Mechanism Design with an Informed Principal: Extensions and Generalizations*

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Abstract

Maskin and Tirole (1992) study the problem of mechanism design by an informed principal when the agent cares also about the principal's information. We extend their model by permitting corner solutions. We also omit their requirement that the least-cost separating (Riley) outcome be interim efficient for positive beliefs - an assumption that can be hard to check. Finally, we permit the indifference curves of different types of principals to be tangent or to coincide, as occurs naturally in certain contexts.

1 Introduction

In traditional mechanism design (Hurwicz 1972, Myerson 1979), an uninformed principal proposes a menu of contracts to an informed agent. But in many real-world cases, privately informed players also have bargaining power and thus can propose mechanisms as well (Myerson 1983). This is the problem of mechanism design with an informed principal.

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While Myerson (1983) takes a cooperative approach to this topic, Maskin and Tirole (1992, henceforth "MT") use the noncooperative approach that is more familiar in applied theory. MT focus on the "common values" case, in which the principal's information appears in the payoff functions of both players and the agent is uninformed.[1] This model has been applied to security design (Chemla and Hennesy 2014), bank bailouts (Phillipon and Skreta 2012), external investment funding (Morellec and Schurhoff 2011), dividend stickiness (Guttman, Kadan, and Kandel 2010), monopoly pricing (Ottaviani and Prat 2001), job market matching (Inderst 2005), and innovation funding (Bouvard 2014), among other topics.

MT's results have clearly had an applied impact. But their use has been hampered by several key assumptions that are often either implausible or hard to verify. Our contribution is to relax several of these assumptions under risk neutrality, thus permitting MT's framework to be more widely applied.

As a motivating example, we apply our results to the asset sale ("AS") game of DeMarzo, Frankel, and Jin (2020a, henceforth "DFJ"). In DFJ, a seller with one-dimensional private information about the values of $n > 1$ assets wishes to sell some or all of these assets to raise cash. For a variety of reasons that we will now explain, MT's original results cannot be applied directly to DFJ. Here the "RSW allocation" corresponds to the least-cost separating outcome of signaling models - also known as the Riley (1979) outcome - in which each successively higher type takes the best outcome for her that does not tempt any lower type to imitate her. In DFJ and in many other contexts, the RSW allocation is the only outcome to survive the Cho-Kreps Intuitive Criterion (Cho and Kreps 1987).

1. MT's main result - their Theorem 1 - relies on their Proposition 5, which assumes two strong properties:

    (a) MT [20, p. 11] assume the RSW allocation is not on the boundary of the feasible

[1] In an extension (MT, sec. 8), they consider the case of two-sided asymmetric information, while an earlier paper studies the simpler private values case (Maskin and Tirole 1990).

set. If action sets are bounded, this typically requires an Inada-like condition to prevent players from choosing points on the boundary. Such conditions often fail in risk-neutral settings. For instance, in DFJ's AS game, the RSW allocation is on the boundary of the feasible set.[2]

(b) MT [20, p. 5] assume the indifference curves of different types are never tangent. This is not a natural assumption in the asset sale context, where the value of some assets may be insensitive to a seller's private information.[3]

We prove MT's Proposition 5 (which appears below as Claim 8) without the above two assumptions. Our relaxation of assumption 1(b) relies on a mild monotonicity property: Assumption 1 below. Without Assumption 1, our results still hold under MT's assumption of nontangent indifference curves.

2. MT's Theorem 1 [20, p. 19] assumes that there exist beliefs of the agent, which put positive weight on each type of principal, for which the RSW allocation is interim efficient: there is no other allocation that is better for some type of the principal, no worse for any type of principal, and at least as good for the agent. This assumption lets MT rely on known properties of sequential equilibria (MT [20, p. 20]). This assumption can be hard to verify: we were unable to do so for DFJ's AS game. Hence, we prove MT's Theorem 1 without it. The key to our proof is to replace sequential equilibrium with the more general solution concept of correlated equilibrium (Aumann [2]) - a change that, in our setting, does not alter the set of equilibrium outcomes.

[2] In particular, the seller sells either all or none of all assets but one.

[3] For instance, suppose a seller holds a zero-coupon bond with a face value of 1 that is secured by an underlying cash flow $Y \in \{1,2\}$ about which she has private information. Let $q \in [0,1]$ be the share sold of the bond, p be the price paid, and δ be the seller's discount factor. Then the seller's payoff from the pair (q,p) is $p-\delta q$: the slope of her indifference curve in (q,p) space is δ, regardless of what she knows about Y. The indifference curves of different sellers thus coincide.

Our main result is that MT's results carry over to DFJ's asset sale game, for any number of assets and types. In particular, under the intuitive criterion, the RSW outcome is the unique outcome but there may exist other, nonintuitive outcomes. These results are shown without relying on the above three assumptions, but with the addition of Assumption 1.

While the above result is general, it does not indicate when nonintuitive equilibria exist and what they look like. To address this, we then focus on the 2x2 case: two assets and two types. In this setting, the RSW allocation is the unique equilibrium outcome if the high type seller is not too likely. Otherwise, there also exists a continuum of other possibilities, none of which are intuitive. However, many of these nontintuitive outcomes also involve separation and thus have the same general form as the RSW allocation. Indeed, the set of parameters for which pooling can occur is a proper subset of the set for which there are non-RSW separating allocations. In this sense, MT's approach seems to favor separation over pooling (although pooling, when it can occur, is Pareto optimal).

Subsequently, we apply our results for the 2x2 case to DFJ's ex-post security design game (DFJ section 3.3-3.4). In this game, a seller seeks to sell rights to a cash flow about which the seller has private information. The seller sees her information and designs and sells a single security whose payment to the investor is a monotone function of the cash flow.[4]

While there can be multiple equilibria using MT's approach, under DFJ's Hazard Rate Ordering property they all possess the qualitative properties of the RSW outcome: the seller sells standard debt with a face value that is nonincreasing in her type. However, in contrast to the RSW outcome, if the high type is sufficiently likely then there are outcomes in which the low type seller is paid more than the fair value of her security. This loosens her incentive compatibility constraint, which lets the high type set a higher face value of her security relative to the RSW outcome. Indeed, if his prior probability is high enough, this face value can equal that of the low type: the two types can sell the same standard debt

[4]Let $S(y)$ denote the payment received by the security holder for each realization y of the underlying cash flow. The security is monotone if $S(0) = 0$ and both $S(y)$ and $y - S(y)$ are nondecreasing in y.

security. Again, none of these non-RSW outcomes are intuitive.

The rest of this paper is as follows. Section 1.1 reviews the payoffs and information structure of DFJ's AS game. Section 1.2 shows that MT's results carry over to DFJ's setting. Section 1.3 explicitly solves the case of two types and two assets and section 1.4 applies these results to security design. Omitted proofs appear in section 1.5.

1.1 DFJ's Asset Sale Game

In DFJ's Asset Sale (AS) game, a privately informed seller is endowed with a fixed portfolio of assets. Selling all her assets is efficient as the seller is relatively impatient. However, as she has private information about her assets' worth, she may have an incentive to retain some shares in order to signal to investors that she is optimistic about the value of her portfolio.

DFJ model the sale of assets as a signaling game. The seller offers a quantity of each asset for sale. Investors use this information to form beliefs about the seller's type. They then assign a price to each asset. DFJ prove that the unique intuitive outcome of the game is the RSW allocation. In this allocation, the lowest-type seller offers her entire portfolio for sale. Successively higher types retain shares of one or more of their assets so as to signal optimism and thus obtain a higher price per share.

DFJ also derive sufficient conditions for a liquidity "pecking-order" in which less informationally-sensitive assets are sold first. If the seller's assets are secured by a common cash flow then, under mild assumptions, these are the seller's more senior assets. If, moreover, the seller can design securities that use her assets as collateral, she optimally pools her assets and sells a single standard debt security whose face value is decreasing in her information.

We now set out the players, payoffs, and information structure of DFJ's AS game. There are a seller and a continuum of investors; all players are risk-neutral and fully rational. The seller owns a portfolio of n assets represented by the vector $a \in \Re^n_+$, where $a_i > 0$ is the number of shares she owns of asset i. A holder of one share of asset is entitled to the random future payout F_i. The seller has private information about these payouts, which is summarized by her type $t \in \{0, 1, ..., T\}$. Conditional on the seller's type, asset i has an expected payout per share of $f_i(t) = E[F_i|t]$. Let $F = (F_i)_{i=1}^n$ denote the vector of random asset payouts and let $f(t) = (f_i(t))_{i=1}^n$ denote the vector of expected payouts conditional on t. The investors have a common, positive prior over the seller's type t.

DFJ assume that (a) the seller's information can be ordered so that higher types t are more optimistic about the expected payout of each asset; (b) an asset's expected conditional payout is never negative (e.g., because of limited liability); and (c) even the most pessimistic seller thinks that her portfolio has a positive expected value:

Assumption 1. *For $t > s$, $f(t) \geq f(s)$. In addition, $f(0) \geq 0$ and $af(0) > 0$.*

The seller is less patient than the investors: she discounts future cash flows at some rate $\delta \in (0, 1)$, while investors' discount factor is normalized to one. Suppose the issuer sells a quantity q_i of each asset i for the price p_i; let $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ be the price and quantity vectors. Then if the issuer's type is t, her payoff is $q \cdot (p - \delta f(t))$ while investors collectively get $q \cdot (f(t) - p)$.

