left(\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - \frac{D''}{\bar{y}\Delta_i'}\right)^2 \end{array}\right] = \bar{y}k_1\left(\Delta_i'\right)^2\left(\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - \frac{D''}{\bar{y}\Delta_i'}\right),
\end{aligned}$$

by L-H. There are now two cases.

Case 1: $\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor \leq \frac{1}{\Delta_i'}$. The last summand in (32), which corresponds to $c = \lfloor D'/\bar{y}\Delta_i' + 1 \rfloor$, is[3]

$$\begin{aligned}
& \int_{z=\left(\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor - 1\right)\Delta_i'}^{\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor \Delta_i'} \eta\left(\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor \Delta_i', \bar{y}z, D'', D'\right) dH(z|t) \\
= & \int_{z=\left(\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor - 1\right)\Delta_i'}^{D'/\bar{y}} \eta\left(\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor \Delta_i', \bar{y}z, D'', D'\right) dH(z|t) \\
& + \eta\left(\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor \Delta_i', D', D'', D'\right)\left[H\left(\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor \Delta_i'|t\right) - H\left(\frac{D'}{\bar{y}}|t\right)\right] \qquad (33) \\
\leq & \bar{y}k_1\left(\Delta_i'\right)^2\left[\frac{D'}{\bar{y}\Delta_i'} + 1 - \left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor\right]. \qquad (34)
\end{aligned}$$

The remainder of the sum in (32) is

$$\sum_{c=\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil + 1}^{\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor - 1} \int_{z=(c-1)\Delta_i'}^{c\Delta_i'} \eta\left(\bar{y}c\Delta_i', \bar{y}z, D'', D'\right) dH(z|t) \leq \bar{y}k_1\left(\Delta_i'\right)^2\left(\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor - \left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - 1\right).$$

Collecting terms, $\left|\Omega^{H\bar{y}i}(t,t,D',D'')\right| \leq k_1\Delta_i'(D'-D'')$. Now take i^* large enough that $k_1\Delta_{i^*}' < \varepsilon$.

---

[3]By Remark 4, line (33) is zero as $\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor \Delta_i' - D' = \bar{y}\Delta_i'\left(\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor - \frac{D'}{\bar{y}\Delta_i'}\right) > 0$. The inequality in line (34) then follows from Lipschitz-H.

Case 2: $\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor > \frac{1}{\Delta_i'}$ or, equivalently, $\left\lfloor \frac{D'}{\bar{y}\Delta_i'} \right\rfloor > \frac{1}{\Delta_i'} - 1$. The final sum on the right hand side of (32) then corresponds to $c = 1/\Delta_i'$. Moreover,

$$\sum_{c=\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil +1}^{1/\Delta_i'} \int_{z=(c-1)\Delta_i'}^{c\Delta_i'} \eta\left(\bar{y}c\Delta_i', \bar{y}z, D'', D'\right) dH(z|t) \leq \bar{y}k_1 \left(\Delta_i'\right)^2 \left[\frac{1}{\Delta_i'} - \left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - 1\right].$$

Thus,

$$\begin{aligned} \left|\Omega^{H\bar{y}i}\left(t,t,D',D''\right)\right| &\leq \bar{y}k_1 \left(\Delta_i'\right)^2 \left[\frac{1}{\Delta_i'} - 1 - \frac{D''}{\bar{y}\Delta_i'}\right] < \bar{y}k_1 \left(\Delta_i'\right)^2 \left[\left\lfloor \frac{D'}{\bar{y}\Delta_i'} \right\rfloor - \frac{D''}{\bar{y}\Delta_i'}\right] \\ &\leq \bar{y}k_1 \left(\Delta_i'\right)^2 \left[\frac{D'}{\bar{y}\Delta_i'} - \frac{D''}{\bar{y}\Delta_i'}\right] = k_1 \Delta_i' \left(D' - D''\right) \end{aligned}$$

as before. Q.E.D.$_{\text{Claim 3}}$

For all $D, D' \in [0, \bar{y}]$ and all $t \in S_i \setminus \{1\}$, define the difference quotients of $v^{H\bar{y}i}(D,t)$ with respect to D and t:

$$\begin{aligned} \Delta_1^{H\bar{y}i}\left(D,D',t\right) &= \frac{v^{H\bar{y}i}(D,t) - v^{H\bar{y}i}(D',t)}{D - D'} \text{ and} && (35) \\ \Delta_2^{H\bar{y}i}(D,t) &= \frac{v^{H\bar{y}i}(D,t+\Delta_i) - v^{H\bar{y}i}(D,t)}{\Delta_i}. && (36) \end{aligned}$$

By parts 1 and 2 of Claim 3, if $D < D'$, then

$$\Delta_1^{H\bar{y}i}\left(D,D',t\right) \in \left(k_0 \left(1 - \Delta_i' \left\lceil \frac{D'}{\bar{y}\Delta_i'} \right\rceil + \Delta_i'\right), k_1 \left(1 - \frac{D}{\bar{y}} + \Delta_i\right)\right) \subset (0,\infty) \qquad (37)$$

while if $D > \bar{y}\Delta_i'$,

$$\Delta_2^{H\bar{y}i}(D,t) \in \left(0, k_1 \min\left\{D, \bar{y}\left(1 - \Delta_i'\right)\right\}\right). \qquad (38)$$

By (22), for any $D \in [0, \bar{y}]$, the partial derivatives of $v^{H\bar{y}}(D,t)$ are given by

$$\begin{aligned} v_2^{H\bar{y}}(D,t) &= -\bar{y} \int_{z=0}^{D/\bar{y}} \frac{\partial H(z|t)}{\partial t} dz \text{ and} && (39) \\ v_1^{H\bar{y}}(D,t) &= 1 - H\left(\frac{D}{\bar{y}}|t\right) = \int_{z=D/\bar{y}}^{1} \frac{\partial H(z|t)}{\partial z} dz. && (40) \end{aligned}$$

For all $D \in [0, \bar{y}]$ and $t \in [0,1]$,

$$\frac{\partial v_2^{H\bar{y}}(D,t)}{\partial D} = -\frac{\partial}{\partial t} H\left(\frac{D}{\bar{y}}|t\right) \in \left[k_0 \frac{D}{\bar{y}}\left(1 - \frac{D}{\bar{y}}\right), k_1\right) \qquad (41)$$

by (39) and L-H;

$$\left|\frac{\partial v_2^{H\bar{y}}(D,t)}{\partial t}\right| \leq k_2 D \tag{42}$$

by (39) and L-H;

$$\frac{\partial v_1^{H\bar{y}}(D,t)}{\partial D} = -\frac{\partial}{\partial D} H\left(\frac{D}{\bar{y}}|t\right) \in \left(-\frac{k_1}{\bar{y}}, -\frac{k_0}{\bar{y}}\right) \tag{43}$$

by (40) and L-H;

$$\frac{k_0 D^2 [3\bar{y} - 2D]}{6\bar{y}^2} < v_2^{H\bar{y}}(D,t) < k_1 D \tag{44}$$

by (39) and L-H;

$$k_0\left(1 - \frac{D}{\bar{y}}\right) < v_1^{H\bar{y}}(D,t) < k_1\left(1 - \frac{D}{\bar{y}}\right) \tag{45}$$

by (40) and L-H; for all $t' \in [0,1]$,

$$\left|v_1^{H\bar{y}}(D,t) - v_1^{H\bar{y}}(D,t')\right| = \left|H\left(\frac{D}{\bar{y}}|t\right) - H\left(\frac{D}{\bar{y}}|t'\right)\right| \leq k_1 |t - t'| \tag{46}$$

by (40) and L-H; for all $D' \in [0,\bar{y}]$,

$$\left|v_1^{H\bar{y}}(D,t) - v_1^{H\bar{y}}(D',t)\right| = \left|H\left(\frac{D}{\bar{y}}|t\right) - H\left(\frac{D'}{\bar{y}}|t\right)\right| < \frac{k_1}{\bar{y}} |D - D'| \tag{47}$$

by L-H. By (44) and (45),

$$0 \leq \frac{k_0 D^2 (3\bar{y} - 2D)}{6\bar{y} k_1 (\bar{y} - D)} < \frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} < \frac{k_1 \bar{y}}{k_0}\left(\frac{D}{\bar{y} - D}\right), \tag{48}$$

and

$$\frac{6\bar{y} k_1 (\bar{y} - D)}{k_0 D^2 (3\bar{y} - 2D)} > \frac{v_1^{H\bar{y}}(D,t)}{v_2^{H\bar{y}}(D,t)} > \frac{k_0}{k_1 \bar{y}}\left(\frac{\bar{y} - D}{D}\right). \tag{49}$$

By (41), (43), (44), (45), and (49),

$$\frac{\partial}{\partial D}\left(\frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)}\right) = \frac{\frac{\partial}{\partial D} v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} - \frac{v_2^{H\bar{y}}(D,t)\,\frac{\partial}{\partial D} v_1^{H\bar{y}}(D,t)}{\left[v_1^{H\bar{y}}(D,t)\right]^2} \in \left[\gamma_1^{\bar{y}}(D), \gamma_2^{\bar{y}}(D)\right] \tag{50}$$

where

$$\gamma_1^{\bar{y}}(D) = \frac{k_0 D}{k_1 \bar{y}}\left[1 + \frac{D k_0 (3\bar{y} - 2D)}{6 k_1 (\bar{y} - D)^2}\right] \geq 0 \tag{51}$$

and

$$\gamma_2^{\bar{y}}(D) = \frac{k_1 \bar{y}}{k_0(\bar{y}-D)}\left[1+\frac{Dk_1}{k_0(\bar{y}-D)}\right] > 0. \tag{52}$$

Note that $\gamma_1^{\bar{y}}(D)$ lies in $(0,\infty)$ if $D \in (0,\bar{y})$, is zero if $D=0$, and is ∞ if $D=\bar{y}$. Moreover, $\gamma_2^{\bar{y}}(D)$ is increasing in D, lies in $(0,\infty)$ if $D\in[0,\bar{y})$, and is ∞ if $D=\bar{y}$. Finally, for $0\leq a\leq b$,

$$\max_{D\in[a,b]} \gamma_2^{\bar{y}}(D) \leq \frac{k_1 \bar{y}}{k_0(\bar{y}-b)}\left[1+\frac{bk_1}{k_0(\bar{y}-b)}\right], \tag{53}$$

which is positive and, if $b<\bar{y}$, finite.

For any real number ℓ, let $(\ell,\infty]$ and $[\ell,\infty]$ denote the sets $(\ell,\infty)\cup\{\infty\}$ and $[\ell,\infty)\cup\{\infty\}$, respectively. Recall that $f^{H\delta\bar{y}}(D,t)$ is defined in (24). For $\underline{t}\in[0,1]$ and $a\in(0,\bar{y}]$, and $k\in(0,\infty]$, we define the following modification of $\mathrm{CP}^{H\delta\bar{y}}$:

> CONTINUOUS INITIAL VALUE PROBLEM WITH PARAMETERS $H,\delta,\bar{y},\underline{t},a,k$ ($\mathrm{CP}_{\underline{t}ak}^{H\delta\bar{y}}$).
>
> The differential equation
>
> $$\frac{dD_{\underline{t}ak}^{H\delta\bar{y}}}{dt} = \max\left\{ f^{H\delta\bar{y}}\left(D_{\underline{t}ak}^{H\delta\bar{y}}(t),t\right), -k\right\} \tag{54}$$
>
> with $D_{\underline{t}ak}^{H\delta\bar{y}}:[\underline{t},1]\to\Re$, together with the initial value $D_{\underline{t}ak}^{H\delta\bar{y}}(\underline{t})=a$.

Clearly, any $D_{0\bar{y}\infty}^{H\delta\bar{y}}$ that solves $\mathrm{CP}_{0\bar{y}\infty}^{H\delta\bar{y}}$ must also be a solution $D^{H\delta\bar{y}}$ to $\mathrm{CP}^{H\delta\bar{y}}$ and vice-versa.

CLAIM 5 *Consider any $\underline{t}\in[0,1]$, H in $\mathscr{H}$, $a\in(0,\bar{y}]$, and $k\in(0,\infty]$.*

1. *If either $a<\bar{y}$ or $k<\infty$ (or both), then there exists a unique solution to $CP_{\underline{t}ak}^{H\delta\bar{y}}$, which is decreasing and differentiable in t and takes values in $(0,a]$.*

2. *Let $\widetilde{\delta},\widehat{\delta}\in(0,1)$ and $\widetilde{H},\widehat{H}\in\mathscr{H}$ satisfy $f^{\widetilde{H}\widetilde{\delta}\bar{y}}\leq f^{\widehat{H}\widehat{\delta}\bar{y}}$, and let $a'\in(0,a]$ and $k'\in[k,\infty]$. Suppose there exist (possibly nonunique) solutions $D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\bar{y}}$ and $D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\bar{y}}$ to $CP_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\bar{y}}$ and $CP_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\bar{y}}$, respectively. Then $D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\bar{y}}(t)\geq D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\bar{y}}(t)$ for all $t\in[\underline{t},1]$.*

3. *If either $a<\bar{y}$ or $k<\infty$ (or both), then the function $D_{\underline{t}ak}^{H\delta\bar{y}}$ is Lipschitz continuous in t with Lipschitz constant* $\min\{k,k_a\}$ *where*

$$k_a = \frac{k_1\bar{y}a}{(1-\delta)k_0(\bar{y}-a)}. \tag{55}$$

4. *If $a < \bar{y}$, then for all $t \in [\underline{t}, 1]$, $D^{H\delta\bar{y}}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}}_{\underline{t}a\infty}(t) \in [0, \bar{y} - a]$.*

PROOF OF CLAIM 5. Part 1. We first show that $\max\left\{f^{H\delta\bar{y}}(D,t), -k\right\}$ is (a) continuous in $t \in [0,1]$ and (b) Lipschitz continuous in $D \in [0,a]$. Since max is a Lipschitz-continuous function, it suffices to show that $f^{H\delta\bar{y}}(D,t)$ has properties (a) and (b) whenever $f^{H\delta\bar{y}}(D,t) \geq -k$. If $k = \infty$, then $D \leq a < \bar{y}$. If instead $k < \infty$ and $f^{H\delta\bar{y}}(D,t) \geq -k$ then by (48), $\frac{k_0 D^2(3\bar{y}-2D)}{6\bar{y}k_1(\bar{y}-D)} < (1-\delta)k$; rearranging, $\frac{D^2}{\bar{y}-D} < \frac{6\bar{y}k_1(1-\delta)k}{k_0(3\bar{y}-2D)} \leq \frac{6k_1(1-\delta)k}{k_0}$ (since $D \leq \bar{y}$). Since $\frac{D^2}{\bar{y}-D}$ is continuous and increasing in D and approaches ∞ as $D \uparrow \bar{y}$, there is a constant $a' < \bar{y}$ such that $D \leq a'$ for any D and t satisfying $f^{H\delta\bar{y}}(D,t) \geq -k$. Collecting cases, $D \leq b = \max\{a, a'\} < \bar{y}$. Hence $1 - H\left(\frac{D}{\bar{y}}|t\right) \geq 1 - H\left(\frac{b}{\bar{y}}|t\right) > 0$; as H is continuously differentiable in t, $f^{H\delta\bar{y}}(D,t)$ is continuous in t and thus satisfies (a). And by (50), (51), and (53), $\left|\frac{\partial}{\partial D} f^{H\delta\bar{y}}(D,t)\right|$ is at most $\frac{k_1\bar{y}}{k_0(\bar{y}-b)(1-\delta)}\left[1 + \frac{bk_1}{k_0(\bar{y}-b)}\right]$, whence $f^{H\delta\bar{y}}(D,t)$ satisfies (b). By the Picard-Lindelöf theorem, there thus exists a unique solution $D^{H\delta\bar{y}}_{\underline{t}ak}$ to $\text{CP}^{H\delta\bar{y}}_{\underline{t}ak}$. The solution $D^{H\delta\bar{y}}_{\underline{t}ak}$ is differentiable in t since the right hand side of (54) is finite. Finally, $f^{H\delta\bar{y}}(D,t) < 0$ for all $D \in (0, \bar{y}]$ and $f^{H\delta\bar{y}}(0,t) = 0$. Hence, $D^{H\delta\bar{y}}_{\underline{t}ak}(t)$ is decreasing in t until and unless it hits zero, where it remains. Thus, by (48), $D^{H\delta\bar{y}}_{\underline{t}ak}(t) \leq a$ for all $t \in [\underline{t}, 1]$. Finally, $D^{H\delta\bar{y}}_{\underline{t}ak}(t) > 0$ by the following lemma. Let $D^{H\delta\bar{y}}_{\underline{t}ak}(t)$ first reach its minimum value of $\underline{D} \geq 0$ at $t = \bar{t} > \underline{t}$.