1.2 Applying MT's Approach to DFJ

We now explain the approach of MT and how it will be applied to the AS model described in section 1.1. We generalize MT by permitting any correlated equilibrium (Aumann [2]) in the last stage of the game, rather than restricting to Nash equilibrium with a public coordinating device. This permits us to prove a version of MT's Theorem 1 that does not require that the RSW allocation be interim efficient with respect to positive beliefs - a condition that we cannot verify in the AS model, and that may be hard to verify in other contexts as well. While in principal the switch to correlated equilibrium might expand the set of equilibrium outcomes, in practice it does not.[5]

[5] In a given game, there can be correlated equilibria that are not Nash equilibria - even if the latter include a public coordinating device (Aumann [2, pp. 70-71]). However, in MT, the principal has enough flexibility

As applied to DFJ's AS game, MT's procedure is as follows. The players consist of a principal and an agent. The principal corresponds to our seller. The agent is best interpreted as a single risk-neutral investor with deep pockets, who replaces our continuum of investors. The payoffs of principal and agent depend on three variables q, ρ, and t, defined as follows.

1. q is a vector of observable and verifiable actions, belonging to a compact, convex set $Q \subset \Re^n$. In our AS model, q is the vector $(q_1, ..., q_n)$ of quantities that the principal sells to the agent, and
$$Q = [0, a_1] \times \cdots \times [0, a_n] \tag{1}$$
is the set of all feasible quantity vectors.

2. $\rho \in \Re$ is a monetary transfer from the agent to the principal. In the AS model it is the revenue $\rho = pq$ that the investors pay the seller. To ensure compactness, we will restrict ρ to lie in $[-\overline{\rho}, \overline{\rho}]$ for some arbitrarily large (but finite) constant $\overline{\rho}$.

3. $t \in \Upsilon = \{0, ..., T\}$ denotes the principal's private information or type, with prior probability distribution $\Pi^{\cdot} = \{\Pi^t\}_{t=0}^T \in \Delta^T$ (the standard T simplex). This is just as in the AS model.

MT define an outcome o as an action-transfer pair (q, ρ):[6]

Definition 1. An *outcome* o is a pair $(q, \rho) \in O$ where
$$O = Q \times \Re \subset \Re^{n+1} \tag{2}$$

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in designing the game that the Nash restriction does not shrink the set of outcomes relative to correlated equilibrium.

[6]In MT, an outcome is denoted μ. We use o since, in our setting, μ is used to denote beliefs. Moreover, MT work with random outcomes, which are probability distributions over pairs (q, ρ). However, as our players are risk-neutral, they care only about the expected action Eq and transfer $E\rho$. Hence there is no gain in generality from working with random outcomes: any result concerning an outcome (q, ρ) will hold also for any random outcome whose expectation is (q, ρ).

is the set of all possible outcomes. Let q^o and ρ^o denote the action and transfer, respectively, of the outcome o: that is, $o = (q^o, \rho^o)$.

The principal and agent have Neumann-Morgenstern (vNM) utility functions $V^t(o)$ and $U^t(o)$, respectively, over outcomes o. As the notation indicates, both payoff functions also depend on the principal's type t. In our AS model, these payoff functions are[7]

$$V^t(o) = \rho^o - \delta q^o f(t) \text{ and } U^t(o) = q^o f(t) - \rho^o. \tag{3}$$

Let

$$o_0 = \left(\overrightarrow{0}, 0\right) \tag{4}$$

denote the reservation outcome in which no assets or cash are transferred and thus, by (3), each player gets a realized payoff of zero. Further define:

Definition 2. An *allocation* is a menu $o^{\cdot} = \{o^t\}_{t=0}^{T} \in A$ of outcomes, one for each type of principal, where

$$A = O^{T+1} \subset \Re^{(n+1)(T+1)} \tag{5}$$

is the set of all possible allocations.

Definition 3. An allocation $o^{\cdot}$ is *incentive-compatible* if, for all types s and t, $V^t(o^t) \geq V^t(o^s)$.

Definition 4. An allocation $\widehat{o}^{\cdot}$ *Pareto-dominates* an allocation $o^{\cdot}$ if $V^t(\widehat{o}^t) \geq V^t(o^t)$ for each type t, where the inequality holds strictly for some t.

Definition 5. Let S be a set of allocations. An *allocation* $o^{\cdot}$ is Pareto optimal in S if $o^{\cdot} \in S$ and $o^{\cdot}$ is not Pareto-dominated by any other allocation in S.

On seeing her type, the principal proposes a mechanism m: a set of actions for each player and a map from action pairs to outcomes. The agent may reject the mechanism, in which case the no-trade outcome o_0 is played. Without loss of generality, we incorporate this option as a special action in the mechanism that always leads to the no-trade outcome:

[7]In MT, the variables q, ρ, and t are called y, t, and i, respectively.

Definition 6. A *mechanism m* consists of a finite set S_m^P of actions for the principal, a finite set of actions S_m^A for the agent, and, for each pair of actions $\left(s^P, s^A\right) \in S_m^P \times S_m^A$, an outcome $o_m^{s^P, s^A} \in O$, such that the agent's action set S_m^A contains a special action s_0^A that prevents trade: if the agent selects s_0^A then, for any principal's action $s^P \in S_m^P$, the outcome $o_m^{s^P, s_0^A}$ selected by m is the no-trade outcome o_0.

Let M denote the set of mechanisms m. Eliminating an explicit "accept/reject" stage lets us specify MT's game simply as follows.

Stage 1. The principal privately observes her type t and announces a mechanism m in the set M.

Stage 2. The principal and agent simultaneously choose actions $s^P \in S_m^P$, $s^A \in S_m^A$. Their realized payoffs are $V^t\left(o_m^{s^P, s^A}\right)$ and $U^t\left(o_m^{s^P, s^A}\right)$, respectively.

Following the principal's choice of a mechanism m, the agent forms interim beliefs $\mathring{\Pi}^{\cdot}(m)$ about the principal's type, where $\mathring{\Pi}^t(m)$ is the probability that she assigns to type t.

Our solution concept for stage 2 is an adaptation of strategic form correlated equilibrium (SFCE) (Forges [10]) to our setting. Why must SFCE be altered? An SFCE is defined for a static, one-shot game in which a player's beliefs about her opponents' types are given by an exogenous prior distribution. Hence, if player 1 believes player 2 has probability zero of being some type t, then type t indeed cannot occur so the SFCE may assign a suboptimal action to type t. This feature is not suitable in our setting, where the agent's beliefs are given by her interim beliefs $\mathring{\Pi}^{\cdot}(m)$. In particular, if the mechanism m is unexpected, the agent's interim beliefs are arbitrary and so may assign zero probability to a type t who has positive ex-ante probability - even if it was type t who chose m! Hence, a SFCE would let a type t deviate to some mechanism m and then play suboptimally in the continuation game, thus violating the principle of sequential rationality.

We fix this by requiring optimality type-by-type for the principal, regardless of the agent's beliefs. More precisely, let

$$\Phi_m = \Delta^{\left|\left(S_m^P\right)^{T+1} \times S_m^A\right|} \tag{6}$$

be the set of distributions of pure action profiles (where the principal's action $s_{\cdot}^{P} \in \left(S_m^P\right)^{T+1}$ specifies the action $s_t^P \in S_m^P$ to be taken by each type t).

Definition 7 ($CEIG_m^{\mathring{\Pi}^{\cdot}}$)**.** A distribution $\pi_m \in \Phi_m$ is a *correlated equilibrium of the incomplete-information game with mechanism m and interim beliefs* $\mathring{\Pi}^{\cdot}$, or "$CEIG_m^{\mathring{\Pi}^{\cdot}}$", if and only if

1. each type of principal is willing to play each action when told to do so:

$$\sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) V^t\left(o_m^{s_t^P, s^A}\right) \geq \sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) V^t\left(o_m^{\hat{s}_t^P, s^A}\right) \text{ for any } t, s_{\cdot}^P, \hat{s}_{\cdot}^P \in \left(S_m^P\right)^{T+1}; \tag{7}$$

2. and the agent is willing to play each action when told to do so:

$$\sum_{s_{\cdot}^P \in \left(S_m^P\right)^{T+1}} \pi_m\left(s_{\cdot}^P, s^A\right) u_{m,\mathring{\Pi}^{\cdot}}^A\left(s^A, s_{\cdot}^P\right) \geq \sum_{s_{\cdot}^P \in \left(S_m^P\right)^{T+1}} \pi_m\left(s_{\cdot}^P, s^A\right) u_{m,\mathring{\Pi}^{\cdot}}^A\left(\hat{s}^A, s_{\cdot}^P\right) \text{ for any } s^A, \hat{s}^A \in S_m^A \tag{8}$$

   where

$$u_{m,\mathring{\Pi}^{\cdot}}^A\left(s_{\cdot}^P, s^A\right) = \sum_{t=0}^{T} \mathring{\Pi}^t U^t\left(o_m^{s_t^P, s^A}\right) \tag{9}$$

   is the agent's expected payoff from the pure strategy profile $\left(s_{\cdot}^P, s^A\right)$ under the beliefs $\mathring{\Pi}^t$.

Definition 8. A *profile* Σ *of the full game* is a triplet $\left(p_{\cdot}^{\cdot}, \mathring{\Pi}^{\cdot}(\cdot), \pi_{\cdot}\right)$ where p_m^t is the probability that the principal will choose the mechanism $m \in M$ given her type t; $\mathring{\Pi}^t(m)$ is the agent's posterior probability that the principal's type is t given that she chose m; and $\pi_m\left(s_{\cdot}^P, s^A\right)$ is the probability, conditional on the principal choosing mechanism m, that the pure action profile $\left(s_{\cdot}^P, s^A\right)$ will be played in stage 2.

Definition 9. An *equilibrium of the full game* is a profile $\Sigma = \left(p_{\cdot}^{\cdot}, \mathring{\Pi}^{\cdot}(\cdot), \pi_{\cdot}\right)$ that satisfies the following three conditions:

1. for any mechanism m (including those not chosen in equilibrium), π_m is a $CEIG_m^{\mathring{\Pi}(m)^{\cdot}}$;

2. Bayes's Rule is used whenever possible: for any mechanism m that some type of principal chooses in equilibrium, the agent's posterior beliefs are determined by Bayes's Rule:

$$\mathring{\Pi}^t(m) = \frac{\Pi^t p_m^t}{\sum_{s=0}^T \Pi^s p_m^s}; \text{ and} \tag{10}$$

3. for each type t, each mechanism m in the support of $p_\cdot^t$ is an optimal choice for the type-t principal given the map $\pi_\cdot$ from mechanisms to correlated equilibria.

1.2.1 Results

For any mechanism m and beliefs $\mathring{\Pi}^\cdot \in \Delta^T$, let

$$\phi_m\left(\mathring{\Pi}^\cdot\right) \subset \Phi_m \tag{11}$$

be the set of associated correlated equilibria of stage 2,[8] and let

$$\mathring{U}_m(\pi_m) = \sum_{s_\cdot^P \in (S_m^P)^{T+1}} \sum_{s^A \in S_m^A} \pi_m\left(s_\cdot^P, s^A\right) u_m^A\left(s_\cdot^P, s^A\right) \tag{12}$$

and

$$\mathring{V}_m^t(\pi_m) = \sum_{s_\cdot^P \in (S_m^P)^{T+1}} \sum_{s^A \in S_m^A} \pi_m\left(s_\cdot^P, s^A\right) u_{m,t}^P\left(s_\cdot^P, s^A\right) \tag{13}$$

be the resulting payoffs of the agent and of the type-t principal, respectively, when they play π_m in the mechanism m. Let

$$\psi_m\left(\mathring{\Pi}^\cdot\right) = \left\{\left(\mathring{V}_m^\cdot(\pi_m), \mathring{U}_m(\pi_m)\right) : \pi_m \in \phi_m\left(\mathring{\Pi}^\cdot\right)\right\} \subset \Re^{T+1} \times \Re \tag{14}$$

denote the set of payoff vectors $\left(\mathring{V}^\cdot, \mathring{U}\right)$ that can occur in correlated equilibria π_m given the mechanism m and interim beliefs $\mathring{\Pi}^\cdot$.

Claim 1. For any mechanism m and interim beliefs $\mathring{\Pi}^\cdot$, the agent's expected payoff $\mathring{U}_m(\pi_m)$ (evaluated using $\mathring{\Pi}^\cdot$) in any correlated equilibrium $\pi_m \in \phi_m\left(\mathring{\Pi}^\cdot\right)$ is nonnegative.

[8]The set Φ_m of distributions over pure action profiles is defined in (6).

Proof. The agent can get zero by always "rejecting" in stage 2; hence, her expected stage-2 continuation payoff $\mathring{U}_m(\pi_m)$ in any correlated equilibrium π_m in $\phi_m\left(\mathring{\Pi}^{\cdot}\right)$ must be nonnegative. □

A simple but important type of mechanism m is a Direct Revelation Mechanism DRM.

Definition 10. A *Direct Revelation Mechanism (DRM)* is a mechanism m with the following properties. The agent's action set S_m^A consists of two actions: accept or reject. The principal's action set S_m^P is just her type space Υ. If the agent rejects then, for any action "t" of the principal, the reservation outcome o_0 is implemented: each player gets a payoff of zero. If the agent accepts and the principal chooses "t", then the DRM implements some prespecified outcome o^t.

For the principal, choosing the action $t \in \Upsilon$ is interpreted as claiming to be of type t. A DRM m is fully specified as the allocation $o^{\cdot} = \{o^t\}_{t=0}^{T}$ that is implemented when the agent accepts and the principal truthfully reports her type. Thus, we will also refer to a DRM as the allocation $o^{\cdot}$ that it is designed to implement. We will also say that the principal chooses the outcome o^t in stage 2; this means that the principal reports the type "t".

A DRM equilibrium $o^{\cdot}$ is an equilibrium in which the allocation $o^{\cdot}$ is implemented as a DRM. More precisely:

Definition 11. A *DRM equilibrium $o^{\cdot}$* is an equilibrium in which, as long as there has been no prior deviation, each type of principal chooses the same DRM $o^{\cdot}$ in stage 1 while, in stage 2, the agent chooses "accept" and the principal of each type t truthfully reports her type: the allocation $o^{\cdot}$ is implemented.

Fix an equilibrium $\Sigma = \left(p_{\cdot}^{\cdot}, \mathring{\Pi}^{\cdot}(\cdot), \pi_{\cdot}\right)$. By risk neutrality, Σ gives the principal (resp., agent)a type-contingent expected payoff of $V^t\left(o_\Sigma^t\right)$ (resp., $U^t\left(o_\Sigma^t\right)$) when the principal's type is t, where[9]

$$o_{\Sigma}^{t}=\sum_{m\in M}p_{m}^{t}\sum_{s_{\cdot}^{P}\in\left(S_{m}^{P}\right)^{T+1}}\sum_{s^{A}\in S_{m}^{A}}\pi_{m}\left(s_{\cdot}^{P},s^{A}\right)o_{m}^{s_{t}^{P},s^{A}} \tag{15}$$

is the expected outcome for this type. We will refer to $o_{\Sigma}^{\cdot}$ as the expected allocation of Σ.

Definition 12. The allocation $o^{\cdot}$ is an *equilibrium allocation* if there exists an equilibrium Σ whose expected allocation $o_{\Sigma}^{\cdot}$ is $o^{\cdot}$.

We next show that for every equilibrium Σ, there is a DRM equilibrium that implements the expected allocation $o_{\Sigma}^{\cdot}$ of Σ. Since a DRM equilibrium is also an equilibrium, this means that $o^{\cdot}$ is an equilibrium allocation if and only if there is a DRM equilibrium that implements it. This will help us characterize the set of equilibrium allocations.

Claim 2. For any equilibrium Σ, there is a DRM equilibrium $o_{\Sigma}^{\cdot}$.

Proof. Section 1.5. □

The following definition is from MT [20, p. 10].

Definition 13. An allocation $\overline{o}^{\cdot}$ is *weakly interim efficient (WIE)* if and only if (a) it is incentive compatible -

$$V^{t}\left(\overline{o}^{t}\right)\geq V^{t}\left(\overline{o}^{s}\right)\text{ for all types }t\text{ and }s \tag{16}$$

- and there exists no allocation $o^{\cdot}$ that (a) Pareto dominates $\overline{o}^{\cdot}$:

$$V^{t}\left(o^{t}\right)\geq V^{t}\left(\overline{o}^{t}\right)\text{ for all types }t\text{, strictly for some }t; \tag{17}$$

(b) is incentive-compatible:

$$V^{t}\left(o^{t}\right)\geq V^{t}\left(o^{s}\right)\text{ for all types }t\text{ and }s; \tag{18}$$

[9] For any outcomes $o=(q,\rho)$ and $o'=(q',\rho')$, and any probability π, the expected outcome $\pi o+(1-\pi)o'$ refers to the outcome in which the portfolio $\pi q+(1-\pi)q'$ is transferred in return for the payment $\pi\rho+(1-\pi)\rho'$ for sure. By risk neutrality, this deterministic outcome is payoff-equivalent to a lottery in which outcome o occurs with probability π and outcome o' occurs with probability $1-\pi$.

and (c) gives the agent at least as high a payoff as $\overline{o}^{\cdot}$ does for each type of principal:

$$U^t\left(o^t\right) \geq U^t\left(\overline{o}^t\right) \text{ for all types } t. \tag{19}$$

MT [20, p. 10] use also the following equivalent formulation.

Claim 3. An allocation $\overline{o}^{\cdot}$ is WIE if and only if it solves the following maximization problem for some vector of positive weights $\{w^t\}_{t=0}^{T} \in \Re_{++}^{T+1}$:

Program I. $\max_{o^{\cdot}} \sum_{t=0}^{T} w^t V^t\left(o^t\right)$ subject to (18) and (19).

Proof. Section 1.5. □

Recall that there is no trade in the reservation allocation $o_0^{\cdot}$: each player gets zero.