LEMMA 6 *The minimum face value $\underline{D}$ is nonzero.*

PROOF OF LEMMA 6: For $t \in [\underline{t}, \bar{t}]$, the function $D^{H\delta\bar{y}}_{\underline{t}ak}(t)$ has a strictly decreasing inverse $t^{H\delta\bar{y}}_{\underline{t}ak}$ that satisfies the following inverse problem:

INVERSE CONTINUOUS INITIAL VALUE PROBLEM WITH PARAMETERS $H, \bar{y}, \underline{t}, a, k$ ($\text{ICP}^{H\delta\bar{y}}_{\underline{t}ak}$). The differential equation

$$\frac{dt^{H\delta\bar{y}}_{\underline{t}ak}}{dD} = -\max\left\{(1-\delta)\frac{v_1^{H\bar{y}}\left(D, t^{H\delta\bar{y}}_{\underline{t}ak}(D)\right)}{v_2^{H\bar{y}}\left(D, t^{H\delta\bar{y}}_{\underline{t}ak}(D)\right)}, \frac{1}{k}\right\} \tag{56}$$

with $t^{H\delta\bar{y}}_{\underline{t}ak} : [\underline{D}, a] \to [\underline{t}, \bar{t}]$, together with the terminal value $t^{H\delta\bar{y}}_{\underline{t}ak}(a) = \underline{t}$.

By (49), $\frac{dt^{H\delta\bar{y}}_{\underline{t}ak}}{dD} \leq -(1-\delta)\frac{k_0}{k_1\bar{y}}\left(\frac{\bar{y}-D}{D}\right)$. Hence, as $t^{H\delta\bar{y}}_{\underline{t}ak}(a) = \underline{t}$,

$$\begin{aligned} \bar{t} &= t^{H\delta\bar{y}}_{\underline{t}ak}(\underline{D}) = t^{H\delta\bar{y}}_{\underline{t}ak}(a) - \int_{D=\underline{D}}^{a} \frac{dt^{H\delta\bar{y}}_{\underline{t}ak}}{dD} dD \\ &\geq \underline{t} + \frac{(1-\delta)k_0}{k_4\bar{y}} \int_{D=\underline{D}}^{\bar{y}} \left(\frac{\bar{y}-D}{D}\right) dD = \underline{t} + \frac{(1-\delta)k_0}{k_4} F\left(\frac{\underline{D}}{\bar{y}}\right), \end{aligned}$$

where $F(x)$ denotes $x - \ln x - 1$, which is finite and differentiable for all finite $x > 0$. F is decreasing in $x \in (0,1)$: $F'(x) = 1 - 1/x < 0$. Thus, $F\left(\frac{\underline{D}}{\bar{y}}\right) < \frac{k_4}{(1-\delta)k_0}(\bar{t} - \underline{t})$, whence $\underline{D} > \bar{y}F^{-1}\left(\frac{k_4}{(1-\delta)k_0}(\bar{t}-\underline{t})\right)$. Moreover, $F(1) = 0$ and $\lim_{x\downarrow 0} F(x) = \infty$, so the inverse F^{-1} is decreasing in $x \in (0,\infty)$ and satisfies $F^{-1}(0) = 1$ and $\lim_{x\to\infty} F^{-1}(x) = 0$. Hence, as $\bar{t} - \underline{t} > 0$, $\underline{D} > 0$. Q.E.D.$_{\text{Lemma 6}}$

Part 2. Suppose not. Since $D^{\widehat{H}\widehat{\delta}\bar{y}}_{\underline{t}ak}(\underline{t}) = a \geq a' = D^{\widetilde{H}\widetilde{\delta}\bar{y}}_{\underline{t}a'k'}(\underline{t})$ and both solutions are continuous in t, there must be $\underline{t} \leq t_0 < t_1 \leq 1$ such that $D^{\widehat{H}\widehat{\delta}\bar{y}}_{\underline{t}ak}(t_0) = D^{\widetilde{H}\widetilde{\delta}\bar{y}}_{\underline{t}a'k'}(t_0)$ and, for all $t \in (t_0, t_1]$, $D^{\widehat{H}\widehat{\delta}\bar{y}}_{\underline{t}ak}(t) < D^{\widetilde{H}\widetilde{\delta}\bar{y}}_{\underline{t}a'k'}(t)$. Hence, by (54),

$$0 < D^{\widetilde{H}\widetilde{\delta}\bar{y}}_{\underline{t}a'k'}(t_1) - D^{\widehat{H}\widehat{\delta}\bar{y}}_{\underline{t}ak}(t_1) = \int_{t=t_0}^{t_1} \left[\begin{array}{c} \max\left\{ f^{\widetilde{H}\widetilde{\delta}\bar{y}}\left(D^{\widetilde{H}\widetilde{\delta}\bar{y}}_{\underline{t}a'k'}(t), t\right), -k' \right\} \\ -\max\left\{ f^{\widehat{H}\widehat{\delta}\bar{y}}\left(D^{\widehat{H}\widehat{\delta}\bar{y}}_{\underline{t}ak}(t), t\right), -k \right\} \end{array} \right] dt$$

which is impossible: since $f^{H\delta\bar{y}}(D,t)$ is decreasing in D and $k' \geq k$, the integrand is nonpositive for all t in $(t_0, t_1]$.

Part 3 follows from (48) and the fact that $D^{H\delta\bar{y}}_{\underline{t}ak} \leq a$.

Part 4. By part 2, $\Gamma_a(t) \overset{d}{=} D^{H\delta\bar{y}}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}}_{\underline{t}a\infty}(t) \geq 0$. By (24),

$$\Gamma'_a(t) = \max\left\{ \begin{array}{c} \left[f^{H\delta\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}\bar{y}k_a}(t), t\right) - f^{H\delta\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a\infty}(t), t\right)\right], \\ -k_a - f^{H\delta\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a\infty}(t), t\right) \end{array} \right\}.$$

Both entries in the max are nonpositive by (48), (24), and the fact that $D^{H\delta\bar{y}}_{\underline{t}a\infty}(t) \leq a$. Accordingly, for all $t \in [\underline{t}, 1]$, $\left| D^{H\delta\bar{y}}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}}_{\underline{t}a\infty}(t) \right| \leq \Gamma_a(\underline{t}) = \bar{y} - a$. Q.E.D.$_{\text{Claim 5}}$

Before addressing the discrete case, we prove some useful bounds:

CLAIM 7 *Let $w, w', \zeta e, \zeta e'$ be in $(0, \bar{y}]$ and satisfy $w' \geq w$, $\zeta e' \geq \zeta e$, $w > \zeta e$, and $w' > \zeta e'$. Then*

$$0 \leq \Delta_2^{H\bar{y}i}\left(\zeta e', t\right) - \Delta_2^{H\bar{y}i}\left(\zeta e, t\right) \leq k_1\left(\zeta e' - \zeta e\right). \tag{57}$$

Moreover, if $\min\left\{w, w', \zeta, \zeta'\right\} > \bar{y}\Delta_i'$ *then*

$$0 \geq \Delta_1^{H\bar{y}i}\left(\zeta e', w', t\right) - \Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right) \geq -k_1\left[\frac{\max\left\{w' - w, \zeta e' - \zeta e\right\}}{\bar{y}} + \Delta_i'\right] \tag{58}$$

and

$$\frac{\Delta_2^{H\bar{y}i}\left(\zeta e, t\right)}{\Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right)} \in \left(0, \frac{k_1\zeta e}{k_0\left(1 - \Delta_i'\left\lceil\frac{w}{\bar{y}\Delta_i'}\right\rceil + \Delta_i'\right)}\right) \subset (0, \infty). \tag{59}$$

PROOF OF CLAIM 7. By (26), for any $\zeta e \in [0, \bar{y}]$ and $t \in S_i \backslash \{1\}$,

$$\begin{aligned} & v^{H\bar{y}i}\left(\zeta e, t + \Delta_i\right) - v^{H\bar{y}i}\left(\zeta e, t\right) \\ = \;& \bar{y}\int_{z=0}^{\frac{\zeta e}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right) - H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t + \Delta_i\right)\right]dz. \end{aligned} \tag{60}$$

As the integrand is nonnegative, (60) is nondecreasing in ζe, so $\Delta_2^{H\bar{y}i}\left(\zeta', t\right) - \Delta_2^{H\bar{y}i}\left(\zeta, t\right) \geq 0$. By L-$H$, for any $z \in [0, \bar{y}]$, $H(z|t) - H(z|t + \Delta_i) < k_1\Delta_i$. Equation (57) then follows from (36) and (60).

By (26),

$$\begin{aligned} \Delta_1^{H\bar{y}i}\left(\zeta e', w', t\right) - \Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right) &= \frac{v^{H\bar{y}i}\left(\zeta e', t\right) - v^{H\bar{y}i}\left(w', t\right)}{\zeta e' - w'} - \frac{v^{H\bar{y}i}\left(\zeta e, t\right) - v^{H\bar{y}i}\left(w, t\right)}{\zeta e - w} \\ = \;\frac{\bar{y}}{w - \zeta e}\int_{z=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right)dz &- \frac{\bar{y}}{w' - \zeta e'}\int_{z=\frac{\zeta e'}{\bar{y}}}^{\frac{w'}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right)dz. \end{aligned}$$

Define the change of variables $z' = \frac{\zeta}{\bar{y}} + \left(\frac{w - \zeta}{w' - \zeta'}\right)\left(z - \frac{\zeta'}{\bar{y}}\right)$. When $z = \frac{\zeta'}{\bar{y}}$, $z' = \frac{\zeta}{\bar{y}}$, and when $z = \frac{w'}{\bar{y}}$, $z' = \frac{w}{\bar{y}}$. Moreover, $dz = \frac{w' - \zeta'}{w - \zeta}dz'$ and $z = \frac{\zeta'}{\bar{y}} + \left(\frac{w' - \zeta'}{w - \zeta}\right)\left(z' - \frac{\zeta}{\bar{y}}\right)$ which we denote $\psi(z')$. So $\int_{z=\frac{\zeta'}{\bar{y}}}^{\frac{w'}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right)dz = \int_{z'=\frac{\zeta}{\bar{y}}}^{\frac{w}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{\psi(z')}{\Delta_i'}\right\rfloor|t\right)\frac{w' - \zeta'}{w - \zeta}dz'$. Renaming z' to z and simplifying,

$$\Delta_1^{H\bar{y}i}\left(\zeta e', w', t\right) - \Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right) = \frac{\bar{y}}{w - \zeta e}\int_{z=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right) - H\left(\Delta_i'\left\lfloor\frac{\psi(z)}{\Delta_i'}\right\rfloor|t\right)\right]dz. \tag{61}$$

We can write

$$z-\psi(z)=\frac{1}{w-\zeta e}\left(\left(z-\frac{\zeta e'}{\bar{y}}\right)[w-\zeta e]-\left(z-\frac{\zeta e}{\bar{y}}\right)\left[w'-\zeta e'\right]\right). \tag{62}$$

As the right hand side is linear in z, it reaches its maximum and minimum at the endpoints of the interval of integration. At the lower endpoint (at $z=\frac{\zeta}{\bar{y}}$), the right hand side of (62) equals $\frac{\zeta-\zeta'}{\bar{y}}$, while at the upper endpoint (at $z=\frac{w}{\bar{y}}$), it equals $\frac{w-w'}{\bar{y}}$. Thus, $-\overline{w}\leq z-\psi(z)\leq-\underline{w}$ where $\underline{w}=\bar{y}^{-1}\min\left\{w'-w,\zeta'-\zeta\right\}$ and $\overline{w}=\bar{y}^{-1}\max\left\{w'-w,\zeta'-\zeta\right\}$. As $\underline{w}$ and $\overline{w}$ are both nonnegative, $z\leq\psi(z)$, which by (61) establishes the first inequality in (58). Finally, by (61) and L-H,

$$\begin{aligned}
&\Delta_1^{H\bar{y}i}\left(\zeta e',w',t\right)-\Delta_1^{H\bar{y}i}\left(\zeta e,w,t\right)\\
\geq\;&\frac{\bar{y}}{w-\zeta e}\int_{z'=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{\psi(z)-\overline{w}}{\Delta_i'}\right\rfloor|t\right)-H\left(\Delta_i'\left\lfloor\frac{\psi(z)}{\Delta_i'}\right\rfloor|t\right)\right]dz\\
\geq\;&\frac{\bar{y}k_1}{w-\zeta e}\int_{z'=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[\Delta_i'\left\lfloor\frac{\psi(z)-\overline{w}}{\Delta_i'}\right\rfloor-\Delta_i'\left\lfloor\frac{\psi(z)}{\Delta_i'}\right\rfloor\right]dz\\
\geq\;&-\frac{\bar{y}k_1}{w-\zeta e}\int_{z'=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[\overline{w}+\Delta_i'\right]dz=-k_1\left[\overline{w}+\Delta_i'\right].
\end{aligned}$$

This establishes the second inequality in (58). Finally, (59) follows from (37) and (38). Q.E.D.$_{\text{Claim 7}}$

CLAIM 8 *Fix H, $\bar{y}$, and i. Define*

$$\phi(D,t)=v^{H\bar{y}i}(D,t+\Delta_i)-\delta v^{H\bar{y}i}(D,t). \tag{63}$$

For any $D\in(\bar{y}\Delta_i',\bar{y}]$ and any $t\in S_i$ there exists a unique solution $D^*=D^*(D)$, which lies in $(\bar{y}\Delta_i',D)$, to

$$\phi(D^*,t)=(1-\delta)v^{H\bar{y}i}(D,t). \tag{64}$$

Moreover, $D^*(D)$ is increasing in D and $D-D^*(D)$ is nondecreasing in D. Finally,

$$\frac{\partial\phi}{\partial D}\geq k_0(1-\delta)\left(1-\frac{D}{\bar{y}}\right). \tag{65}$$

PROOF OF CLAIM 8. We first show three properties.

1. $\phi(\bar{y}\Delta_i', t) < (1-\delta) v^{H\bar{y}i}(D,t)$. Proof: by (26), for all t in S_i, $v^{H\bar{y}i}(\bar{y}\Delta_i', t) = \bar{y}\Delta_i'$. Hence, $\phi(\bar{y}\Delta_i', t) = (1-\delta)\bar{y}\Delta_i'$. Moreover, $v^{H\bar{y}i}(D,t)$ is strictly increasing in $D \in [0,\bar{y}]$ by part 2 of Claim 3, so $v^{H\bar{y}i}(D,t) > \bar{y}\Delta_i'$. The result then follows from (63) and (64).

2. $\phi(D,t) > (1-\delta) v^{H\bar{y}i}(D,t)$. Proof: for $D \in (\bar{y}\Delta_i', \bar{y}]$, $v^{H\bar{y}i}(D,t)$ is increasing in t by part 1 of Claim 3. The result then follows from (63) and (64).

3. ϕ is continuous and increasing in $D \in [0,\bar{y}]$. Proof: $v^{H\bar{y}i}(D,t)$ is continuous in D by (20), so $\phi(D,t)$ also is continuous in D. By (26),

$$\phi(D,t) = (1-\delta) D - \bar{y}\left[\int_{z=0}^{\frac{D}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor |t+\Delta_i\right) - \delta H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor |t\right)\right] dz\right].$$

By L-H, $1-H(z|t) > k_0(1-z)$ for all $z \in [0,1]$ (and thus, substituting $z=0$, $k_0 < 1$), so letting $z_0 = \Delta_i'\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor \le \frac{D}{\bar{y}}$,

$$\begin{aligned}\frac{\partial\phi}{\partial D} &= 1-\delta - [H(z_0|t+\Delta_i) - \delta H(z_0|t)] \ge (1-\delta)[1-H(z_0|t)] \\ &\ge (1-\delta) k_0 (1-z_0) \ge k_0(1-\delta)\left(1-\frac{D}{\bar{y}}\right).\end{aligned}$$

This establishes the result as well as equation (65).