Definition 14. An allocation $\widehat{o}^{\cdot}$, with payoffs

$$\widehat{V}^t \stackrel{d}{=} V^t\left(\widehat{o}^t\right) \text{ for all } t, \tag{20}$$

is an *RSW allocation* if and only if, for all t, it solves:

Program II^t. $\widehat{V}^t = \max_{o^{\cdot}} V^t\left(o^t\right)$ subject to

$$V^s\left(o^s\right) \geq V^s\left(o^r\right) \text{ for all types } r,s; \tag{21}$$

$$U^s\left(o^s\right) \geq U^s\left(o_0^s\right) \text{ for all types } s. \tag{22}$$

Claim 4. There exists an RSW allocation $\widehat{o}^{\cdot}$. Any RSW allocation $\widehat{o}^{\cdot}$ is WIE.

Proof. The proof is the same as in MT [20, p. 11]. □

For convenience, we now restate the Recursive Linear Program (RLP). Recall that Q, defined in (1), is the set of all quantity vectors q.

RLP Let $Q_0 = Q$ and, for each $t = 1, \ldots, T$, let

$$Q_t = \{q \in Q : \text{ for all } s < t,\ q\left[f(t) - \delta f(s)\right] \leq u^*(s)\} \tag{23}$$

be the set of all quantity vectors that can be assigned to type t which, if accompanied by revenue $\rho = qf(t)$, do not tempt any lower type to imitate type t. For each $t = 0, ..., T$, let

$$q^*(t) \in \arg\max_{q \in Q_t} [qf(t)], \tag{24}$$

$$\rho^*(t) = q^*(t) f(t), \text{ and} \tag{25}$$

$$u^*(t) = \rho^*(t) - \delta q^*(t) f(t). \tag{26}$$

RLP is thus a maximization program that produces a seller's payoff function u^*, a quantity function q^*, and a revenue function ρ^*, where u^* and ρ^* are unique by DFJ's PROPOSITION 1.[10] Any solution to RLP yields an associated allocation $o^{\cdot}$ where $o^t = (q^*(t), \rho^*(t))$. In any such allocation, the payoff $q^*(t) f(t) - \rho^*(t)$ of investors is zero by (25).

Crucially, RSW and RLP are equivalent in our setting, in the following sense.

Claim 5. There is a one-to-one correspondence between RSW allocations and solutions to RLP, which is as follows.

1. Let $\widehat{o}^{\cdot}$ be an RSW allocation. Then the agent's payoff $U^t(\widehat{o}^t)$ is identically zero and the functions $q^*(t) = q^{\widehat{o}^t}$, $\rho^*(t) = \rho^{\widehat{o}^t}$, and $u^*(t) = V^t(\widehat{o}^t)$ solve RLP.
2. Let (u^*, ρ^*, q^*) solve RLP. Then the allocation $o^{\cdot}$ with $o^t = (q^t, \rho^t) = (q^*(t), \rho^*(t))$ is an RSW allocation.

Proof. Section 1.5. □

The following definition is from MT [20, p. 10]. Roughly speaking, an allocation is interim efficient with respect to some beliefs if there is no other allocation that is better for some type of the principal, no worse for any type of principal, and at least as good for the agent under the given beliefs.

---

[10]Multiple quantity functions q^* may solve RLP. E.g., if the two assets i and j have the same conditional expected values (if $f_i(t) = f_j(t)$ for all t) then the players care only about their total holdings of the two assets combined. Hence, RLP cannot tell us the number of shares a seller of a given type t will sell of each asset, although it may pin down the sum $q_i(t) + q_j(t)$.

Definition 15. An allocation $\overline{o}^{\cdot}$ is *interim efficient (IE) relative to beliefs* $\Pi^{\cdot}$ if and only if (1) it is incentive compatible -

$$V^t\left(\overline{o}^t\right) \geq V^t\left(\overline{o}^s\right) \text{ for all types } t \text{ and } s \tag{27}$$

- and (2) there exists no allocation $o^{\cdot}$ that (a) Pareto dominates $\overline{o}^{\cdot}$:

$$V^t\left(o^t\right) \geq V^t\left(\overline{o}^t\right) \text{ for all types } t\text{, strictly for some } t; \tag{28}$$

(b) is also incentive-compatible:

$$V^t\left(o^t\right) \geq V^t\left(o^s\right) \text{ for all types } t \text{ and } s; \tag{29}$$

and (c) is at least as good for the agent under the beliefs $\Pi^{\cdot}$:

$$\sum_{t=0}^{T} \Pi^t U^t\left(o^t\right) \geq \sum_{t=0}^{T} \Pi^t U^t\left(\overline{o}^t\right). \tag{30}$$

That is, MT [20, p. 10] use also the following equivalent formulation.

Claim 6. An allocation $\overline{o}^{\cdot}$ is IE with respect to the beliefs $\Pi^{\cdot}$ if and only if it solves the following maximization problem for some vector of positive weights $\{w^t\}_{t=0}^{T} \in \Re_{++}^{T+1}$:

Program I. $\max_{o^{\cdot}} \sum_{t=0}^{T} w^t V^t\left(o^t\right)$ subject to (29) and (30).

Proof. Section 1.5. □

The following is MT's Proposition 3, specialized to our model.

Claim 7. For any WIE allocation $\overline{o}^{\cdot}$, the set of beliefs $\Pi\left(\overline{o}^{\cdot}\right)$ for which $\overline{o}^{\cdot}$ is IE is nonempty and convex.

Proof. Section 1.5. □

Theorem 1 in MT relies on the following claim (their Proposition 5). The proof in MT relies on their assumption that the indifferent curves of different types are never tangent. In our setting, two adjacent types may have identical preferences. In this case, their indifference surfaces in (q, ρ) space are identical. Hence, we use a different proof, which relies on Assumption 1.

Claim 8. In any equilibrium of the contract proposal game, the payoff of the type t principal is at least her RSW payoff $V^t\left(\widehat{o}^t\right)$.

Proof. Section 1.5. □

We now extend the Intuitive Criterion (Cho and Kreps [6]) to our setting. This criterion restricts the investor's beliefs when the principal deviates to a mechanism m that no type's strategy puts positive weight on: for all t, $p_m^t = 0$. Recall that Υ is the set of all types t and $\mathring{V}_m^t(\pi_m)$, defined in (13), is the type-t principal's payoff in the correlated equilibrium π_m of the mechanism m. For a fixed equilibrium $\Sigma = \left(p_{\cdot}^{\cdot}, \mathring{\Pi}^{\cdot}(\cdot), \pi_{\cdot}\right)$, the payoff V^t of a type-t principal is $V^t\left(o_\Sigma^t\right)$ where o_Σ^t is the expected outcome for this type, defined in (15). Consider the following restriction on the agent's interim beliefs $\mathring{\Pi}^{\cdot}(m)$ in the equilibrium Σ, following the principal's choice of mechanism m, where, for any set $S \subset \Upsilon$ of types,

$$\phi_m^S = \bigcup_{\mathring{\Pi}^{\cdot} \in \Delta^T:\ \mathrm{supp}\left(\mathring{\Pi}^{\cdot}\right) \subset S} \phi_m\left(\mathring{\Pi}^{\cdot}\right) \subset \Phi_m$$

denotes the set of correlated equilibria for interim beliefs $\mathring{\Pi}^{\cdot}$ whose support lies in S.[11]

Int_m^Σ Fix an equilibrium Σ. For any mechanism m, let $J_m \subset \Upsilon$ denote the (possibly empty) set of types t for whom the payoff of the principal in Σ exceeds her maximum payoff in all correlated equilibria of m for any interim beliefs of the agent:

$$J_m = \left\{ t \in \Upsilon : V^t\left(o_\Sigma^t\right) > \max_{\pi_m \in \phi_m^\Upsilon} \mathring{V}_m^t(\pi_m) \right\}. \tag{31}$$

Suppose there exists a type t in $\Upsilon - J_m$ of principal whose payoff in Σ is no higher than her minimum payoff in any correlated equilibrium of m that results from beliefs that put zero weight on any type in J_m:

$$V^t\left(o_\Sigma^t\right) \le \min_{\pi_m \in \phi_m^{\Upsilon - J_m}} \mathring{V}_m^t(\pi_m). \tag{32}$$

Then on seeing m, the agent's interim beliefs $\mathring{\Pi}^{\cdot}(m)$ put zero weight on any type in J_m: $\mathrm{supp}\left[\mathring{\Pi}^{\cdot}(m)\right] \subset \Upsilon - J_m$.

[11] The set $\phi_m\left(\mathring{\Pi}^{\cdot}\right)$ of correlated equilibria of m for beliefs $\mathring{\Pi}^{\cdot}$ is defined at equation (11).

Int_m^Σ holds automatically if m is offered by some type in equilibrium. Why? No type in J_m will choose m, and m is chosen with positive probability in Σ, so by Bayes's Rule the agent's interim beliefs on seeing m must put zero weight on J_m: Int_m^Σ holds.

We use Int_m^Σ to define intuitive beliefs, and an intuitive equilibrium.

Definition 16 (The Intuitive Criterion)**.** Let $\Sigma = \left(p_\cdot^\cdot, \mathring{\Pi}^\cdot(\cdot), \pi_\cdot\right)$ be an equilibrium. The beliefs $\mathring{\Pi}^\cdot(\cdot)$ and equilibrium Σ are intuitive if and only if Int_m^Σ holds for every mechanism $m \in M$.

Without strengthening Assumption 1, we cannot prove the following result, which is needed to apply MT's Theorem 1 (stated below as Proposition 1) to our model:

Claim 9. Let $\widehat{o}$ be an RSW allocation. Then $\widehat{o}$ is IE with respect to some positive beliefs $\widehat{\Pi}^\cdot \in \Re_{++}^{T+1}$.[12]

We will instead rely only on the following weaker property:

Claim 10. Let $\widehat{o}$ be an RSW allocation. Then $\widehat{o}$ is IE relative to some nonnegative beliefs $\widehat{\Pi}^\cdot \in \Re_{+}^{T+1}$.

Proof. Follows from Claims 21 and 4. □

We now state and prove our analogue to MT's Theorem 1. The result shows also that the RSW allocation is intuitive. Recall that $\widehat{V}^t$, defined in (20), is the RSW payoff of a principal of type t.

Proposition 1. *An allocation $o^\cdot$ is the expected allocation of some equilibrium if and only if (a) it is incentive compatible:*

$$V^t\left(o^t\right) \geq V^t\left(o^s\right) \text{ for all types } t \text{ and } s; \tag{33}$$

(b) it is "profitable in expectation" for the agent under the prior beliefs $\Pi^\cdot$:

$$\sum_{t=0}^{T} \Pi^t U^t\left(o^t\right) \geq \sum_{t=0}^{T} \Pi^t U^t\left(o_0^t\right); \tag{34}$$

[12] That is, $\Pi^t > 0$ for each type t.

and (c) it gives each type of principal at least her RSW payoff:

$$V^t\left(o^t\right) \geq \widehat{V}^t \text{ for each type } t. \tag{35}$$

Moreover, if $o^{\cdot}$ is an RSW allocation, there is an intuitive equilibrium with expected allocation $o^{\cdot}$.

Proof. Section 1.5. □

Proposition 1 shows that any RSW allocation $\widehat{o}^{\cdot}$ is intuitive: it is the expected allocation of an intuitive equilibrium. The converse also holds:

Proposition 2. *Any expected allocation of an intuitive equilibrium is an RSW allocation.*

Proof. Section 1.5. □

Hence, the sets of intuitive and RSW allocations coincide. Together with Claim 5, this establishes that the set of intuitive allocations of the above game coincides with the set of solutions to RLP which, in turn, are the set of intuitive outcomes of our AS game.

We have shown that under the intuitive criterion, the results of our AS game are robust to the choice of mechanism: it does not matter whether the assets are sold using our procedure or that of MT. However, without the intuitive criterion, MT's approach may permit additional outcomes by Proposition 1.

1.3 The MT Approach: 2x2 Case

We next apply the results of section 1.2 to the case of two assets $i = 1,2$ and two types $t = 1,2$. Recall that $f_i(t)$ is the expected payout of asset $i = 1,2$ conditional on the seller's type being $t = 1,2$. In order to apply the results in section 4 of DFJ's online appendix (DeMarzo, Frankel, and Jin 2020b, henceforth "DFJ2"), we assume generic parameters.[13] Thus, by Assumption 1 (monotonicity),

$$f_i(2) > f_i(1) > 0 \text{ for each asset } i. \tag{36}$$

[13] We also start the type numbering at $t = 1$ as in DFJ2 rather than at $t = 0$ as in section 1.2.

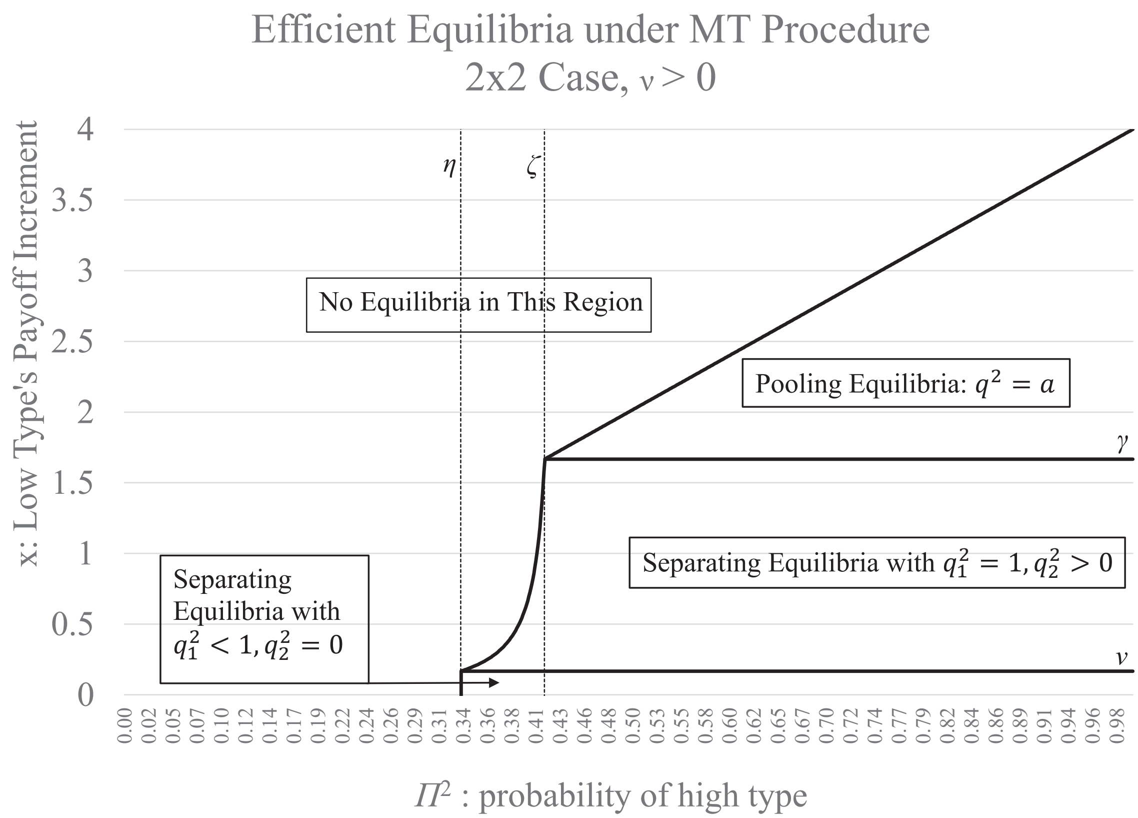

Figure 1: the case $\nu > 0$.

Swapping asset indices if needed, we can also assume w.l.o.g. that

$$f_2(2)/f_2(1) > f_1(2)/f_1(1). \tag{37}$$

That is, an increase in the seller's type raises the expected value of asset 2 proportionally more than that of asset 1: IIS holds. Finally, we normalize the number of shares of each asset to one: the seller's endowment vector a is $(1,1)$.

We first preview our results graphically. Figure 1 illustrates the case in which case the type 2 seller retains all of asset 2 in the RSW allocation.[14] Figure 2 illustrates the case in which the type 2 seller sells all of asset 1 in the RSW allocation. We now explain these figures.

[14]The notation ν, η, ζ, and γ will be defined later.

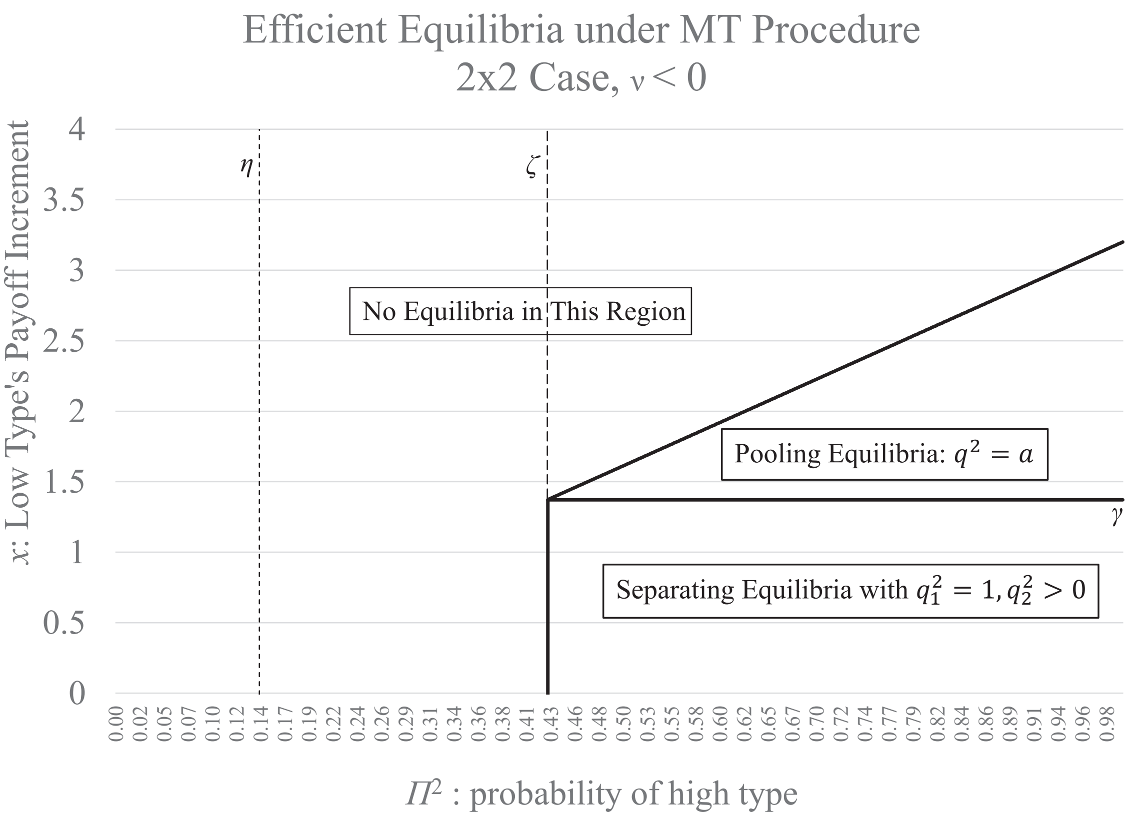

Figure 2: the case $\nu \leq 0$.