Facts 1-3 imply that for any $D \in (\bar{y}\Delta_i', \bar{y}]$, there exists a unique $D^* = D^*(D)$ satisfying (64), and that it lies in $(\bar{y}\Delta_i', D)$. Moreover, $D^*(D)$ is increasing in D.

Finally, let $\widehat{D}_0 > D$ and let $\widehat{D}^* = D^*\left(\widehat{D}_0\right)$. To show that $D - D^*(D)$ is nondecreasing in D, we must show that $\widehat{D}_0 - \widehat{D}^* \ge D - D^*$ or, equivalently, that $\widehat{D}_0 - D \ge \widehat{D}^* - D^*$. By (63),

$$\phi(D,t) = v^{H\bar{y}i}(D, t+\Delta_i) - \delta v^{H\bar{y}i}(D,t) = (1-\delta) v^{H\bar{y}i}(D,t) + v^{H\bar{y}i}(D,t+\Delta_i) - v^{H\bar{y}i}(D,t).$$

Hence, by (36), (57), and (64), and since $\widehat{D}^* > D^*$,

$$\begin{aligned}(1-\delta)\left[v^{H\bar{y}i}\left(\widehat{D}_0, t\right) - v^{H\bar{y}i}(D,t)\right] &= \phi\left(\widehat{D}^*, t\right) - \phi(D^*, t) \\ &\ge (1-\delta)\left[v^{H\bar{y}i}\left(\widehat{D}^*, t\right) - v^{H\bar{y}i}(D^*, t)\right],\end{aligned}$$

whence by part 2 of Claim 3,

$$\frac{\widehat{D}^* - D^*}{\widehat{D}_0 - D} \leq \frac{\frac{v^{H\bar{y}i}\left(\widehat{D}_0,t\right)-v^{H\bar{y}i}(D,t)}{\widehat{D}_0-D}}{\frac{v^{H\bar{y}i}\left(\widehat{D}^*,t\right)-v^{H\bar{y}i}(D^*,t)}{\widehat{D}^*-D^*}},$$

which is in $(0,1]$ by (35) and (58). Thus, $\widehat{D}^* - D^* \leq \widehat{D}_0 - D$ as claimed. Q.E.D.$_{\text{Claim 8}}$

For any real number ℓ, let $(\ell,\infty]$ and $[\ell,\infty]$ denote the sets $(\ell,\infty)\cup\{\infty\}$ and $[\ell,\infty)\cup\{\infty\}$, respectively. For any constants $\underline{t} \in S_i$, $a \in (0,\bar{y}]$, and $k \in (0,\infty]$, consider the following initial value problem, where $S_i^{\underline{t}}$ denotes the set of types $t \geq \underline{t}$ in S_i:

DISCRETE INITIAL VALUE PROBLEM WITH PARAMETERS $H,\bar{y},\underline{t},a,k$ ($\mathrm{DP}_{\underline{t}ak}^{H\delta\bar{y}i}$). The condition

$$D_{\underline{t}ak}^{H\delta\bar{y}i}(t+\Delta_i) = \max\left\{D_{\underline{t}ak}^{H\delta\bar{y}i*}(t+\Delta_i), D_{\underline{t}ak}^{H\delta\bar{y}i}(t) - k\Delta_i\right\} \tag{66}$$

with $D_{\underline{t}ak}^{H\delta\bar{y}i} : S_i^{\underline{t}} \to \Re$, where $D_{\underline{t}ak}^{H\delta\bar{y}i*}(t+\Delta_i)$ is the (by Claim 8) unique solution $D^* \in \left(\bar{y}\Delta_i', D_{\underline{t}ak}^{H\delta\bar{y}i}(t)\right)$ to

$$v^{H\bar{y}i}(D^*,t+\Delta_i) - \delta v^{H\bar{y}i}(D^*,t) = (1-\delta)\, v^{H\bar{y}i}\left(D_{\underline{t}ak}^{H\bar{y}i}(t),t\right), \tag{67}$$

together with the initial value $D_{\underline{t}ak}^{H\delta\bar{y}i}(\underline{t}) = a > \bar{y}\Delta_i'$.

Clearly, any $D_{0\bar{y}\infty}^{H\delta\bar{y}i}$ that solves $\mathrm{DP}_{0\bar{y}\infty}^{H\delta\bar{y}i}$ must also be a solution $D^{H\delta\bar{y}i}$ to $\mathrm{DP}^{H\delta\bar{y}i}$ and vice-versa.

CLAIM 9 *For any* $\underline{t} \in S_i$, $a \in (\bar{y}\Delta_i', \bar{y}]$, *and* $k \in (0,\infty]$*:*

1. *There exists a unique solution* $D_{\underline{t}ak}^{H\delta\bar{y}i}$ *to* $DP_{\underline{t}ak}^{H\delta\bar{y}i}$. *This function is decreasing in* $t \in S_i^{\underline{t}}$ *and takes values in* $(\bar{y}\Delta_i', a]$.

2. *Let* $a' \in (\bar{y}\Delta_i', a]$ *and* $k' \in [k,\infty]$. *Then* $D_{\underline{t}ak}^{H\delta\bar{y}i}(t) \geq D_{\underline{t}a'k'}^{H\delta\bar{y}i}(t)$ *for all types* $t \in S_i^{\underline{t}}$.

3. *Let* $k_a^i = \frac{k_1\bar{y}(\bar{y}+a)}{k_0(1-\delta)\left(\bar{y}\left(1-\Delta_i'\right)-a\right)}$. *For all types* $t \in S_i^{\underline{t}}$, $0 > D_{\underline{t}ak}^{H\delta\bar{y}i*}(t+\Delta_i) - D_{\underline{t}ak}^{H\delta\bar{y}i}(t) \geq -\Delta_i k_a$ *and hence* $0 > D_{\underline{t}ak}^{H\delta\bar{y}i}(t+\Delta_i) - D_{\underline{t}ak}^{H\delta\bar{y}i}(t) \geq -\Delta_i \min\{k,k_a\}$.

4. *For all* $t \in S_i^{\underline{t}}$, $D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t) \in [0, \bar{y} - a]$.

PROOF OF CLAIM 9. Part 1. Follows from Claim 8.

Part 2. Clearly, $D^{H\delta\bar{y}i}_{\underline{t}ak}(\underline{t}) - D^{H\delta\bar{y}i}_{\underline{t}a'k'}(\underline{t}) = a - a' \geq 0$. And if, for some $t \in S_i^{\underline{t}} \setminus \{1\}$, $D^{H\delta\bar{y}i}_{\underline{t}ak}(t) \geq D^{H\delta\bar{y}i}_{\underline{t}a'k'}(t)$, then $D^{H\delta\bar{y}i}_{\underline{t}ak}(t) - k\Delta_i \geq D^{H\delta\bar{y}i}_{\underline{t}a'k'}(t) - k'\Delta_i$ and, by Claim 8, $D^{H\delta\bar{y}i*}_{\underline{t}ak}(t+\Delta_i) \geq D^{H\delta\bar{y}i*}_{\underline{t}a'k'}(t+\Delta_i)$, so $D^{H\delta\bar{y}i}_{\underline{t}ak}(t+\Delta_i) \geq D^{H\delta\bar{y}i}_{\underline{t}a'k'}(t+\Delta_i)$.

Part 3. Let $D' = D^{H\delta\bar{y}i}_{\underline{t}ak}(t)$, $D'' = D^{H\delta\bar{y}i}_{\underline{t}ak}(t+\Delta_i)$, and $D^* = D^{H\delta\bar{y}i*}_{\underline{t}ak}(t+\Delta_i)$. By Claim 8, $D^* - D' < 0$ and $\min\{D', D'', D^*\} > \bar{y}\Delta_i'$. We will show that $D^* - D' \geq -\Delta_i k_a$ which, by (66), implies $0 > D'' - D' \geq -\Delta_i \min\{k, k_a\}$. The result is trivial when $a = \bar{y}$ since $k_{\bar{y}} = \infty$. Suppose $a < \bar{y}$. By (36), (38), (63), and (64),

$$
\begin{aligned}
\phi\left(D', t\right) - \phi\left(D^*, t\right) &= \phi\left(D', t\right) - (1-\delta)\, v^{H\bar{y}i}\left(D', t\right) \\
&= v^{H\bar{y}i}\left(D', t+\Delta_i\right) - v^{H\bar{y}i}\left(D', t\right) \leq k_1 D' \Delta_i \leq k_1 a \Delta_i.
\end{aligned}
$$

But by (65), $\phi(D', t) - \phi(D^*, t) \geq [D' - D^*]\, k_0 (1-\delta)\left(1 - \frac{a}{\bar{y}}\right)$. Combining the two inequalities and using (55) yields the result.

Part 4. Fix $\underline{t} \in [0,1]$ and $a \in (0, \bar{y})$, whence $k_a \in (0, \infty)$. By part 2, $D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) \geq D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)$ for all types $t \in S_i^{\underline{t}}$. Let $\Gamma_a(t) = D^{H\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t) \geq 0$. As $D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t+\Delta_i) = D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i)$,

$$
\Gamma_a(t+\Delta_i) - \Gamma_a(t) = \max\left\{
\begin{array}{c}
D^{H\delta\bar{y}i*}_{\underline{t}\bar{y}k_a}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - \left[D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right], \\
-k_a\Delta_i - \left[D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right]
\end{array}
\right\}
$$

for any $t < 1$ in $S_i^{\underline{t}}$ by (66). By Claim 8, $D' - D^*(D')$ is nondecreasing in D', so

$$
D^{H\delta\bar{y}i*}_{\underline{t}\bar{y}k_a}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - \left[D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right] \leq 0,
$$

and by part 3, $D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t) \geq -\Delta_i k_a$. Hence, $\Gamma_a(t+\Delta_i) \in [0, \Gamma_a(t)]$, so for all $t \in S_i^{\underline{t}}$, $\left|D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right| \leq \Gamma_a(\underline{t}) = \bar{y} - a$. Q.E.D.$_{\text{Claim 9}}$

As already noted, we extend any function defined on S_i to any $t \in [0,1]$ by evaluating it at $\tau_t^i = \Delta_i \lfloor t/\Delta_i \rfloor$.

CLAIM 10 *Fix* $\underline{t} \in [0,1)$*,* $a \in (\bar{y}\Delta_i', \bar{y}]$*, and* $k \in (0,\infty]$ *such that either* $a < \bar{y}$ *or* $k < \infty$ *(or both). For any* $a_0, a_1 \in (0,a]$ *and any type* $t \in [\underline{t},1]$*,* $D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)$ *converges to zero as* $i \to \infty$ *and* $|a_0 - a_1| \to 0$ *(in either order), uniformly in* $H \in \mathscr{H}$*,* $\bar{y} \in (0,\mathbf{y}]$*, and* t*.*

PROOF OF CLAIM 10. By (67), for any $t \in [0,1]$,

$$(1-\delta)\,\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t), \tau_t^i\right)\Delta D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t) + \Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), \tau_t^i\right) = 0 \tag{68}$$

where we define

$$\Delta D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t) = \frac{D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)}{\Delta_i}. \tag{69}$$

Equation (68) can be rewritten

$$\Delta D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t) = -\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t), \tau_t^i\right)},$$

whence $D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i) = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - \frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i),\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau_t^i\right)}\Delta_i$ and thus, by (66),

$$D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i) = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - \Delta_i \min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t), \tau_t^i\right)}, k\right\},$$

and so

$$\begin{aligned}\Delta D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) &\stackrel{d}{=} \frac{D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)}{\Delta_i} \\ &= -\min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i), D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t), \tau_t^i\right)}, k\right\}.\end{aligned}$$

LEMMA 11 *In the above formula for* $\Delta D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)$*, we may replace* $D^{H\delta\bar{y}i*}_{\underline{t}a_0k}$ *by* $D^{H\delta\bar{y}i}_{\underline{t}a_0k}$*. That is,*

$$\Delta D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) = -\min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i), D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t), \tau_t^i\right)}, k\right\}. \tag{70}$$

PROOF OF LEMMA 11. Let $w = w' = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)$. Let $\zeta e = D^{H\delta\bar{y}i*}_{\underline{t}a_0k}(t+\Delta_i)$ and $\zeta e' = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i)$. By Claim 8 and part 1 of Claim 9, $\min\{w,\zeta,w',\zeta'\} > \bar{y}\Delta_i'$. Hence, by Claim 7, $\Delta_1^{H\bar{y}i}(\zeta',w',\tau_t^i) \le \Delta_1^{H\bar{y}i}(\zeta,w,\tau_t^i)$ and $\Delta_2^{H\bar{y}i}(\zeta',\tau_t^i) \ge \Delta_2^{H\bar{y}i}(\zeta,\tau_t^i)$. Thus,

$$\frac{\Delta_2^{H\bar{y}i}(\zeta e',\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta e',w',\tau_t^i)} \ge \frac{\Delta_2^{H\bar{y}i}(\zeta e,\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta e,w,\tau_t^i)}, \tag{71}$$

with equality when $\zeta e' = \zeta e$. Accordingly, there are two cases. If $\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}(\zeta,\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta,w,\tau_t^i)} \le k$, then $\zeta e' = \zeta e$, which implies (70). If $\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}(\zeta,\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta,w,\tau_t^i)} \ge k$, then $\Delta D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) = -k$ but by (71), $\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}(\zeta',\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta',w',\tau_t^i)} \ge k$ as well, so (70) holds. Q.E.D.$_{\text{Lemma 11}}$

For all $t' \in [\underline{t},1]$, since $D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\underline{t}) = a_0$,

$$\begin{aligned}
D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t') &= a_0 + \Delta_i \sum_{t\in S_i^{\underline{t}},\, t<\tau^i_{t'}} \Delta D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) \\
&= a_0 - \Delta_i \sum_{t\in S_i^{\underline{t}},\, t<\tau^i_{t'}} \min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),t\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),t\right)},k\right\} \\
&= a_0 - \int_{t=\underline{t}}^{\tau^i_{t'}} \min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau_t^i\right)},k\right\}dt, \quad (72)
\end{aligned}$$

By (54) and (24), for all $t' \in [\underline{t},1]$,

$$D^{H\delta\bar{y}}_{\underline{t}a_1k}(t') = a_1 - \int_{t=\underline{t}}^{t'} \min\left\{\frac{1}{1-\delta}\frac{v_t\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),t\right)}{v_D\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),t\right)},k\right\}dt. \tag{73}$$

Define the analogous quantities to $\Delta_1^{H\bar{y}i}$ and $\Delta_2^{H\bar{y}i}$ with $v^{H\bar{y}i}$ replaced by $v^{H\bar{y}}$: for any $D',D'' \in [0,\bar{y}]$, let

$$\begin{aligned}
\Delta_1^{H\bar{y}}(D',D'',t) &= \frac{v^{H\bar{y}}(D',t) - v^{H\bar{y}}(D'',t)}{D'-D''} \\
&\in \left(k_0\left(1-\frac{\max\{D',D''\}}{\bar{y}}\right),k_1\left(1-\frac{\min\{D',D''\}}{\bar{y}}\right)\right) \text{ and} \quad (74) \\
\Delta_2^{H\bar{y}}(D',t,\Delta_i) &= \frac{v^{H\bar{y}}(D',t+\Delta_i) - v^{H\bar{y}}(D',t)}{\Delta_i} \in \left(\frac{k_0(D')^2[3\bar{y}-2D']}{6\bar{y}^2},k_1D'\right), \quad (75)
\end{aligned}$$

where the bounds follow from (44) and (45) and imply that

$$\frac{\Delta_2^{H\bar{y}}\left(D',t,\Delta_i\right)}{\Delta_1^{H\bar{y}}\left(D',D'',t\right)} \in \left(\frac{k_0\left(D'\right)^2\left[3\bar{y}-2D'\right]}{6\bar{y}k_1\left(\bar{y}-\min\left\{D',D''\right\}\right)}, \frac{\bar{y}k_1D'}{k_0\left(\bar{y}-\max\left\{D',D''\right\}\right)}\right). \tag{76}$$

If $a<\bar{y}$ then by (59), (76), (48), and (55), for all $D,D' \in (\bar{y}\Delta_i', a]$ and all $t\in[0,1]$, the ratios $\frac{\Delta_2^{H\bar{y}i}\left(D',\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D',D'',\tau_t^i\right)}$, $\frac{\Delta_2^{H\bar{y}}\left(D',t,\Delta_i\right)}{\Delta_1^{H\bar{y}}\left(D',t,\tau_t^i\right)}$, $\frac{v_t^{H\bar{y}}\left(D',\tau_t^i\right)}{v_D^{H\bar{y}}\left(D',\tau_t^i\right)}$, and $\frac{v_t^{H\bar{y}}\left(D',t\right)}{v_D^{H\bar{y}}\left(D',t\right)}$ are all at most $(1-\delta)k_a$. Let

$$\kappa = (1-\delta)\min\{k,k_a\} < \infty. \tag{77}$$

It follows that by (72), (73) and the triangle inequality that for all $t'\in[\underline{t},1]$,

$$\begin{aligned}(1-\delta)\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t'\right)-D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(t'\right)\right| &= \left|\begin{array}{l}\int_{t=\underline{t}}^{\tau_{t'}^i}\min\left\{\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t),\tau_t^i\right)},\kappa\right\}dt\\ -\int_{t=\underline{t}}^{t'}\min\left\{\frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t),t\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t),t\right)},\kappa\right\}dt+a_1-a_0\end{array}\right|\\ &\le |a_1-a_0|+A_1+A_2+A_3+A_4+A_5+A_6\end{aligned}$$

where

$$A_1 = \left|\int_{t=\tau_{t'}^i}^{t'}\min\left\{\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right),\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right),D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right),\tau_t^i\right)},\kappa\right\}dt\right|,$$

$$A_2 = \int_{t=\underline{t}}^{t'}\left|\begin{array}{c}\min\left\{\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t),\tau_t^i\right)},\kappa\right\}\\ -\min\left\{\frac{\Delta_2^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),\tau_t^i,\Delta_i\right)}{\Delta_1^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t),\tau_t^i\right)},\kappa\right\}\end{array}\right|dt,$$

$$A_3 = \int_{t=\underline{t}}^{t'}\left|\begin{array}{c}\min\left\{\frac{\Delta_2^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),\tau_t^i,\Delta_i\right)}{\Delta_1^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t),\tau_t^i\right)},\kappa\right\}\\ -\min\left\{\frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),\tau_t^i\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t),\tau_t^i\right)},\kappa\right\}\end{array}\right|dt,$$

$$A_4 = \int_{t=\underline{t}}^{t'}\left|\min\left\{\frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right),\tau_t^i\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right),\tau_t^i\right)},\kappa\right\}-\min\left\{\frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right),\tau_t^i\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right),\tau_t^i\right)},\kappa\right\}\right|dt,$$

$$A_5 = \int_{t=\underline{t}}^{t'}\left|\min\left\{\frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right),\tau_t^i\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right),\tau_t^i\right)},\kappa\right\}-\min\left\{\frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(t\right),\tau_t^i\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(t\right),\tau_t^i\right)},\kappa\right\}\right|dt,$$

and

$$A_6 = \int_{t=\underline{t}}^{t'} \left| \min\left\{ \frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), \tau_t^i\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), \tau_t^i\right)}, \kappa \right\} - \min\left\{ \frac{v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t\right)}{v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t\right)}, \kappa \right\} \right| dt.$$

Clearly, $A_1 \leq \kappa\Delta_i$. For A_2, A_3, A_4, and A_5, we require the following claim.

LEMMA 12 *For any* $a, b, c, d \geq 0$, $\max\{|a-b|, |c-d|\}$ *is an upper bound on both*

$$|\min\{a, c\} - \min\{b, d\}|$$

and $|\max\{a, c\} - \max\{b, d\}|$.

PROOF OF LEMMA 12. We prove the result for $|\min\{a, c\} - \min\{b, d\}|$; the proof for $|\max\{a, c\} - \max\{b, d\}|$ is identical with the keyword min replaced max throughout. First, assume $a \geq b$ and $c \geq d$. Then $|\min\{a, c\} - \min\{b, d\}| = \min\{a, c\} - \min\{b, d\}$. And $\max\{|a-b|, |c-d|\} = \max\{a-b, c-d\}$. But

$$\min\{a, c\} - \min\{b, d\} \leq \max\{a-b, c-d\}$$

since

$$\begin{aligned} \min\{a, c\} &\leq \min\{\max\{a, b+c-d\}, \max\{c, d+a-b\}\} \\ &= \min\{b + \max\{a-b, c-d\}, d + \max\{a-b, c-d\}\} \\ &= \min\{b, d\} + \max\{a-b, c-d\}. \end{aligned}$$

The other cases (in which $b > a$ or $c > d$ or both) are analogous. Q.E.D.$_{\text{Lemma 12}}$

This Lemma leads to the following useful bound.

LEMMA 13 *For any* $a, b, \kappa \in (0, \infty)$ *and* $c, d \in [0, \infty)$,

$$\left| \min\left\{ \frac{a}{c}, \kappa \right\} - \min\left\{ \frac{b}{d}, \kappa \right\} \right| \leq \left| \frac{a-b}{\max\{c, a/\kappa\}} \right| + \frac{b}{\max\{d, b/\kappa\}} \frac{\max\{|c-d|, |a-b|/\kappa\}}{\max\{c, a/\kappa\}}. \tag{78}$$

PROOF OF LEMMA 13. For any $a,b \geq 0$ and $c,d > 0$,

$$\left|\frac{a}{c}-\frac{b}{d}\right| = \left|\frac{ad-bc}{cd}\right| \leq \left|\frac{ad-bd}{cd}\right| + \left|\frac{bd-bc}{cd}\right| = \left|\frac{a-b}{c}\right| + \frac{b}{d}\left|\frac{d-c}{c}\right|. \tag{79}$$

Moreover, for any $a > 0$ and $c, \kappa \geq 0$,

$$\min\left\{\frac{a}{c},\kappa\right\} = \kappa \min\left\{\frac{a}{c\kappa},1\right\} = \kappa \min\left\{\frac{a}{c\kappa},\frac{a}{a}\right\} = \frac{a\kappa}{\max\{c\kappa,a\}} = \frac{a}{\max\{c,a/\kappa\}}. \tag{80}$$

By (79) and (80) and using Lemma (in that order),

$$\begin{aligned}\left|\min\left\{\frac{a}{c},\kappa\right\} - \min\left\{\frac{b}{d},\kappa\right\}\right| &= \left|\frac{a}{\max\{c,a/\kappa\}} - \frac{b}{\max\{d,b/\kappa\}}\right| \\ &\leq \left|\frac{a-b}{\max\{c,a/\kappa\}}\right| + \frac{b}{\max\{d,b/\kappa\}}\left|\frac{\max\{d,b/\kappa\}-\max\{c,a/\kappa\}}{\max\{c,a/\kappa\}}\right| \\ &\leq \left|\frac{a-b}{\max\{c,a/\kappa\}}\right| + \frac{b}{\max\{d,b/\kappa\}}\frac{\max\{|c-d|,|a-b|/\kappa\}}{\max\{c,a/\kappa\}}.\end{aligned}$$

Q.E.D.$_{\text{Lemma 13}}$

Let $D'_t = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i) \in (\bar{y}\Delta'_i, a_0)$, $D''_t = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) \in (\bar{y}\Delta'_i, a_0]$, $a_t = \Delta^{H\bar{y}i}_2\left(D'_t, \tau^i_t\right)$, $b_t = \Delta^{H\bar{y}}_2\left(D'_t, \tau^i_t, \Delta_i\right)$, $c_t = \Delta^{H\bar{y}i}_1\left(D'_t, D''_t, \tau^i_t\right)$, and $d_t = \Delta^{H\bar{y}}_1\left(D'_t, D''_t, \tau^i_t\right)$. By (78),

$$A_2 \leq \int_{t=\underline{t}}^{t'}\left|\frac{a_t-b_t}{\max\{c_t,a_t/\kappa\}}\right|dt + \int_{t=\underline{t}}^{t'}\left(\frac{b_t}{\max\{d_t,b_t/\kappa\}}\frac{\max\{|c_t-d_t|,|a_t-b_t|/\kappa\}}{\max\{c_t,a_t/\kappa\}}\right)dt,$$

By (35), (36), (74), (75), and parts 3 and 4 of Claim 3, for any $\varepsilon > 0$ there is an $i^* < \infty$ such that if $i > i^*$, $\max\{|a_t-b_t|,|c_t-d_t|\} < \varepsilon$. By (75), $b_t \in \left[\frac{k_0(D'_t)^2}{6\bar{y}}, k_1\bar{y}\right]$. By part 1 of Claim 3, $a_t > k_0\frac{(D')^2(3\bar{y}-2D')}{6\bar{y}^2} > k_0\frac{(D')^2}{6\bar{y}}$ where $D' = D'_t - 2\bar{y}\Delta'_i < \bar{y}$. Hence, $\min\{a_t,b_t\} > \frac{k_0\left(D'_t-2\bar{y}\Delta'_i\right)^2}{6\bar{y}}$. By (35), (74), and part 2 of Claim 3, $\min\{c_t,d_t\} \geq k_0\left(1-\frac{D''_t}{\bar{y}}\right)$ so since $D'_t \geq D''_t - k\Delta_i$, both $\max\{c_t,a_t/\kappa\}$ and $\max\{d_t,b_t/\kappa\}$ are at least

$$k_0\max\left\{1-\frac{D''_t}{\bar{y}},\frac{(D'_t-2\bar{y}\Delta'_i)^2}{6\bar{y}\kappa}\right\} \geq k_0\max\left\{1-\frac{D''_t}{\bar{y}},\frac{(D''_t-k\Delta_i-2\bar{y}\Delta'_i)^2}{6\bar{y}\kappa}\right\},$$

which is bounded below by a strictly positive constant κ' for large enough i as $\bar{y} > 0$. Collecting these bounds, $A_2 \leq \varepsilon\kappa_2[t'-\underline{t}]$ where $\kappa_2 = \frac{1}{\kappa'}\left[1+\frac{k_1\bar{y}}{\kappa'}\max\{1,1/\kappa\}\right] \in (0,\infty)$.

By (78), redefining $a_t = \Delta_2^{H\bar{y}}\left(D_t', \tau_t^i, \Delta_i\right)$, $b_t = v_t^{H\bar{y}}\left(D_t', \tau_t^i\right)$, $c_t = \Delta_1^{H\bar{y}}\left(D_t', D_t'', \tau_t^i\right)$, and $d_t = v_D^{H\bar{y}}\left(D_t'', \tau_t^i\right)$,

$$A_3 \le \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left(\frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}} \right) dt.$$

By (44), $b_t \le k_1 \bar{y}$. By the Mean Value Theorem, there is a $t \in \left[\tau_t^i, \tau_t^i + \Delta_i\right]$ such that $v_t^{H\bar{y}}\left(D_t', t\right) = a_t$. Thus, by (42),

$$|a_t - b_t| = \left| v_t^{H\bar{y}}\left(D_t', t\right) - b_t \right| = \left| v_t^{H\bar{y}}\left(D_t', t\right) - v_t^{H\bar{y}}\left(D_t', \tau_t^i\right) \right| \le k_2 \bar{y} \Delta_i.$$

Also by the Mean Value Theorem, there is a $D \in [D_t', D_t'']$ such that $v_D^{H\bar{y}}\left(D, \tau_t^i\right) = c_t$. By part 3 of Claim 9 and (77),

$$\left| D_t'' - D_t' \right| \le \Delta_i \min\{k, k_a\} = \frac{\kappa}{(1-\delta)} \Delta_i. \tag{81}$$

Hence, $|c_t - d_t| = \left| v_D^{H\bar{y}}\left(D, \tau_t^i\right) - v_D^{H\bar{y}}\left(D_t'', \tau_t^i\right) \right| \le \frac{k_1}{\bar{y}} \left| D_t'' - D_t' \right| \le \frac{k_1 \kappa}{\bar{y}(1-\delta)} \Delta_i$. By (44), (45), (74), and (75), $\min\{a_t, b_t\} \ge \frac{k_0 (D_t')^2}{6\bar{y}}$, and $\min\{c_t, d_t\} \ge k_0 \left(1 - \frac{D_t''}{\bar{y}}\right)$ so as shown in the prior paragraph, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds, $A_3 \le \Delta_i \kappa_3 [t' - \underline{t}]$ where $\kappa_3 = \frac{\bar{y}}{\kappa'} \left[k_2 + \frac{k_1}{\kappa'} \max\left\{ \frac{k_1 \kappa}{\bar{y}(1-\delta)}, \frac{k_2 \bar{y}}{\kappa} \right\} \right] \in (0, \infty)$.

By (78), redefining $a_t = v_t^{H\bar{y}}\left(D_t', \tau_t^i\right)$, $b_t = v_t^{H\bar{y}}\left(D_t'', \tau_t^i\right)$, and $c_t = d_t = v_D^{H\bar{y}}\left(D_t'', \tau_t^i\right)$,

$$A_4 \le \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left(\frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{|a_t - b_t|/\kappa}{\max\{c_t, a_t/\kappa\}} \right) dt,$$

By (44), $b_t \le k_1 \bar{y}$. By (47) and (81),

$$|a_t - b_t| = \left| v_t^{H\bar{y}}\left(D_t', \tau_t^i\right) - v_t^{H\bar{y}}\left(D_t'', \tau_t^i\right) \right| \le \frac{k_1 \kappa}{(1-\delta)\bar{y}} \Delta_i.$$

By (44) and (45), $\min\{a_t, b_t\} \ge \frac{k_0 (D_t')^2}{6\bar{y}}$, and $\min\{c_t, d_t\} \ge k_0 \left(1 - \frac{D_t''}{\bar{y}}\right)$ so as shown in the prior paragraph, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds, $A_4 \le \Delta_i \kappa_4 [t' - \underline{t}]$, where $\kappa_4 = \frac{k_1}{\kappa'(1-\delta)} \left[\frac{\kappa}{\bar{y}} + \frac{k_1}{\kappa'} \right] \in (0, \infty)$.

By (78), redefining $a_t = v_t^{H\bar{y}}\left(D_t'', \tau_t^i\right)$, $b_t = v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right), \tau_t^i\right)$, $c_t = v_D^{H\bar{y}}\left(D_t'', \tau_t^i\right)$, and $d_t = v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right), \tau_t^i\right)$,

$$A_5 \le \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left(\frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}} \right) dt.$$

By (44), $b_t \leq k_1\bar{y}$. By (42), $|a_t - b_t| \leq k_2\bar{y}\left|D_t'' - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right)\right|$. By (47),

$$|c_t - d_t| \leq \frac{k_1}{\bar{y}}\left|D_t'' - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right)\right|.$$

By (44), (45), $\min\{a_t, b_t\} \geq \frac{k_0(D_t')^2}{6\bar{y}}$ and $\min\{c_t, d_t\} \geq k_0\left(1 - \frac{D_t''}{\bar{y}}\right)$ so as shown above, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds and using $D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) = D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(\tau_t^i\right)$,

$$\begin{aligned}
A_5 &\leq \kappa_5\int_{t=\underline{t}}^{t'}\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(\tau_t^i\right) - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right)\right|dt \leq \kappa_5\left[t' - \underline{t}\right]\max_{t\in[\underline{t},t']}\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(\tau_t^i\right) - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right)\right| \\
&\leq \kappa_5\left[t' - \underline{t}\right]\max_{t\in[\underline{t},t']}\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t)\right|,
\end{aligned}$$

where $\kappa_5 = \frac{\bar{y}}{\kappa'}\left[k_2 + \frac{k_1}{\kappa'}\max\left\{\frac{k_1}{\bar{y}}, \frac{k_2\bar{y}}{\kappa}\right\}\right] \in (0, \infty)$.