In any equilibrium allocation $o^{\cdot}$, let x denote the difference between the low type's equilibrium payoff $V^1(o^1)$ and her RSW payoff $\widehat{V}^1$. We show that in any equilibrium allocation other than the RSW allocation, x is positive (Claims 13 and 14): the low type must improve on her RSW payoff. The low type's payoff increment x is depicted on the vertical axis of Figures 1 and 2, while the prior probability Π^2 of the high type appears on the horizontal axis.

Let S_x denote the set of equilibrium allocations that raise the low type's payoff by exactly $x > 0$ vis-a-vis the RSW allocation. For simplicity, we focus on the set $\overline{S}_x$ of *efficient* elements of S_x: the elements in S_x that give the greatest social welfare, which we define as the sum of the unconditional expected payoffs of seller and agent. If S_x is nonempty, then $\overline{S}_x$ is a singleton.[15]

This unique efficient allocation has two key properties. First, the low type sells her whole portfolio but gets more than the fair value of this portfolio: $\rho^1 > af(1)$. This has the effect of loosening her incentive compatibility constraint, so the high type can sell more of her portfolio. In this way, more gains from trade are realized than in the RSW allocation so welfare is higher. Second, as in the RSW allocation, the high type sells all of asset 1 before she sells any of asset 2.[16]

These properties allow us to divide the efficient allocations (over all possible type-1 payoff increments x) into three categories, all of which appear in Figure 1.[17] For a range of payoff increments x there are pooling allocations, in which the high type also sells her entire portfolio: $q^2 = a$. For an intermediate range of x, there are separating allocations in which the high type sells all of asset 1 and some of asset 2 ($q_1^2 = 1$ and $q_2^2 > 0$). Finally, in Figure 1 there is a low range of increments x for which there are separating allocations in which the high type sells some of asset 1 but none of asset 2 ($q_1^2 > 0$ and $q_2^2 = 0$).[18]

[15] See Claim 16 and Proposition 3.

[16] These are, respectively, properties P1 and P4 in Claim 15.

[17] These results appear in Claim 16 and Proposition 3.

[18] As shown, each type of allocation exists only if the prior probability Π^2 of the high type exceeds a

This third set of allocations is present in Figure 1 but not in Figure 2. Why? As noted, any efficient allocation yields higher welfare than the RSW allocation. And as also noted, the high type must sell all of asset 1 before any of asset 2, as in the RSW allocation. It follows that she cannot sell fewer shares of asset 1 than in the RSW allocation, as she cannot sell more sales of asset 2 to offset this. Finally, Figure 2 concerns the case in which the high type sells all of asset 1 in the RSW allocation. Hence, in this case, she must also sell all of asset 1 in any efficient allocation.

These figures show that in the 2x2 model with MT's sale procedure, pooling is less likely than separation. More precisely, for any prior probability Π^2 of the high type that permits pooling, there is a continuum of separating equilibria. But in Figure 1, there is also a positive-measure set of prior probabilities (the interval $\Pi^2 \in [\eta, \zeta]$) for which separating equilibria exist but pooling equilibria do not. These properties are indeed general, as shown in Claim 16 and Proposition 3 below.

The final result of this section, Claim 17, characterizes the set of globally efficient equilibrium allocations: those equilibrium allocations that give the highest social welfare for any payoff increment x. This set consists of the pooling equilibria depicted in Figures 1 and 2, together with the upper, curved boundary of the region of separating equilibria for Π^2 between η and ζ in Figure 1.[19]

To proceed, let us define the following set

Definition 17. Let S denote the set of allocations $o^{\cdot}$ that are incentive-compatible for the

threshold that is nondecreasing in x. Intuitively, the low type seller's gain x equals the agent's loss when she encounters this type. In order to accept a larger such loss x, the probability $1 - \Pi^2$ of encountering the low type must be lower.

[19]In stating this result, we do not mean to imply that the globally efficient equilibrium will be selected. The equilibrium selection literature has not provided any convincing rationale for this "Pareto criterion"; see, e.g., the skeptical discussion in Fudenberg and Tirole [11, pp. 20-22]. While we also focus on efficient equilibria (for a given payoff increment) in Claim 16 and Proposition 3, our purpose there is to not to make predictions but rather to shrink the equilibrium set to manageable size while still giving a flavor of the wide variety of outcomes that can occur.

principal:

$$V^1\left(o^1\right) \geq V^1\left(o^2\right) \text{ and} \tag{38}$$

$$V^2\left(o^2\right) \geq V^2\left(o^1\right); \tag{39}$$

that give the agent a nonnegative unconditional expected payoff:

$$U\left(o^{\cdot}\right) = \sum_{t=1}^{2} \Pi^t U^t\left(o^t\right) \geq 0 \tag{40}$$

where $\Pi^t > 0$ is the prior probability of type t; and that give each type of principal at least her RSW payoff:

$$V^1\left(o^1\right) \geq \widehat{V}^1 \text{ and} \tag{41}$$

$$V^2\left(o^2\right) \geq \widehat{V}^2. \tag{42}$$

Claim 11.

1. S is the set of equilibrium allocations $o^{\cdot}$.
2. S contains the RSW allocation $\widehat{o}^{\cdot}$.
3. S is compact.

Proof. Parts 1 and 2 follow from Proposition 1. As for part 3, each q^t lies in $[0,1]^2$ whence:

1. by (36), (41), and (42),

$$\rho^t \geq \delta q^t f(t) + \widehat{V}^t \geq \widehat{V}^t > -\infty \text{ for } t = 1,2; \tag{43}$$

2. by (40), $\sum_{t=1}^{2} \Pi^t \rho^t \leq \sum_{t=1}^{2} \Pi^t a f(t)$ whence, for $s = 1,2$,

$$\begin{aligned} \rho^s &\leq \left(\Pi^s\right)^{-1}\left[-\Pi^{3-s}\rho^{3-s} + \sum_{t=1}^{2} \Pi^t a f(t)\right] \\ &\leq \left(\Pi^s\right)^{-1}\left[-\Pi^{3-s}\widehat{V}^{3-s} + \sum_{t=1}^{2} \Pi^t a f(t)\right] < \infty \text{ by (43).} \end{aligned}$$

Hence, S is compact. □

In the next claim, we solve analytically for the RSW allocation and show it is unique. Under (36), the model we study is a special case of A2NM described in section 4 of DFJ2. Thus, the solution to RLP in this section is identical to that characterized by Claims 10 and 11 in DFJ2. And by Claim 5, for each type t, the principal's RSW payoff $\widehat{V}_t$ equals her RLP payoff $u^*(t)$ while the reservation payoff $U^t\left(o_0^t\right)$ of the agent is zero by (3) and (3). This implies the first part of the following claim. Following DFJ2, for any parameter $x \in [0,1]$, define

$$\Delta^x = (\Delta_1^x, \Delta_2^x) = f(2) - xf(1). \tag{44}$$

We will often write Δ and Δ_i as shorthand for Δ^δ and Δ_i^δ,respectively:

$$\Delta = (\Delta_1, \Delta_2) = f(2) - \delta f(1) \text{ where } \Delta_i = f_i(2) - \delta f_i(1). \tag{45}$$

The following facts will be useful.

$$\Delta_2^1 \Delta_1 - \Delta_1^1 \Delta_2 = (1-\delta)\left[f_1(1) f_2(2) - f_2(1) f_1(2)\right] > 0 \text{ by (37);} \tag{46}$$

$$\Delta_i - \delta \Delta_i^1 = (1-\delta) f_i(2) > 0 \text{ by (36);} \tag{47}$$

Also define[20]

$$\gamma = \delta\left(a - \widehat{q}^2\right)\Delta^1 \in \left(0, a\Delta^1\right) \tag{48}$$

and[21]

$$\nu = \gamma - \delta \Delta_2^1 < \gamma, \tag{49}$$

as well as

$$\eta = \delta \Delta_1^1 / \Delta_1 \tag{50}$$

and

$$\zeta = \gamma / a\Delta_1. \tag{51}$$

[20]The second equality holds by (63) and the set inclusion by (36), (55), and (56).

[21]The inequality holds by (36).