Now redefine $a_t = v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), \tau_t^i\right)$, $b_t = v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t\right)$, $c_t = v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), \tau_t^i\right)$, and $d_t = v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t\right)$. By (78),

$$A_6 \leq \int_{t=\underline{t}}^{t'}\left|\frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}}\right|dt + \int_{t=\underline{t}}^{t'}\left(\frac{b_t}{\max\{d_t, b_t/\kappa\}}\frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}}\right)dt.$$

By (44), $b_t \leq k_1\bar{y}$. By (42), $|a_t - b_t| \leq k_2\bar{y}\Delta_i$. By (46), $|c_t - d_t| \leq k_1\Delta_i$. By (44) and (45), $\min\{a_t, b_t\} \geq \frac{k_0(D_t')^2}{6\bar{y}}$ and $\min\{c_t, d_t\} \geq k_0\left(1 - \frac{D_t''}{\bar{y}}\right)$ so as shown above, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds, $A_6 \leq \Delta_i\kappa_6[t' - \underline{t}]$ where $\kappa_6 = \frac{\bar{y}}{\kappa'}\left[k_2 + \frac{k_1}{\kappa'}\max\left\{k_1, \frac{k_2\bar{y}}{\kappa}\right\}\right] \in (0, \infty)$.

Summarizing our findings and since $\lim_{i\to\infty}\Delta_i = 0$ and $t' \in [\underline{t}, 1]$, for all $\varepsilon > 0$ there is an $i^* < \infty$ such that if $i > i^*$, then

$$(1 - \delta)\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t'\right) - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(t'\right)\right| \leq |a_1 - a_0| + \kappa''\varepsilon + \kappa_5\left[t' - \underline{t}\right]\max_{t\in[\underline{t},t']}\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t)\right|$$

where $\kappa'' = \kappa + (\kappa_2 + \kappa_3 + \kappa_4 + \kappa_6)[1 - \underline{t}]$. So for any $t'' \in [\underline{t}, t']$,

$$\begin{aligned}
&(1 - \delta)\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t''\right) - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(t''\right)\right| \\
\leq\ & |a_1 - a_0| + \kappa''\varepsilon + \kappa_5\left[t'' - \underline{t}\right]\max_{t\in[\underline{t},t'']}\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t)\right| \\
\leq\ & |a_1 - a_0| + \kappa''\varepsilon + \kappa_5\left[t' - \underline{t}\right]\max_{t\in[\underline{t},t']}\left|D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t)\right|
\end{aligned}$$

and therefore,

$$
\begin{aligned}
& (1-\delta)\max_{t\in[\underline{t},t']}\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)-D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)\right| \\
\leq\; & |a_1-a_0|+\kappa''\varepsilon+\kappa_5\left[t'-\underline{t}\right]\max_{t\in[\underline{t},t']}\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)-D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)\right|.
\end{aligned}
$$

Now for $t'\in[\underline{t},\underline{t}+b]$ where $b=\frac{1-\delta}{2\kappa_5}>0$, $(1-\delta)-\kappa_5[t'-\underline{t}]\geq\frac{1-\delta}{2}$, so

$$
\max_{t\in[\underline{t},t']}\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)-D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)\right|\leq\frac{2}{1-\delta}\left(|a_1-a_0|+\kappa''\varepsilon\right),
$$

whence $\max_{t\in[\underline{t},\underline{t}+b]}\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)-D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)\right|\leq\frac{2}{1-\delta}\left(|a_1-a_0|+\kappa''\varepsilon\right)$. In particular,

$$
\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\underline{t}+b)-D^{H\delta\bar{y}}_{\underline{t}a_1k}(\underline{t}+b)\right|\leq\frac{2}{1-\delta}\left(|a_1-a_0|+\kappa''\varepsilon\right).
$$

Let $a_2=D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\underline{t}+b)$ and $a_3=D^{H\delta\bar{y}}_{\underline{t}a_1k}(\underline{t}+b)$. Since $D^{H\delta\bar{y}i}_{\underline{t}a_0k}$ and $D^{H\delta\bar{y}}_{\underline{t}a_1k}$ are decreasing functions, $\max\{a_2,a_3\}<a$ so in the above reasoning we can use the same constant a and thus the same constant k_a and thus the same κ''. Accordingly,

$$
\begin{aligned}
\max_{t\in[\underline{t}+b,\underline{t}+2b]}\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)-D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)\right| &\leq \frac{2}{1-\delta}\left(|a_3-a_2|+\kappa''\varepsilon\right)\\
&\leq \frac{2}{1-\delta}\left(\frac{2}{1-\delta}\left(|a_1-a_0|+\kappa''\varepsilon\right)+\kappa''\varepsilon\right)\\
&= \left(\frac{2}{1-\delta}\right)^2|a_1-a_0|+\left[\frac{2}{1-\delta}+\left(\frac{2}{1-\delta}\right)^2\right]\kappa''\varepsilon.
\end{aligned}
$$

Iterating this reasoning $n=\left\lceil\frac{1-\underline{t}}{b}\right\rceil$ (which does not depend on i) times, we obtain

$$
\max_{t\in[\underline{t},1]}\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t)-D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)\right|\leq\left(\frac{2}{1-\delta}\right)^n|a_1-a_0|+\kappa''\varepsilon\sum_{i=1}^{n}\left[\left(\frac{2}{1-\delta}\right)^i\right].
$$

Since the constants multiplying $|a_1-a_0|$ and ε are independent of i, $t\in[\underline{t},1]$, $\bar{y}\in(0,\mathbf{y}]$, and $H\in\mathscr{H}$, the result follows. Q.E.D.$_{\text{Claim 10}}$

CLAIM 14 *For all* $\underline{t}\in[0,1)$, *there exist unique solutions* $D^{H\delta\bar{y}}_{\underline{t}\bar{y}\infty}$ *and* $D^{H\delta\bar{y}i}_{\underline{t}\bar{y}\infty}$ *to* $CP^{H\delta\bar{y}}_{\underline{t}\bar{y}\infty}$ *and* $DP^{H\delta\bar{y}i}_{\underline{t}\bar{y}\infty}$, *respectively. They are decreasing in t and, in the case of* $D^{H\delta\bar{y}}_{\underline{t}\bar{y}\infty}$, *continuous in* $t\in[\underline{t},1]$.

PROOF OF CLAIM 14. Define, for all $t \in [\underline{t}, 1]$,

$$D_{\underline{t}}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}}(t), \tag{82}$$

and for all $t \in S_i^{\underline{t}}$, $D_{\underline{t}}^{H\delta\bar{y}i}(t) = \lim_{a\uparrow\bar{y}} D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) = \lim_{a\uparrow\bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t)$. The four limits exist by part 2 of Claims 5 and 9 and the Monotone Convergence Theorem. The pair of limits that appears in each equation are equal by part 4 of Claims 5 and 9. Hence, $D_{\underline{t}}^{H\delta\bar{y}}$ and $D_{\underline{t}}^{H\delta\bar{y}i}$ exist and are unique. Moreover, by part 2 of Claims 5 and 9, any solutions $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ and $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ to $\mathrm{CP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ and $\mathrm{DP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ must satisfy $D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}} \geq D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}} \geq D_{\underline{t}a\infty}^{H\delta\bar{y}}$ and $D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i} \geq D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i} \geq D_{\underline{t}a\infty}^{H\delta\bar{y}i}$ for any $a \in (0, \bar{y}]$, and thus must coincide with $D_{\underline{t}}^{H\delta\bar{y}}$ and $D_{\underline{t}}^{H\delta\bar{y}i}$, respectively. It remains to show that $D_{\underline{t}}^{H\delta\bar{y}}$ and $D_{\underline{t}}^{H\delta\bar{y}i}$ solve $\mathrm{CP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ and $\mathrm{DP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$: that

$$D_{\underline{t}}^{H\delta\bar{y}}(\underline{t}) = \bar{y} \text{ and, for } t \in (\underline{t}, 1), \ \frac{dD_{\underline{t}}^{H\delta\bar{y}}(t)}{dt} = f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t), t\right) \tag{83}$$

and

$$D_{\underline{t}}^{H\delta\bar{y}i}(\underline{t}) = \bar{y} \text{ and, for } t \in S_i^{\underline{t}} \setminus \{1\}, D_{\underline{t}}^{H\delta\bar{y}i}(t + \Delta_i) = D_{\underline{t}}^{H\delta\bar{y}i*}(t + \Delta_i) \tag{84}$$

where $D_{\underline{t}}^{H\delta\bar{y}i*}(t+\Delta_i)$ is the (by Claim 8) unique solution $D^* \in \left(\bar{y}\Delta_i', D_{\underline{t}}^{H\delta\bar{y}i}(t)\right)$ to

$$v^{H\bar{y}i}(D^*, t + \Delta_i) - \delta v^{H\bar{y}i}(D^*, t) = (1-\delta) v^{H\bar{y}i}\left(D_{\underline{t}}^{H\delta\bar{y}i}(t), t\right). \tag{85}$$

First, $D_{\underline{t}}^{H\delta\bar{y}}(\underline{t}) = \lim_{a\uparrow\bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}}(\underline{t}) = \lim_{a\uparrow\bar{y}} a = \bar{y}$. Second, for $t \in (\underline{t}, 1)$, we must show that $dD_{\underline{t}}^{H\delta\bar{y}}(t)/dt = f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t), t\right)$. By (50) and (24), $f^{H\delta\bar{y}}(\zeta, t)$ is continuous in t so, by (54), $f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t), t\right)$ is the limit of the derivatives $dD_{\underline{t}a\infty}^{H\delta\bar{y}}(t)/dt$ as a goes to $\bar{y}$. We now invoke the following well known result.[4]

THEOREM 15 *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of real-valued functions on* $t \in [\underline{t}, 1]$. *Suppose that they converge uniformly to some function* f. *Assume also that the sequence of derivatives* $(f_n')_{n=1}^{\infty}$ *converges, uniformly in* $t \in [\underline{t}, 1]$, *to some continuous function. Then* f *is differentiable and* $\lim_{n\to\infty} f_n'(t) = f'(t)$ *for all* $t \in [\underline{t}, 1]$..

[4]See, e.g., Theorem 9.13 in Apostol (1981, p. 229).

Let $f_n = D^{H\delta\bar{y}}_{\underline{t}a_n\infty}$ where $(a_n)_{n=1}^{\infty}$ is an increasing sequence that converges to $\bar{y}$. By part 4 of Claim 5, $(f_n)_{n=1}^{\infty}$ converges to $f = D^{H\delta\bar{y}}_{\underline{t}}$, uniformly in $t \in [\underline{t}, 1]$. Fix $w \in (\underline{t}, 1)$. We will show that the sequence of derivatives $(f_n')_{n=1}^{\infty}$ converges, uniformly in $t \in [w, 1]$, to f'. The result then follows by taking $w \to \underline{t}$. By (48) and (24), letting $b = \frac{1}{1-\delta}\frac{k_0}{6k_1}$ and $x = f_n(w)$,

$$\begin{aligned} x &= f_n(\underline{t}) + \int_{t=\underline{t}}^{w} f_n'(t)\,dt = a_n + \int_{t=\underline{t}}^{w} f^{H\delta\bar{y}}(f_n(t), t)\,dt < a_n - \frac{b}{\bar{y}} \int_{t=\underline{t}}^{w} [f_n(t)]^2\,dt \\ &< a_n - \frac{b}{\bar{y}} \int_{t=\underline{t}}^{w} x^2 dt = a_n - \frac{b}{\bar{y}} x^2 (w - \underline{t}) \end{aligned}$$

so x lies below the higher root of $bx^2(w-\underline{t}) + \bar{y}x - \bar{y}a_n = 0$, which (since $a_n \le \bar{y}$) is at most $\frac{-\bar{y}+\sqrt{\bar{y}^2+4b(w-\underline{t})\bar{y}^2}}{2b(w-\underline{t})}$. If we set this equal to $\bar{y}(1-c)$ and solve for c, we obtain $\frac{c}{(1-c)^2} = b(w-\underline{t})$. Hence, for all $t \in [w, 1]$,

$$\frac{f_n(t)}{\bar{y}} \le \frac{f_n(w)}{\bar{y}} < 1 - c \tag{86}$$

where c is positive and independent of n. Thus, for all $t \in [w, 1]$ and all n,

$$|f_n'(t)| = -f^{H\delta\bar{y}}(f_n(t), t) < \frac{1}{1-\delta} \frac{k_1 \bar{y}}{k_0} \frac{1-c}{c}$$

which is a finite constant that is independent of n. Thus, the functions $(f_n)_{n=1}^{\infty}$, as well as their limit (call it f_∞), are Lipschitz on $t \in [w, 1]$ with the same Lipschitz constant. As $f^{H\delta\bar{y}}$ is continuous and f_∞ is Lipschitz continuous, the function

$$\lim_{n\to\infty} f_n'(t) = \lim_{n\to\infty} f^{H\delta\bar{y}}(f_n(t), t) = f^{H\delta\bar{y}}\left(\lim_{n\to\infty} f_n(t), t\right) = f^{H\delta\bar{y}}(f_\infty(t), t)$$

is continuous in $t \in [w, 1]$. Moreover, convergence of $f_n'(t)$ is uniform since, for all $n, n' \ge 1$, by (50), (24), and (86), letting $k_5 = \frac{k_1}{k_0 c(1-\delta)}\left[1 + \frac{k_1(1-c)}{k_0 c}\right] \in (0, \infty)$,

$$\begin{aligned} |f_n'(t) - f_{n'}'(t)| &= \left| f^{H\delta\bar{y}}(f_n(t), t) - f^{H\delta\bar{y}}(f_{n'}(t), t) \right| \le k_5 |f_n(t) - f_{n'}(t)| \\ &= k_5 \left| D^{H\delta\bar{y}}_{\underline{t}a_n\infty}(\underline{t}) - D^{H\delta\bar{y}}_{\underline{t}a_{n'}\infty}(\underline{t}) \right| \le \bar{y} - \min\{a_n, a_{n'}\}, \end{aligned}$$

where the last inequality is from part 4 of Claim 5. Hence, $(f_n')_{n=1}^{\infty}$ is a uniform (in $t \in [w, 1]$) Cauchy sequence and thus converges uniformly. This proves existence and uniqueness.

By part 1 of claim 9, $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ is decreasing in t. It remains to show that $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ is decreasing and continuous in $t \in [\underline{t}, 1]$. By L-H and (24), $f^{H\delta\bar{y}}(D,t)$ is finite and negative for all $D < \bar{y}$. Thus, by (83), $D_{\underline{t}}^{H\delta\bar{y}}$ is decreasing and continuous in $t \in (\underline{t}, 1]$. By (83), for continuity at $t = \underline{t}$ we must show that $\lim_{t \to \underline{t}} D_{\underline{t}}^{H\delta\bar{y}}(t) = \bar{y}$. By (82), $D_{\underline{t}}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)$. By $\mathrm{CP}_{\underline{t}ak}^{H\delta\bar{y}}$, $D_{\underline{t}ak}^{H\delta\bar{y}}(\underline{t}) = a$. By the triangle inequality, for any $a \in (0, \bar{y})$,

$$\left| D_{\underline{t}}^{H\delta\bar{y}}(t) - \bar{y} \right| = \left| D_{\underline{t}}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| + \left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - a \right| + |a - \bar{y}| .$$

By parts 2 and 4 of Claim 5, $\left| D_{\underline{t}}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \le \left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \le |\bar{y} - a|$. Finally, $\left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - a \right| = \left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(\underline{t}) \right| \le \frac{k_1\bar{y}a}{(1-\delta)k_0(\bar{y}-a)} |t - \underline{t}|$ by part 3 of Claim 5. Collecting terms and letting $a = \bar{y} - \sqrt{t - \underline{t}} \le \bar{y}$, $\left| D_{\underline{t}}^{H\delta\bar{y}}(t) - \bar{y} \right| \le \left[2 + \frac{k_1\bar{y}^2}{(1-\delta)k_0} \right] \sqrt{t - \underline{t}}$ which goes to zero as $t \downarrow \underline{t}$. Q.E.D.$_{\text{Claim 14}}$

In light of Claim 14, part 5 of Theorem 1 is implied by part 2 of Claim 5, setting $\underline{t} = 0$, $a = a' = \bar{y}$, and $k = k' = \infty$.

We now turn to the convergence of the discrete solutions to the continuous one. By part 2 of Claims 5 and 9, for all $a \in (0, \bar{y})$,

$$D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) \le D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t) \le D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)$$

and thus, by the triangle inequality,

$$\left| D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t) \right| \le \max \left\{ \begin{array}{l} \left| D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| + \left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) \right|, \\ \left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) \right| + \left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \end{array} \right\} . \tag{87}$$

Fix $\varepsilon > 0$. By Claim 10, there is an i^*, independent of $\bar{y}$, H, and t, such that for $i > i^*$, $\left| D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right|$ and $\left| D_{\underline{t}\bar{y}k}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k}^{H\delta\bar{y}}(t) \right|$ are each less than $\varepsilon/2$. By part 4 of Claim 5, for all $a \in [\bar{y} - \varepsilon/2, \bar{y})$, $\left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \le \varepsilon/2$. But (87) holds for all $a \in (0, \bar{y})$, so it holds in particular for $a \in [\bar{y} - \varepsilon/2, \bar{y})$. Thus, for all $\varepsilon > 0$ there is an i^*, independent of $\bar{y}$, H, and t, such that for $i > i^*$, $\left| D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t) \right| \le \varepsilon$. We now set $\underline{t} = 0$ to obtain the desired result. Together with Claim 14, this proves that the unique solution $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ to $\mathrm{DP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ converges to the unique solution $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ to $\mathrm{CP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ as $i \to \infty$, uniformly in $H \in \mathscr{H}$, $\bar{y} \in [0, \mathbf{y}]$, and $t \in [0, 1]$.

We next prove that $p^{H\delta\bar{y}}$ and thus $u^{H\delta\bar{y}}$ is both continuous and decreasing in the type t. First, $p^{H\delta\bar{y}}(t) = v^{H\bar{y}}\left(D^{H\delta\bar{y}}_{t\bar{y}\infty}(t), t\right)$. By (22) and L-$H$, $v^{H\bar{y}}$ is continuous in both arguments. Since $D^{H\delta\bar{y}}_{t\bar{y}\infty}$ is also continuous in t, so is $p^{H\delta\bar{y}}$. By (25),

$$\begin{aligned}
\frac{d}{dt}\left[v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right] &= v_1^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\frac{dD^{H\delta\bar{y}}}{dt} + v_2^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right) \\
&= \delta v_1^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\frac{dD^{H\delta\bar{y}}}{dt}
\end{aligned}$$

which is negative by (25), (44), and (45). Hence, $p^{H\delta\bar{y}}$ is decreasing in t.

We now turn to convergence of $p^{H\delta\bar{y}i}$ to $p^{H\delta\bar{y}}$ and thus of $u^{H\delta\bar{y}i}$ to $u^{H\delta\bar{y}}$. For any $t \in [0,1]$, recall that $\tau_t^i = \Delta_i \lfloor t/\Delta_i \rfloor$. As $D^{H\delta\bar{y}i}(t) = D^{H\delta\bar{y}i}\left(\tau_t^i\right)$,

$$\begin{aligned}
\left|p^{H\delta\bar{y}i}(t) - p^{H\delta\bar{y}}(t)\right| &= \left|p^{H\delta\bar{y}i}\left(\tau_t^i\right) - p^{H\delta\bar{y}}(t)\right| \\
&= \left|v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right| \leq A_1' + A_2' + A_3'
\end{aligned}$$

where

$$\begin{aligned}
A_1' &= \left|v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right)\right|, \\
A_2' &= \left|v^{H\bar{y}}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), \tau_t^i\right)\right|, \text{ and} \\
A_3' &= \left|v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right|.
\end{aligned}$$

By (22), (26), and since $H\left(0|\tau_t^i\right)$ is zero,

$$\frac{A_1'}{\bar{y}} \leq \int_{z=0}^{D^{H\delta\bar{y}i}(t)/\bar{y}} \left|H\left(z|\tau_t^i\right) - H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|\tau_t^i\right)\right| dz \leq k_1\Delta_i',$$

where the second inequality is by L-H. By (22), $A_2' \leq 2\left|D^{H\delta\bar{y}i}(t) - D^{H\delta\bar{y}}(t)\right|$. By (22) and L-$H$,

$$A_3' \leq \bar{y}\int_{z=0}^{D^{H\delta\bar{y}}(t)/\bar{y}} \left|H(z|t) - H\left(z|\tau_t^i\right)\right| dz \leq \bar{y}k_1\left|t - \tau_t^i\right| \leq \bar{y}k_1\Delta_i.$$

Hence, by the prior result, for all $\varepsilon > 0$ there is an $i^* < \infty$ such that if $i > i^*$,

$$\left|p^{H\delta\bar{y}i}(t) - p(t)\right| \leq \bar{y}k_1\Delta_i' + 2\left|D^{H\delta\bar{y}i}(t) - D^{H\delta\bar{y}}(t)\right| + \bar{y}k_1\Delta_i < \varepsilon,$$

for all $t \in [0,1]$, $\bar{y} \in (0, \mathbf{y}]$, and $H \in \mathscr{H}$, as claimed. Hence, $p^{H\delta\bar{y}i}$ converges uniformly to $p^{H\delta\bar{y}}$.

We now show uniform convergence of $\Pi^{H\delta\bar{y}i}$ to $\Pi^{H\delta\bar{y}}$:

$$\left|\Pi^{H\delta\bar{y}i}(t)-\Pi^{H\delta\bar{y}}(t)\right| \leq \left|E^{i}[\bar{y}Z|t]-E^{\infty}[\bar{y}Z|t]\right|+\left|u^{H\delta\bar{y}i}(t)-u^{H\delta\bar{y}}(t)\right|$$

and for $z\in[(c-1)\Delta_i',c\Delta_i')$, $c=\left\lfloor\frac{z}{\Delta_i'}\right\rfloor+1$, so

$$\begin{aligned}
\left|E^{i}[\bar{y}Z|t]-E^{\infty}[\bar{y}Z|t]\right| &= \left|\sum_{c=1}^{1/\Delta_i'}\bar{y}c\Delta_i'\left[H\left(c\Delta_i'|t\right)-H\left((c-1)\Delta_i'|t\right)\right]-\int_{z=0}^{1}\bar{y}zdH(z|t)\right| \\
&= \bar{y}\left|\sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\left(\left\lfloor\frac{z}{\Delta_i'}\right\rfloor+1\right)\Delta_i'dH(z|t)-\int_{z=0}^{1}zdH(z|t)\right| \\
&\leq \bar{y}\Delta_i'\int_{z=0}^{1}\left|\left\lfloor\frac{z}{\Delta_i'}\right\rfloor+1-\frac{z}{\Delta_i'}\right|dH(z|t)\leq\bar{y}\Delta_i'. \qquad (88)
\end{aligned}$$

Thus, $E^{i}[\bar{y}Z|t]$ converges uniformly to $E^{\infty}[\bar{y}Z|t]$ and hence $\Pi^{H\delta\bar{y}i}$ converges uniformly to $\Pi^{H\delta\bar{y}}$.

As there is no mention of G in the statement of the problems $\mathrm{DP}^{H\delta\bar{y}i}$ and $\mathrm{CP}^{H\delta\bar{y}}$, any solutions $D^{H\delta\bar{y}i}$ and $D^{H\delta\bar{y}}$ to these problems for one distribution G are also solutions for any other distribution $\widehat{G}$ that satisfies our assumptions.[5] Since, moreover, the solutions $D^{H\delta\bar{y}i}$ and $D^{H\delta\bar{y}}$ are unique by parts 1 and 2 of the theorem, they must be independent of G. Hence, convergence of $D^{H\delta\bar{y}i}$, $p^{H\delta\bar{y}i}$, $u^{H\delta\bar{y}i}$, and $\Pi^{H\delta\bar{y}i}$ is uniform in G as well.

We now show that $Eu^{GH\delta\bar{y}i}$ converges uniformly to $Eu^{GH\delta\bar{y}}$. Since G has no atoms, $G(0)=0$; hence,

$$Eu^{GH\delta\bar{y}i}=\sum_{c=1}^{1/\Delta_i}u^{H\delta\bar{y}i}(c\Delta_i)\left[G(c\Delta_i)-G((c-1)\Delta_i)\right]=\int_{t=0}^{1}u^{H\delta\bar{y}i}\left(\tau_t^i+\Delta_i\right)dG(t)$$

and thus, $Eu^{GH\delta\bar{y}i}-Eu^{GH\delta\bar{y}}=A_1''+A_2''-A_3''$ where

$$A_1''=\int_{t=0}^{1}\left[u^{H\delta\bar{y}i}\left(\tau_t^i+\Delta_i\right)-u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right)\right]dG(t),$$

$A_2''=\int_{t=0}^{1}u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right)dG(t)$, and $A_3''=\int_{t=0}^{1}u^{H\delta\bar{y}}(t)\,dG(t)$. As shown above, for all $\varepsilon>0$ there is an i^* such that for all models $i>i^*$, parameters $\bar{y}$ in $(0,\mathbf{y}]$, and conditional

[5] In particular, $\widehat{G}$ must have support $[0,1]$.

distribution functions H in $\mathscr{H}$,

$$\left|A_1''\right| \leq \int_{t=0}^{1} \left| u^{H\delta\bar{y}i}\left(\tau_t^i + \Delta_i\right) - u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right)\right| dG(t) < \int_{t=0}^{1} \frac{\varepsilon}{2} dG(t) = \frac{\varepsilon}{2}.$$

Moreover, since $u^{H\delta\bar{y}}$ is a nonincreasing function of t, it follows that

$$u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right) \leq u^{H\delta\bar{y}}(t) \leq u^{H\delta\bar{y}}\left(\tau_t^i\right),$$

so $A_2'' \leq A_3'' \leq A_4''$ where

$$A_4'' = \int_{t=0}^{1} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) = \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) + \int_{t=\Delta_i}^{1} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t).$$

But $\tau_{t+\Delta_i}^i = \tau_t^i + \Delta_i$. Hence, letting $t' = t - \Delta_i$ and renaming t' to t,

$$\int_{t=\Delta_i}^{1} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) = \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right) dG(t + \Delta_i).$$

Now let i^* also be large enough that $\Delta_{i^*} < \frac{\varepsilon}{2(1-\delta)\mathbf{y}[2k_3+k_4]}$. Then by Lipschitz-$G$ and since, by (21), $u^{H\delta\bar{y}} \in [0, (1-\delta)\bar{y}]$, for all $i > i^*$,

$$\begin{aligned}
&\left|A_2'' - A_3''\right| \leq \left|A_2'' - A_4''\right| \\
&= \left| \int_{t=0}^{1} u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right) dG(t) - \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) - \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right) dG(t + \Delta_i)\right| \\
&\qquad \leq \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) + \int_{t=1-\Delta_i}^{1} u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right) dG(t) \\
&\qquad\quad + \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right) \left|\psi(t) - \psi(t + \Delta_i)\right| dt \\
&\qquad\qquad \leq (1-\delta)\bar{y}k_3\Delta_i + (1-\delta)\bar{y}k_3\Delta_i + (1-\delta)\bar{y}k_4\Delta_i < \frac{\varepsilon}{2}.
\end{aligned}$$

For all $i > i^*$, G in $\mathscr{G}$, H in $\mathscr{H}$, and $\bar{y} \in (0, \mathbf{y}]$, $\left|Eu^{GH\delta\bar{y}i} - Eu^{GH\delta\bar{y}}\right| < \varepsilon$ as claimed.

We now show that $E^i[\bar{y}Z]$ converges uniformly to $E^\infty[\bar{y}Z]$. Combined with the preceding result, this will show that $E\Pi^{GH\delta\bar{y}i}$ converges uniformly to $E\Pi^{GH\delta\bar{y}}$. First,

$$E^i[\bar{y}Z] = \sum_{c=1}^{1/\Delta_i} E^i[\bar{y}Z|t = c\Delta_i]\left[G(c\Delta_i) - G((c-1)\Delta_i)\right] = \int_{t=0}^{1} E^i\left[\bar{y}Z|\tau_t^i + \Delta_i\right] dG(t)$$

so by the triangle inequality, $\left|E^i[\bar{y}Z] - E^\infty[\bar{y}Z]\right| \leq A_1''' + A_2'''$ where

$$A_1''' = \int_{t=0}^{1} \left|E^i\left[\bar{y}Z|\tau_t^i + \Delta_i\right] - E^\infty\left[\bar{y}Z|\tau_t^i + \Delta_i\right]\right| dG(t)$$

and $A_2''' = \int_{t=0}^{1} \left| E^{\infty}\left[\bar{y}Z|\tau_t^i + \Delta_i\right] - E^{\infty}\left[\bar{y}Z|t\right]\right| dG(t)$. By (88), $A_1''' \leq \bar{y}\Delta_i'$. Integrating by parts, $E^{\infty}[Z|t] = 1 - \int_{z=0}^{1} H(z|t)\,dz$. Thus, by L-$H$,

$$\begin{aligned}\left|E^{\infty}\left[\bar{y}Z|t=\tau_t^i+\Delta_i\right]-E^{\infty}\left[\bar{y}Z|t\right]\right| &\leq \bar{y}\int_{z=0}^{1}\left|H\left(z|\tau_t^i+\Delta_i\right)-H(z|t)\right|dz \\ &\leq \bar{y}k_1\int_{z=0}^{1}\left|\tau_t^i+\Delta_i-t\right|dz \leq \bar{y}k_1\Delta_i,\end{aligned}$$

so $A_2''' \leq \bar{y}k_1\Delta_i$. Hence, $E^i\left[\bar{y}Z\right]$ converges uniformly to $E^{\infty}\left[\bar{y}Z\right]$ as claimed.

As for homogeneity in $\bar{y}$, using (24), equation (25) for $\bar{y}=1$ can be rewritten as

$$\frac{dD^{H\delta 1}}{dt} = \frac{1}{1-\delta}\frac{\int_{z=0}^{D^{H\delta 1}}\frac{\partial H(z|t)}{\partial t}dz}{1-H\left(D^{H\delta 1}|t\right)}.$$

Hence, for any solution $D^{H\delta\bar{y}}$ to CP$^{H\delta\bar{y}}$, $D^{H\delta 1} = \bar{y}^{-1}D^{H\delta\bar{y}}$ is a solution to CP$^{H\delta 1}$. But both solutions $D^{H\delta\bar{y}}$ and $D^{H\delta 1}$ are unique as shown above. Hence, $D^{H\delta\bar{y}}$ must equal $\bar{y}D^{H\delta 1}$. Thus, the expected payout $p^{H\delta\bar{y}}(t) = v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t),t\right)$ must equal $v^{H\bar{y}}\left(\bar{y}D^{H\delta 1}(t),t\right)$. But by (22), for any D, $v^{H\bar{y}}(D,t)$ equals $\bar{y}v^{H1}\left(\frac{D}{\bar{y}},t\right)$, so $p^{H\delta\bar{y}}(t) = \bar{y}p^{H\delta 1}(t)$, $u^{H\delta\bar{y}}(t) = \bar{y}u^{H\delta 1}(t)$, and $Eu^{GH\delta\bar{y}} = \bar{y}Eu^{GH\delta 1}$ as claimed. These properties hold for $\Pi^{H\delta\bar{y}}$ and $E\Pi^{GH\delta\bar{y}}$ as well since $E^{\infty}\left[\bar{y}Z|t\right]$ is homogeneous of degree 1 in $\bar{y}$.

As for model i, by (20),

$$v^{H\bar{y}i}(D,t) = \bar{y}v^{H1i}\left(\frac{D}{\bar{y}},t\right), \tag{89}$$

so if $D^{H\delta 1i}$ solves DP$^{H\delta 1i}$, then $\bar{y}^{-1}D^{H\delta 1i}$ solves DP$^{H\delta\bar{y}i}$. But both solutions $D^{H\delta\bar{y}i}$ and $D^{H\delta 1i}$ are unique as shown above. Hence, $D^{H\delta\bar{y}i}$ must equal $\bar{y}D^{H\delta 1i}$. Thus, by (89), the expected payout $p^{H\delta\bar{y}i}(t) = v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t),t\right)$ must equal $\bar{y}v^{H1i}\left(D^{H\delta 1i}(t),t\right) = \bar{y}p^{H\delta 1i}(t)$ and so $u^{H\delta\bar{y}i}(t)$ equals $\bar{y}u^{H\delta 1i}(t)$ as claimed. These properties hold for $\Pi^{H\delta\bar{y}i}$ and $E\Pi^{GH\delta\bar{y}i}$ as well since $E^i\left[\bar{y}Z|t\right]$ is homogeneous of degree 1 in $\bar{y}$. This completes the proof of Theorem 1. Q.E.D.