Claim 12. There is a unique RSW allocation $\widehat{o} = \left\{ \left(\widehat{\rho}^1, \widehat{q}^1\right), \left(\widehat{\rho}^2, \widehat{q}^2\right) \right\}$, with the following properties.

$$\widehat{q}^1 = a = (1,1); \tag{52}$$

$$\widehat{\rho}^1 = af(1); \tag{53}$$

$$\widehat{V}^1 = (1-\delta) af(1) = V^1\left(\widehat{o}^2\right) = \widehat{q}^2 \Delta; \tag{54}$$

$$\widehat{q}_1^2 = \min\left\{1, \frac{\widehat{V}^1}{\Delta_1}\right\} \in (0,1]; \tag{55}$$

$$\widehat{q}_2^2 = \max\left\{0, \frac{\widehat{V}^1 - \Delta_1}{\Delta_2}\right\} \in [0,1); \tag{56}$$

$$\widehat{\rho}^2 = \widehat{q}^2 f(2); \tag{57}$$

$$\widehat{V}^2 = (1-\delta)\widehat{q}^2 f(2). \tag{58}$$

The following properties also hold:

$$\widehat{V}^2 > V^2\left(\widehat{o}^1\right) = a\Delta; \tag{59}$$

$$\widehat{V}^1 - \widehat{V}^2 = \delta \widehat{q}^2 \Delta^1 \tag{60}$$

$$= \delta\left\{\min\left\{1, \frac{\widehat{V}^1}{\Delta_1}\right\}\Delta_1^1 + \max\left\{0, \frac{\widehat{V}^1 - \Delta_1}{\Delta_2}\right\}\Delta_2^1\right\}; \tag{61}$$

$$\widehat{V}^1 - V^2\left(\widehat{o}^1\right) = \delta a \Delta^1; \tag{62}$$

$$\widehat{V}^2 - V^2\left(\widehat{o}^1\right) = \gamma = \delta\left(a - \widehat{q}^2\right)\Delta^1 = \delta a \Delta^1 - \left(\widehat{V}^1 - \widehat{V}^2\right); \tag{63}$$

$$\nu = \delta \Delta_1^1 - \left(\widehat{V}^1 - \widehat{V}^2\right); \tag{64}$$

$$\widehat{q}_1^2 = \min\left\{1, \frac{\widehat{V}^1 - \widehat{V}^2}{\delta \Delta_1^1}\right\}; \tag{65}$$

$$\widehat{q}_2^2 = \max\left\{0, \frac{\widehat{V}^1 - \widehat{V}^2 - \delta\Delta_1^1}{\delta \Delta_2^1}\right\} = \max\left\{0, \frac{-\nu}{\delta \Delta_2^1}\right\}; \tag{66}$$

$$\widehat{V}^1 \lesseqqgtr \Delta_1 \text{ as } \widehat{V}^1 - \widehat{V}^2 \lesseqqgtr \delta \Delta_1^1 \text{ as } \nu \gtreqqless 0; \tag{67}$$

$$\widehat{V}^1 = a\Delta - a\Delta^1; \tag{68}$$

$$\zeta > \eta. \tag{69}$$

Proof. Section 1.5. □

An immediate consequence of Claim 12 is that in the RSW allocation, the type-2 seller retains all of asset 2 (resp., sells all of asset 1) if ν is positive (resp., negative):

Corollary 1. *If*

$$\nu > 0, \tag{70}$$

then

$$\widehat{V}^1 < \Delta_1 \tag{71}$$

and type 2 retains all of asset 2 in the RSW allocation:

$$\widehat{q}^2 = \left(\frac{\widehat{V}^1}{\Delta_1}, 0\right) = \left(\frac{\widehat{V}^1 - \widehat{V}^2}{\delta\Delta_1^1}, 0\right). \tag{72}$$

If instead

$$\nu \leq 0, \tag{73}$$

then

$$\widehat{V}^1 \geq \Delta_1 \tag{74}$$

and type 2 sells all of asset 1 in the RSW allocation:[22]

$$\widehat{q}^2 = \left(1, \frac{|\nu|}{\delta\Delta_2^1}\right). \tag{75}$$

Proof. Under (70), equation (71) follows from (67), while (72) is implied by (55), (56), (65), (66), and (71). If instead (73) holds, then (74) follows from (67), while (75) results from by (65), (66), and (57). □

We next consider elements of S in which each type of seller gets her RSW payoff. We show that any such allocation must coincide with the RSW allocation $\widehat{o}$, which is unique by Claim 12. The nontrivial part of the proof is to show that the agent's payoff must be zero not only ex-ante, but also ex-post.

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Claim 13. Let $o^{\cdot}$ be an equilibrium allocation in which each type of seller gets her RSW allocation payoff: $V^t\left(o^t\right) = \widehat{V}^t$ for $t = 1,2$. Then $o^{\cdot}$ is the unique RSW allocation $\widehat{o}^{\cdot}$.

Proof. Section 1.5. □

We now turn to elements of S in which some type of seller gets more than her RSW payoff, a property that we write formally:

P0 For some $s \in \{1,2\}$, $V^s\left(o^s\right) > \widehat{V}^s$.

In any such allocation, the agent strictly prefers (not) to buy from the high (low) type seller, while the low type seller gets more than her RSW payoff:

Claim 14. In any equilibrium allocation $o^{\cdot}$ with property P0, $U^1\left(o^1\right) < 0$, $U^2\left(o^2\right) > 0$, and $x = V^1\left(o^1\right) - \widehat{V}^1 > 0$.

Proof. Section 1.5. □

We now give a necessary and sufficient condition for there to exist an equilibrium allocation $o^{\cdot}$ in which the type 1 seller's payoff $V\left(o^1\right)$ exceeds her RSW payoff $\widehat{V}^1$ by exactly some given $x > 0$. We also characterize the efficient such allocation: the one that maximizes the joint surplus

$$J\left(o^{\cdot}\right) = \sum_{t=1}^{2} \Pi^t \left[V^t\left(o^t\right) + U^t\left(o^t\right)\right] = (1-\delta) \sum_{t=1}^{2} \Pi^t q^t f(t) \tag{76}$$

of seller and agent.[23] Let

$$S_x = \left\{o^{\cdot} \in S : V^1\left(o^1\right) = \widehat{V}^1 + x\right\}$$

[23] In MT [20], the agent is ignored in the definition of efficiency. In practice, by (40), this would restrict us to allocations in which her expected payoff is zero. However, such allocations generally do not maximize joint surplus. Intuitively, if the type 2 seller gets more than her RSW payoff, we can have her transfer $x = \varepsilon\left(a - q^2\right)$ more shares to the agent (for some small $\varepsilon > 0$) in return for an additional payment equal to type 1's opportunity cost $\delta x f(1)$. This raises joint surplus without tempting the type 1 seller to imitate; however, it also raises the agent's payoff and thus violates MT's notion of efficiency.

denote the set of equilibrium allocations $o^{\cdot}$ in which the type-1 seller gets exactly x more than her RSW payoff and let

$$\overline{S}_x = \{o^{\cdot} \in S_x : \forall \widehat{o^{\cdot}} \in S_x, J(\widehat{o^{\cdot}}) \leq J(o^{\cdot})\}$$

be the efficient frontier of S_x: the set of allocations in S_x that maximize the joint surplus J. We first show that each element of $\overline{S}_x$ satisfies four key properties in addition to P0:

Claim 15. Fix $x > 0$. If S_x is nonempty then so is $\overline{S}_x$. Moreover, any allocation $o^{\cdot}$ in $\overline{S}_x$ has the following four properties:

P1 The low type sells her entire portfolio ($q^1 = a$) in return for the payment $\rho^1 = af(1) + x$.

P2 The low type's IC constraint binds: $V^1(o^2) = V^1(o^1)$.

P3 The high type either sells her whole portfolio or gets her RSW payoff: either $q^2 = a$ or $V^2(o^2) = \widehat{V}^2$.

P4 The high type sells all of asset 1 before selling any units of asset 2: either $q_1^2 = 1$ or $q_2^2 = 0$.

We will now explicitly characterize the set $\overline{S}_x$ of efficient allocations for any given increase $x > 0$ in the type 1 seller's payoff vis-a-vis the RSW allocation. The notation (ρ^t, q^t) will refer to the outcome in a given allocation $o^{\cdot}$ that corresponds to type $t = 1, 2$. As property P4 pins down the outcome (ρ^1, q^1) of the type 1 seller, outcomes in $\overline{S}_x$ can differ only with respect to the type 2 seller's outcome (ρ^2, q^2).

We now turn to the formal results that underpin Figures 1 and 2. We first consider the subset

$$\overline{S}_x^{\text{pool}} = \{o^{\cdot} \in \overline{S}_x : q^2 = q^1 = a\}$$

of efficient allocations in which each type sells their entire portfolio. By P2, this implies equal transfers as well: $\rho^1 = \rho^2$. Hence, by P1, $\overline{S}_x^{\text{pool}}$ is the set of pooling outcomes.

Claim 16. Fix $x > 0$. The set $\overline{S}_x^{\text{pool}}$ is nonempty if and only if[24]

$$\Pi^2 \geq \zeta \tag{77}$$

and x lies in the interval

$$\left[\gamma, \Pi^2 a \Delta^1\right] \tag{78}$$

(which is nonempty by (77)). In this case, $\overline{S}_x^{\text{pool}}$ contains a single allocation, in which the agent pays $\rho^t = af(1) + x$ to each type t of seller in return for her whole portfolio.