4 Relaxing Monotonicity

We now show that the monotonicity assumed in ASSUMPTION A is not necessary for our results, in the case of two assets and two types with generic parameters. Why? By genericity, one type must expect her portfolio to have a higher total payout than the other. Swapping indices if needed, we can assume this is type 2. Also by genericity, an increase in the issuer's type from 1 to 2 must raise the expected value of one asset proportionally more than that of the other. Again swapping indices if needed, we can assume that asset 2 has this property. With respect to this labeling of types and assets, Increasing Informational Sensitivity (IIS) holds: an increase in the issuer's type from 1 to 2 raises the expected value of asset 2 proportionally more than that of asset 1. We then show that IIS can be used as an alternative to monotonicity in establishing a unique intuitive equilibrium of the 2x2 model.

In this equilibrium, type 1 sells her entire portfolio, while type 2 retains first asset 2 and then, if needed, asset 1, until type 1 is just willing not to imitate her. Hence, as in our base model with the IIS assumption, the equilibrium displays the Pecking Order property: the asset with the higher informational sensitivity is retained first. However, there is one contrast. Without monotonicity, asset 1 may be worth *less* to type 2 than to type 1.[6] If so, the issuer uses only asset 2 to signal her type: both types sell asset 1 in its entirety. Intuitively, type 1 gains more from retaining asset 1 than type 2 does. Hence, if type 2 retains some of asset 1, then she must retain even more of asset 2 in order to credibly signal her type. And such inefficient signalling is ruled out by the Intuitive Criterion, which we assume.

The model is as follows; proofs appear below in section 4.1. There are two assets $i = 1,2$ and two types $t = 1,2$. Each type has positive prior probability. For simplicity, we normalize the number of shares of each asset to one: the issuer's endowment vector a is $(1,1)$. Let

$$f_i(t) > 0 \tag{90}$$

denote the expected payout of asset $i = 1,2$ conditional on the issuer's type being $t = 1,2$. We restrict to generic parameters. Hence we can assume, w.l.o.g., that type 2 has a higher

[6]As shown below in Claim 16, this is not true of asset 2: it is worth more to type 2 than to type 1, as in the base model.

conditional expected portfolio value than type 1:

$$f_1(2) + f_2(2) > f_1(1) + f_2(1) > 0. \tag{91}$$

(If not, swap type indices.) Genericity also lets us assume, w.l.o.g., that

$$f_2(2)/f_2(1) > f_1(2)/f_1(1) > 0. \tag{92}$$

(If not, swap asset indices.) Intuitively, (92) says that an increase in the issuer's type raises the expected value of asset 2 proportionally more than that of asset 1.

Under our assumptions, asset 1 may be worth less to type 2 than it is to type 1: $f_1(2)$ may be less than $f_1(1)$. On the other hand, asset 2 must be worth more to type 2 than it is to type 1:

CLAIM 16 *Assume (91) and (92). Then*

$$f_2(2) > f_2(1). \tag{93}$$

Proof. If instead $f_2(1) \geq f_2(2)$ then $f_2(2)/f_2(1) \leq 1$ whence, by (92), $f_1(2)/f_1(1) < 1$ so $f_1(1) > f_1(2)$. Combining the first and last inequalities yields $f_1(1) + f_2(1) > f_1(2) + f_2(2)$, which contradicts (91). ■

The timing is as follows. On seeing her type t, the issuer chooses quantities $q = (q_1, q_2) \in [0,1]^2$ to offer for sale as well as price caps $\overline{p} = (\overline{p}_1, \overline{p}_2)$. A competitive set of deep-pocketed, uninformed investors see this choice $(q, \overline{p})$ and form posterior beliefs $\mu(t|q, \overline{p})$ about the probability of each type t. The issuer then sells q_i units of each asset $i = 1, 2$ to the investors for some price $p_i \leq \overline{p}_i$; let $p = (p_1, p_2)$ denote the price vector. The payoff of an issuer of type t is then $q[p - \delta f(t)]$: her issuance revenue pq less the present discounted value $\delta p f(t)$ of the assets she sells.

A perfect Bayesian equilibrium (PBE) of this game is a profile

$$e = (q(\cdot), \overline{p}(\cdot), \mu(\cdot|\cdot,\cdot), p(\cdot,\cdot))$$

of strategies and beliefs that satisfies the following properties. The notation $V \wedge V'$ and $V \vee V'$ denotes componentwise minimum and maximum, respectively, for any vectors V and V' of equal length.

Profit Maximization. The issuer's behavior is optimal given the investors' price function: for each $t = 1, 2$, $(q(t), \overline{p}(t)) \in \arg\max_{q,\overline{p}} (q[p(q,\overline{p}) - \delta f(t)])$.

Competitive Pricing. Given the investors' beliefs, an asset's price equals its conditional expected value or the price cap, whichever is lower:

$$p(q,\overline{p}) = \overline{p} \wedge \left[\sum_{t=1}^{2} f(t)\, \mu(t|q,\overline{p}) \right].$$

Rational Updating. Investors' beliefs $\mu(\cdot|q,\overline{p})$ are given by Bayes's Rule if some type chooses $(q,\overline{p})$ in equilibrium.

Given a profile e, we now define two ancillary functions that will be useful. The first is the (reduced-form) price function $p(t) = p(q(t), \overline{p}(t))$, which gives the prices offered to each type in equilibrium. The second is the outcome of the PBE, $u(t) = p(t)q(t) - \delta q(t) f(t)$, which gives each type's payoff: her revenue less her cost of parting with the securities that she sells.

In order to obtain a unique PBE, we will impose the Intuitive Criterion, which is as follows. Fix a PBE e with outcome $u(\cdot)$. Suppose an issuer of type t deviates to $(q,\overline{p})$. The deviation must harm her if her equilibrium payoff $u(t)$ exceeds her maximum payoff $q[\overline{p} - \delta f(t)]$ from the deviation - or, equivalently, if her opportunity cost $u(t) + \delta q f(t)$ of deviating exceeds her maximum deviation revenue $q\overline{p}$. The Intuitive Criterion states that on seeing a deviation $(q,\overline{p})$, investors put zero weight on any type t who is definitely harmed by the deviation if there is some other type s who is not definitely harmed:

The Intuitive Criterion. A PBE e with outcome $u(\cdot)$ is intuitive if, for any deviation $(q,\overline{p})$ and any type t such that $u(t) + \delta q f(t) > \overline{p}q$, investors' posterior weight $\mu(t|q,\overline{p})$ on type t is zero if there exists a type $s \neq t$ for whom $u(s) + \delta q f(s) \leq \overline{p}q$.

We will now construct a particular PBE $e^* = (q^*(\cdot), \overline{p}^*(\cdot), \mu^*(\cdot|\cdot,\cdot), p^*(\cdot,\cdot))$, with associated price function $p^*(\cdot)$ and outcome $u^*(\cdot)$, and then show that it is the unique intuitive PBE. Any price caps may be chosen as long as they satisfy $\overline{p}^*(t) \geq f(1) \vee f(2)$

for each type t; this will imply that they do not bind in equilibrium. Type $t = 1$ sells her whole portfolio: $q^*(1) = a$. As for type 2, let

$$A_2 = \{q : u^*(1) \geq q[f(2) - \delta f(1)]\} \tag{94}$$

denote the set of quantity vectors that type 2 can choose without attracting imitation by type 1, where type 1's payoff $u^*(1)$ is given below. Let $q^*(2)$ be any vector in A_2 that maximizes type 2's symmetric-information payoff $(1-\delta)qf(2)$ in A_2 (and thus also maximizes the value $qf(2)$ of the securities she sells). Investors' beliefs are given by Bayes's Rule in equilibrium: they believe that the issuer is of type 1 (resp., 2) if they see the issuance choice $(q^*(1), \overline{p}^*(1))$ (resp., $(q^*(2), \overline{p}^*(2))$). Hence Rational Updating holds. We also impose Competitive Pricing, whence the choices of types 1 and 2 lead to price vectors $p(1) = f(1)$ and $p(2) = f(2)$, respectively. Accordingly, the issuer's equilibrium payoffs are

$$u^*(1) = (1-\delta)q^*(1)f(1) = (1-\delta)af(1) \tag{95}$$

and

$$u^*(2) = (1-\delta)q^*(2)f(2). \tag{96}$$

Following a deviation $(q, \overline{p})$, beliefs $\mu(\cdot|q, \overline{p})$ put positive weight only on types t that have the lowest opportunity cost of deviating: those in the set $T(q) = \arg\min_{t'}[u(t') + \delta q f(t')]$. This ensures that beliefs are intuitive. In particular, if the set $T(q)$ is a singleton $\{t\}$, then investors believe the issuer is of type t. If $T(q)$ contains both types, investors believe the issuer is of that type t for which the deviation revenue $q[\overline{p} \wedge f(t)]$ under symmetric information is minimized. If, this deviation revenue is the same for the two types, then investors' beliefs do not matter. We can assume, e.g., that they believe that $t = 1$, although any other beliefs would give the same results.

We first prove some properties of this profile.

CLAIM 17 *The strategy profile e^* has the following properties.*

1. *Type 1's incentive compatibility (IC) constraint binds at type 2's quantity vector* $q^*(2)$*:*
$$u^*(1) = q^*(2)\left[f(2) - \delta f(1)\right]. \tag{97}$$

2. *Type 2 sells all of asset 1 if she sells any of asset 2: if* $q_2^*(2) > 0$*, then* $q_1^*(2) = 1$*.*

3. *A type 2 issuer retains a portion of asset 2:* $q_2^*(2) < 1$*.*

4. *The issuer's revenue and payoff are decreasing in her type:* $q^*(1)\, f(1) > q^*(2)\, f(2) > 0$ *and* $u^*(1) > u^*(2) > 0$*.*

5. *If a type 2 issuer does not value asset 1 more than a type 1 issuer does, then she sells all shares of this asset: if* $f_1(2) \leq f_1(1)$ *then* $q_1^*(2) = 1$*.*

6. *The vector* $q^*(2)$ *uniquely maximizes 2's symmetric-information payoff in* A_2*: for any vector* $q \neq q^*(2)$ *in* A_2*,* $(1-\delta)\, q f(2) < u^*(2)$*.*

The intuition for part 1 is simple: if type 1's IC constraint does not bind, then type 2 can sell a bit more of either asset without being mistaken for type 1, so she will do so. Parts 1 and 2 also yield an algorithm for finding type 2's optimal quantity vector. We start with $q = a$ and gradually lower q_2 until it reaches zero, and then begin lowering q_1. By part 1, we must stop when type 1 is just willing not to imitate and be mistaken for 2: at which equation (97) holds. Hence, an issuer does not sell any of asset 2 until she has liquidated all of asset 1. This is a kind of Pecking Order property in which the asset i with the higher informational sensitivity - as measured by the ratio $f_i(2)/f_i(1)$ - is retained first. Importantly, this property holds only if types are sorted in ascending order of total portfolio value.

By part 5 of the Claim, both types sell asset 1 in its entirety if its value is nonincreasing in the issuer's type. Why? If a type 2 issuer retains part of asset 1 in equilibrium then, by part 2, she must also retain asset 2 in its entirety. Now suppose type 1 imitates type 2. That is, he retains all of asset 2 as well as part of asset 1, without obtaining a higher price for asset 1 than he gets in equilibrium (since $f_1(2) \leq f_1(1)$). Thus, his payoff from the deviation must be less than $(1-\delta) f_1(1)$ which, in turn, is less than his equilibrium payoff of $u^*(1) = (1-\delta)[f_1(1) + f_2(1)]$. Since the deviation is strictly worse for type 1, his IC constraint does not bind, which contradicts part 1 of the Claim.

The following two results establish that e^* is the unique intuitive PBE. More precisely, (a) it is an intuitive PBE, and (b) any other intuitive PBE has identical equilibrium behavior and payoffs to e^*.[7]

PROPOSITION 18 *The profile e^* is an intuitive PBE.*

PROPOSITION 19 *If $e = (q(\cdot), \overline{p}(\cdot), \mu(\cdot|\cdot,\cdot), p(\cdot,\cdot))$ is an intuitive PBE of the above game, with outcome $u(\cdot)$ and price function $p(\cdot)$, then the issuer makes the same quantity choices and receives the same payoffs in e as in e^*: for each type t, $q(t) = q^*(t)$ and $u(t) = u^*(t)$. Moreover, if the quantity $q_i(t) = q_i^*(t)$ of an asset i sold by a type t is positive, the resulting price of asset i is the same in the two equilibria and equals the fair value of the asset: $p_i(t) = p_i^*(t) = f_i(t)$.*

4.1 Proofs

Proof of Claim 17. Part 1. First, $af(2) > af(1)$ by (91) whence $a \notin A_2$ by (94). Hence either $q_1^*(2)$ or $q_2^*(2)$ is less than one; assume w.l.o.g. that $q_2^*(2) < 1$. Thus, if (97) does not hold then for ε in $(0, 1-q_2^*(2)]$, the vector $q' = q^*(2) + (0,\varepsilon)$ is also in A_2. By (94), q' offers type 2 a higher payoff under symmetric information than $q^*(2)$ does. Hence $q^*(2)$ is not optimal for type 2 in A_2, a contradiction.

Part 2. Suppose not: $q_1^*(2) < 1$ and $q_2^*(2) > 0$. We will derive a contradiction. Let $\varepsilon > 0$ be small enough that $q_1^*(2) + \varepsilon < 1$ and $q_2^*(2) - \iota > 0$ where $\iota = \varepsilon \frac{f_1(2) - \delta f_1(1)}{f_2(2) - \delta f_2(1)}$, whose denominator is positive by (93). Consider the alternative quantity vector $q = (q_1, q_2)$

[7] The other PBE may differ in ways that are not payoff relevant. In particular, it may have different nonbinding price caps and different beliefs following deviations (as long as these beliefs are intuitive and serve to deter the deviations). Moreover, if an issuer does not sell any shares of a given asset, then the price of this asset is indeterminate and hence may vary across equilibria.

where $q_1 = q_1^*(2) + \varepsilon$ and q_2 equals the lesser of $q_2^*(2) - \iota$ and one. First, $u^*(1) - q[f(2) - \delta f(1)]$ can be written as the sum of $u^*(1) - q^*(2)[f(2) - \delta f(1)]$, which is non-positive since $q^*(2)$ is in A_2, and $[q^*(2) - q][f(2) - \delta f(1)]$, which by (93) is bounded above by $\varepsilon[f_1(2) - \delta f_1(1)] - \iota[f_2(2) - \delta f_2(1)]$ which equals zero by definition of ι. Hence $u^*(1) \leq q[f(2) - \delta f(1)]$, whence q is in A_2. Moreover, the effect on type 2's payoff from switching from $q^*(2)$ to q (under symmetric information) equals the difference $1-\delta$ in discount factors times the change in revenue $qf(2) - q^*(2)f(2)$ which, in turn, equals $\min\{\varepsilon f_1(2) - \iota f_2(2), \varepsilon f_1(2) + 1 - q_2^*(2)\}$. But both elements in the min are positive: the first by (92) and the second by (90) and since $\varepsilon > 0$ and $q_2^*(2) \leq 1$. We conclude that q, which is in A_2, is better for 2 than $q^*(2)$ under symmetric information, which contradicts the definition of $q^*(2)$.

Part 3. This holds by part 2 and since, as shown in the proof of part 1, $q^*(2) \neq a$.

Part 4. Part 3 implies $af(1) > q^*(2)f(1)$ which, by (97), implies $af(1) > q_2^*(2)f(2)$ as claimed. Multiplying each side by $1-\delta$, we obtain $u^*(1) > u^*(2)$. Clearly, $u^*(2) \geq 0$ since a type 2 issuer can always set $q = (0,0)$. Finally, the vector $(\varepsilon, \varepsilon)$ is in A_2 where $\varepsilon = \frac{u^*(1)}{a[f(2)-\delta f(1)]}$ is positive. (The denominator is positive by (91).) It follows that $u^*(2)$ is not less than $(1-\delta)(\varepsilon,\varepsilon)f(2)$, which is positive by (90). Hence $u^*(2)$ and $\frac{u^*(2)}{1-\delta} = q^*(2)f(2)$ are positive as claimed.

Part 5. Suppose instead that $f_2(2) \leq f_2(1)$ yet $q_1^*(2) < 1$. Then by part 2, $q_2^*(2) = 0$. Substituting this into (97) and using the assumed inequalities, we obtain

$$(1-\delta)af(1) = q_1^*(2)[f_1(2) - \delta f_1(1)] < (1-\delta)f_1(1)$$

which is impossible by (90).

Part 6. Let $\widehat{q}$ denote $q^*(2)$ and let $q \neq \widehat{q}$ be another vector in A_2 that maximizes $(1-\delta)qf(2)$. Then

$$qf(2) = \widehat{q}f(2) \tag{98}$$

whence, by (90), $q_i \neq \widehat{q}_i$ for $i = 1,2$. Let us assume w.l.o.g. that $q_1 > \widehat{q}_1$; if not, relabel q

to $\widehat{q}$ and vice-versa. Since each vector is optimal, part 1 implies that

$$q\left[f(2)-\delta f(1)\right]=\widehat{q}\left[f(2)-\delta f(1)\right] \tag{99}$$

as each side equals $u^*(1)$. Rearranging (98) and (99) we obtain

$$(q_1-\widehat{q}_1)\frac{f_1(2)}{f_2(2)}+q_2-\widehat{q}_2=0$$

and, using (93), $(q_1-\widehat{q}_1)\frac{f_1(2)-\delta f_1(1)}{f_2(2)-\delta f_2(1)}+q_2-\widehat{q}_2=0$, respectively. Since $q_1>\widehat{q}_1$ this implies

$$0=\frac{f_1(2)-\delta f_1(1)}{f_2(2)-\delta f_2(1)}-\frac{f_1(2)}{f_2(2)}=\frac{\delta\left[f_1(2)f_2(1)-f_2(2)f_1(1)\right]}{\left[f_2(2)-\delta f_2(1)\right]f_2(2)}<0$$

(where the last inequality is by (92) and (93)) - a contradiction. Q.E.D.$_{\text{Claim 17}}$

Proof of Proposition 18. Beliefs are intuitive by construction and we have already verified Competitive Pricing and Rational Updating. To show that e^* is an intuitive PBE, it remains to verify Payoff Maximization: that neither type has a (strictly) profitable deviation. By construction, investors' beliefs μ^* following a deviation $(q,\overline{p})$ depend on q but not on $\overline{p}$. Hence, if a deviator chooses binding price caps, this can only lower his payoff from deviating. We can thus assume, w.l.o.g., that a deviator chooses price caps that can never bind (such as the caps $\overline{p}^*$).

We begin with type 1. Let

$$\Gamma(q)=q\left[f(2)-\delta f(1)\right]-u^*(1) \tag{100}$$

denote the change in type 1's payoff from deviating to q and being mistaken for type 2. Let

$$\Lambda(q)=\left[u^*(2)+\delta q f(2)\right]-\left[u^*(1)+\delta q f(1)\right] \tag{101}$$

denote the difference between type 2's and type 1's opportunity cost of deviating to q. By construction of μ^*, investors must believe the issuer is of type 1 (resp., 2) when $\Lambda(q)$ is positive (negative). And when $\Lambda(q)=0$, they think he is of type 1: beliefs are governed by deviation revenue $qf(t)$ which by (101) and part 3 of Claim 17, is lower for type 1 in this case. By (96) and (97),

$$\Gamma(q^*(2))=\Lambda(q^*(2))=0. \tag{102}$$

Now suppose type 1 deviates to $(q,\overline{p})$. If $\Lambda(q) \geq 0$, investors will think the issuer's type is 1: he gets $(1-\delta)\, q f(1)$ which cannot exceed his equilibrium payoff $u^*(1)$ as $q \leq a$. So such a deviation is not profitable. If instead $\Lambda(q) < 0$, investors will think his type is 2 whence his payoff changes by $\Gamma(q)$, which is nonpositive by the following result.

CLAIM 20 *For any q in $[0,1]^2$, if $\Lambda(q) < 0$ then $\Gamma(q) \leq 0$.*

Proof of Claim 20. Suppose not: $\Lambda(q) < 0$ and $\Gamma(q) > 0$. We will derive a contradiction. For brevity let $\widehat{q}$ denote $q^*(2)$. For any $x \in [0,1]$, define

$$\Delta^x = (\Delta_1^x, \Delta_2^x) = f(2) - x f(1). \tag{103}$$

By (90) and (93),

$$\Delta_2^x > 0 \tag{104}$$

for all such x, whence

$$\frac{d}{dx}\frac{\Delta_1^x}{\Delta_2^x} \propto f_1(2)\, f_2(1) - f_2(2)\, f_1(1) < 0 \tag{105}$$

by (92). We will rely on the following lemma.

LEMMA 21 *Let $q' \in [0,1]^2$ be such that $\Gamma(q') \geq 0$ and $\Lambda(q') < 0$. Then $q_1' > \widehat{q}_1$.*

Proof. By (102), $\Gamma(\widehat{q}) = 0 \leq \Gamma(q')$ whence by (100),

$$\left(\widehat{q}_1 - q_1'\right)\frac{\Delta_1^\delta}{\Delta_2^\delta} + \left(\widehat{q}_2 - q_2'\right) \leq 0. \tag{106}$$

Also by (102), $\Lambda(\widehat{q}) = 0 > \Lambda(q')$ whence, by (101),

$$\left(\widehat{q}_1 - q_1'\right)\frac{\Delta_1^1}{\Delta_2^1} + \left(\widehat{q}_2 - q_2'\right) > 0. \tag{107}$$

Subtracting (106) from (107) yields $\left(\widehat{q}_1 - q_1'\right)\left(\frac{\Delta_1^1}{\Delta_2^1} - \frac{\Delta_1^\delta}{\Delta_2^\delta}\right) > 0$, which by (105) yields $q_1' > \widehat{q}_1$. ■

Since $\Gamma(q) > 0$, $q\Delta^\delta > u^*(1) > 0$ by (100) and part 4 of Claim 17. Let $\alpha = u^*(1) / \left[q\Delta^\delta\right] \in (0,1)$ and $\widetilde{q} = \alpha q$. Since $\Lambda(q) < 0$ by assumption, $u^*(1) - u^*(2)$ (which is positive by part 4 of Claim 17) exceeds $\delta q\Delta^1$ and thus also exceeds $\delta\widetilde{q}\Delta^1$ since $\widetilde{q}=\alpha q$ and $\alpha\in(0,1)$. Rearranging, we obtain $\Lambda(\widetilde{q})<0$. Since, moreover, $\Gamma(\widetilde{q})=0$ by (100) and (103), Lemma 21 implies that $\widetilde{q}_1>\widehat{q}_1$. Hence, by (105),

$$(\widetilde{q}_1-\widehat{q}_1)\left(\frac{\Delta_1^0}{\Delta_2^0}-\frac{\Delta_1^\delta}{\Delta_2^\delta}\right)>0. \tag{108}$$

We can also rearrange $\Gamma(\widehat{q})=\Gamma(\widetilde{q})=0$ to obtain

$$(\widetilde{q}_1-\widehat{q}_1)\frac{\Delta_1^\delta}{\Delta_2^\delta}+\widetilde{q}_2-\widehat{q}_2=0. \tag{109}$$

Adding (108) and (109) yields $(\widetilde{q}_1-\widehat{q}_1)\frac{\Delta_1^0}{\Delta_2^0}+\widetilde{q}_2-\widehat{q}_2>0$. This can be rearranged (with the result multiplied by $1-\delta$) to yield $(1-\delta)\widetilde{q}\Delta^0>(1-\delta)\widehat{q}\Delta^0$. Moreover, $\widetilde{q}$ is in A_2 since $\Gamma(\widetilde{q})=0$. This contradicts the definition of $\widehat{q}$ as the vector in A_2 that maximizes type 2's symmetric-information payoff. Q.E.D.$_{\text{Claim 20}}$

We have shown that a type 1 issuer will not deviate. Let us now consider a deviation $(q,\overline{p})$ by a type 2 issuer. There are two cases.

1. $\Lambda(q)\geq 0$. Then on seeing q, investors believe the issuer's type is 1. Hence, the deviation is profitable for the type 2 issuer only if

   $$0<\Omega(q)=q[f(1)-\delta f(2)]-u^*(2).$$

   If this holds then, by (101), $0<\Lambda(q)+\Omega(q)=(1-\delta)qf(1)-u^*(1)$ which is not possible by (90) and since $q\leq a$: the deviation is not profitable.

2. $\Lambda(q)<0$. Then on seeing q, investors believe the issuer's type is 2. Hence, type 2's change in payoffs from this deviation equals $\Gamma(q)$, which is nonpositive by Claim 20: the deviation is not profitable.

As this verifies Payoff Maximization, e^* is an intuitive PBE. Q.E.D.$_{\text{Proposition 18}}$

Proof of Proposition 19. We first show that in any intuitive PBE, an issuer of a given type is paid the expected value (given her type) of the portfolio that she sells. The precise property is as follows.

Fair Pricing. A PBE e with quantity function $q(\cdot)$ and price function $p(\cdot)$ is Fairly Priced if, for each type t, $p(t)\,f(t) = q(t)\,f(t)$.

LEMMA 22 *Any intuitive PBE e is Fairly Priced.*

Proof. Let e be intuitive, and suppose type t chooses $(q,\overline{p})$ in equilibrium and that investors respond with price vector p. First suppose the issuance is underpriced: $pq < qf(t)$. Let S denote the set of types s for which $qf(s) < qf(t)$. There are two cases.

1. S is empty. Then by Competitive Pricing, if the type t issuer deviates to $(q, f(t))$, her revenue cannot be less than $qf(t)$. As this would be a profitable deviation, this case is not possible.

2. S is nonempty. Let $s' \in S$ be a type s for which $qf(s)$ is at a maximum among all s in S: there is no type s in S for which $qf(s)$ exceeds $qf(s')$. Then $qf(s') < qf(t)$ whence, since $pq < qf(t)$, there is a $\lambda \in [0,1)$ such that
$$\frac{q[p-\delta f(t)]}{q[f(t)-\delta f(t)]} < \lambda < \frac{q[p-\delta f(s)]}{q[f(t)-\delta f(s)]}. \tag{110}$$
Let type t deviate to $(\lambda q, f(t))$. By (110), $\lambda q[f(t)-\delta f(t)]$ exceeds $q[p-\delta f(t)]$, which equals $u(t)$. And for each s in S, $\lambda q[f(t)-\delta f(s)]$ is less than $q[p-\delta f(s)]$ which, in turn, is not greater than $u(s)$. Accordingly, for each s in S,
$$u(t)+\delta(\lambda q)\,f(t) < \lambda q f(t) < u(s)+\delta(\lambda q)\,f(s)$$
and hence, since s is intuitive, $\mu(s|(\lambda q, f(t)))$ is zero for each s in S. Hence, by Competitive Pricing, the revenue that results from the deviation $(\lambda q, f(t))$ is $\lambda q f(t)$ whence type t 's payoff from the deviation is $\lambda q[f(t)-\delta f(t)]$ which, as noted, exceeds $u(t)$: the deviation is profitable, which is not possible.

We have shown that in an intuitive PBE e, underpricing cannot occur (except perhaps off the equilibrium path). Hence, $pq \geq qf(t)$ for any type t who chooses $(q,\overline{p})$ in equilibrium. But by Competitive Pricing,
$$pq = q\left(\overline{p} \wedge \left[\sum_{t=1}^{2} f(t)\,\mu(t|q,\overline{p})\right]\right) \leq \sum_{t=1}^{2} qf(t)\,\mu(t|q,\overline{p}).$$

Thus, pq is bounded above by a convex combination of $qf(t)$ for those types t who choose $(q,\overline{p})$ in equilibrium. But by the prior result, pq is also bounded below by $qf(t)$ for each such type t, so it cannot be less than the highest such $qf(t)$. Hence all types who choose $(q,\overline{p})$ in equilibrium have the same value of $qf(t)$, and this also equals the revenue pq that they receive: the equilibrium is Fairly Priced. ■

Moreover, if a PBE is Fairly Priced, then each asset that is sold in equilibrium must be priced correctly.

CLAIM 23 *Suppose a PBE e with quantity function $q(\cdot)$ and price function $p(\cdot)$ is Fairly Priced. Then for each type t, and each asset i for which $q_i(t) > 0$, we have $p_i(t) = f_i(t)$.*

Proof. Suppose not. Then by Competitive Pricing, there is a type t who sells some shares of an asset i with a binding price cap: $q_i(t) > 0$ and $\overline{p}_i(t) < f_i(t)$. Thus, this type's revenue $p_i(t)\,q_i(t)$ from asset i is less than the value $q_i(t)\,f_i(t)$ of shares sold. And by Competitive Pricing, her revenue $p_j(t)\,q_j(t)$ from any other asset j cannot exceed its true value $q_j(t)\,f_j(t)$. Combining these facts, we find that her total issuance revenue $p(t)\,q(t)$ must be less than the total value $q(t)\,f(t)$ of shares she sells, which violates Fair Pricing. ■

Now let e be an intuitive PBE with outcome $u(\cdot)$ and price function $p(\cdot)$. By Lemma 22, e is Fairly Priced, whence

$$u(t) = (1-\delta)\,q(t)\,f(t) \text{ for each type } t. \tag{111}$$

Hence, $u(1) \leq (1-\delta)\,af(1) = u^*(1)$. And as e is a PBE, type 1 cannot prefer to imitate type 2. So since e is Fairly Priced, type 2's choice $q(2)$ must satisfy $q(2)\,[f(2) - \delta f(1)] \leq u(1)$. As $u(1) \leq u^*(1)$, this implies that $q(2)$ lies in A_2. But $u^*(2)$ is type 2's highest payoff $(1-\delta)\,qf(2)$ among all vectors q in A_2, whence $u(2) \leq u^*(2)$ as well.

Now suppose that in e, type 1 deviates to $(q,\overline{p})$ where $q = a$ and $\overline{p} = f(1) \vee f(2)$.[8] By Competitive Pricing and part 4 of Claim 17, his resulting revenue is $a\left[\sum_{t=1}^{2} f(t)\,\mu(t|q,\overline{p})\right] \geq$

[8]Recall that $\vee$ denotes the componentwise maximum.

$af(1)$ whence his payoff from the deviation is at least $u^*(1)$. Thus, for 1 not to deviate, $u(1)$ must equal $u^*(1)$. By (90) and (111), this implies $q(1) = a = q^*(1)$ as claimed.

Now consider type 2. If $u(2) < u^*(2)$, suppose type 2 deviates to $(q, \overline{p})$ where $q = \lambda q^*(2)$ for some $\lambda \in \left(\frac{u(2)}{u^*(2)}, 1\right)$ and $\overline{p} = f(1) \vee f(2)$. By the prior result, $u(1) = u^*(1) = q^*(2)[f(2) - \delta f(1)] > \lambda q^*(2)[f(2) - \delta f(1)]$ and $u(2) < \lambda u^*(2) = (1-\delta)\lambda q^*(2) f(2)$. Thus, $u(1) + \delta\lambda q^*(2) f(1) > \lambda q^*(2) f(2) > u(2) + \delta\lambda q^*(2) f(2)$ whence, since e is intuitive, $\mu(2|q, \overline{p}) = 1$ and hence, by Competitive Pricing, type 2's payoff from the deviation is $\lambda u^*(2) > u(2)$. Thus, for type 2 not to deviate, her payoff $u(2)$ must equal $u^*(2)$. And since $u(1) = u^*(1)$ and by Fair Pricing, $q(2)$ must lie in A_2 (else type 1 will deviate to $q(2)$). So $q = q(2)$ maximizes type 2's symmetric-information payoff $(1-\delta) q f(2)$ in A_2 whence, by part 6 of Claim 17, $q(2)$ must equal $q^*(2)$ as claimed.

Finally, as $q(t)$ equals $q^*(t)$ for each type t, Claim 23 implies that the prices $p_i(t)$ and $p_i^*(t)$ each equal $f_i(t)$ for each asset i for which $q_i(t) = q_i^*(t)$ is positive. Q.E.D.$_{\text{Proposition 19}}$

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