Proof. Section 1.5. □

We now turn to the set

$$\overline{S}_x^{\text{sep}} = \left\{o^{\cdot} \in \overline{S}_x : q^2 \neq q^1 = a\right\}$$

of efficient allocations for which $q^2 \neq q^1$: the separating allocations. In such an allocation, P3 implies that the high type gets her RSW payoff: $V^2\left(o^2\right) = \rho^2 - \delta q^2 f(2) = \widehat{V}^2$, whence

$$\rho^2 = \delta q^2 f(2) + \widehat{V}^2. \tag{79}$$

Hence it suffices to solve for q^2, with the corresponding transfer ρ^2 given by (79). We do so in the following claim. Cases 1(a) and 1(b) correspond, respectively, to the lower and upper regions of separating equilibria in Figure 1, while case 2(a) relates to the single region of separating equilibria in Figure 2.

Proposition 3. *With respect to $\overline{S}_x^{sep}$, there are two cases.*

1. *Case 1. Suppose $\nu > 0$: in the RSW allocation, the type-2 seller retains all of asset 2. There are three subcases.*

   (a) *Case 1(a). If x lies in the interval*

$$(0, \nu] \tag{80}$$

[24]The notation ζ is defined in (51).

then $\overline{S}_x^{sep}$ is nonempty if and only if[25]

$$\Pi^2 \in [\eta, 1], \tag{81}$$

which has positive measure by (36): the probability of the high type cannot be too low. In this case, $\overline{S}_x^{sep}$ contains a single allocation, in which the type 2 seller retains all of asset 2:

$$q^2 = \left(\frac{\widehat{V}^1 - \widehat{V}^2 + x}{\delta \Delta_1^1}, 0 \right). \tag{82}$$

(b) *Case 1(b). If*

$$x \in [\nu, \gamma], \tag{83}$$

then $\overline{S}_x^{sep}$ is nonempty if and only if

$$\Pi^2 \geq \phi(x) \overset{d}{=} x \left[\Delta_1 - \widehat{V}^1 + (x - \nu) \frac{\Delta_2}{\delta \Delta_2^1} \right]^{-1}. \tag{84}$$

The function $\phi(x)$ is continuous and increasing,[26] *takes values in (0,1) when $x \geq \nu$, and satisfies $\phi(\nu) = \eta$, $\phi(\gamma) = \zeta$, and $\lim_{x \to \infty} \phi(x) = \delta \Delta_2^1 / \Delta_2 > \zeta$. If (84) holds, $\overline{S}_x^{sep}$ contains a single allocation, in which the type 2 seller sells all of asset 1:*

$$q^2 = \left(1, \frac{x - \nu}{\delta \Delta_2^1} \right). \tag{85}$$

Finally, the region R defined by (83) and (84) is the union of two regions. The first, region (i), is defined by the two conditions

$$\Pi^2 \in [\eta, \zeta] \tag{86}$$

and

$$x \in \left[\nu, \phi^{-1}\left(\Pi^2\right) \right], \tag{87}$$

[25] The notation η is defined in (50).

[26] Intuitively, $\phi(x)$ is increasing in $x$since, in order for the agent to participate when the type-1 agent's payoff increment x over her RSW payoff is higher, the agent must believe that the agent is more likely to be of type 2.

where

$$\phi^{-1}\left(\Pi^2\right) = \frac{\left[\delta\Delta_2^1\left(\Delta_1 - \widehat{V}^1\right) - \nu\Delta_2\right]\Pi^2}{\delta\Delta_2^1 - \Delta_2\Pi^2}. \tag{88}$$

The second, region (ii), is defined by (83) and

$$\Pi^2 \in [\zeta, 1]. \tag{89}$$

(c) *Case 1(c). If x exceeds the upper endpoint of (83) then $\overline{S}_x^{sep}$ is empty.*

2. *Case 2. Suppose $\nu \leq 0$: in the RSW allocation, the type 2 seller sells all of asset 1. There are two subcases.*

(a) *Case 2(a). If x lies in the interval*

$$[0, \gamma], \tag{90}$$

then $\overline{S}_x^{sep}$ is nonempty if and only if

$$\Pi^2 \geq \zeta, \tag{91}$$

in which case $\overline{S}_x^{sep}$ contains a single allocation, in which the quantities q^2 sold by type 2 are given by (85).

(b) *Case 2(b). If x exceeds the upper endpoint of (90) then $\overline{S}_x^{sep}$ is empty.*

Proof. Section 1.5. □

Finally, we characterize the set

$$\overline{S} = \{o^{\cdot} \in S : \forall \widehat{o} \in S, J(\widehat{o}) \leq J(o^{\cdot})\}$$

of globally efficient equilibrium allocations: those equilibrium allocations that give the highest social welfare for any payoff increment x. This set consists of the pooling equilibria depicted in Figures 1 and 2, together with the upper, curved boundary of the region of separating equilibria for Π^2 between η and ζ in Figure 1.

Claim 17. There are three cases.

1. If $\Pi^2 \geq \zeta$, $\overline{S}$ consists of those pooling allocations in which, for some x in the interval (78), the agent pays $\rho^t = af(1) + x$ to each type of seller for her whole portfolio.

2. If $\nu > 0$ and $\Pi^2 \in [\eta, \zeta)$, $\overline{S}$ is a singleton consisting of the allocation in which q^2 is given by (85) for $x = \phi^{-1}(\Pi^2)$.

3. In all other cases, $\overline{S}$ is empty.[27]

Proof. Section 1.5. □

1.4 The MT Approach and Security Design

We now apply the results of section 1.3 to the problem of security design. Assume a seller owns a stochastic cash flow Y that takes three possible values: $y_2 > y_1 > y_0 = 0$. Suppose the seller's beliefs about Y, conditional on her type t, satisfy the Hazard Rate Ordering (HRO) property (DFJ, section 2.5). Assume the seller can sell any monotone security S: any nondecreasing function $S : \{0, y_1, y_2\} \to \Re$ such that, for each possible Y, $S(Y)$ and $Y - S(Y)$ are both in $[0, Y]$.

How do we apply MT's approach in this setting? In section 1.2 we defined an outcome as a pair (q, ρ) consisting of a payment ρ from agent to seller in return for the vector $q = (q_1, ..., q_n)$ of quantities of the seller's assets. In the present setting, the vector q is replaced by a monotone security S; let $\widehat{O}$ be the set of outcomes (S, ρ). The rest is as in section 1.2, with the set O of outcomes replaced by $\widehat{O}$. Let us call this the MTSD game (for MT - security design).

How do we find the set of equilibria of the MTSD game? As shown in DFJ section 3.3, a monotone security S is equivalent to a portfolio of assets, where the set of assets is given by the maximal tranching of the cash flow Y into prioritized, monotone securities. In the present case with three possible outcomes of the cash flow, this maximal tranching consists

[27] As is $\overline{S}_x^{\text{pool}}$ and $\overline{S}_x^{\text{sep}}$ for all $x > 0$.

of the two securities $F_1(Y) = y_1 1_{Y\in\{y_1,y_2\}}$ and $F_2(Y) = (y_2 - y_1) 1_{Y=y_2}$. Let MTAS refer to the 2x2 MT asset sale game in which these are the two assets that the seller wishes to sell.

As in DFJ section 3.3, there is an isomorphism between the MTSD and MTAS, which is as follows. For any security S in the MTSD game, the quantity of asset $i = 1,2$ in the MTAS game is given by $q_i^S = [S(y_i) - S(y_{i-1})] / (y_i - y_{i-1})$. As shown in DFJ section 3.3, the monotone security S is payoff-equivalent to the portfolio $q^S = \left(q_1^S, q_2^S\right)$ of assets F_1 and F_2: for any realization Y of the cash flow, the security S and the portfolio q^S imply the same ex-post transfer from seller to agent. And any portfolio $q = (q_1, q_2)$ of the assets F_1 and F_2 in the MTAS game is payoff-equivalent to the monotone security $S^q = qF(Y)$ in the MTSD game, where $F(Y)$ is the vector $(F_1(Y), F_2(Y))$ of realized payouts of assets F_1 and F_2.

We now solve the MTSD game by applying the results of section 1.3 to the MTAS game. For each type $t = 1,2$ and each asset $i = 1,2$, let $f_i(t) = E[F_i(Y)|t]$ denote the expected payout of asset F_i when the seller' type is t. Let $f(t) = (f_1(t), f_2(t))$ denote the vector of these expected payouts for type t. The function f satisfies (36) since HRO implies FOSD and since the assets are monotone functions of Y. It satisfies (37) by DFJ's Proposition 5 and by HRO. Consequently, the set of equilibrium allocations $\left(\left(q^1, \rho^1\right), \left(q^2, \rho^2\right)\right)$ identified in section 1.3 in the MTAS game for this function f is isomorphic to the set of equilibrium allocations $\left(\left(S_1, \rho^1\right), \left(S_2, \rho^2\right)\right)$ of the MTSD game: for each type $t = 1,2$, the agent pays the seller ρ^t in return for the monotone security $S_t(Y) = q^t F(Y)$.

Using this equivalence, we can now discuss the main implications of section 1.3 for the MTSD game. As in section 1.3, we restrict to allocations that are efficient contingent on giving the low type a variable amount x more than her RSW payoff. By Claim 12, the RSW allocation is always an equilibrium of the MTAS game. What its equivalent in the MTSD game? By (52), in the RSW allocation, the low type sells her whole portfolio. This is equivalent, in the MTSD game, to standard debt (secured by Y) with face value y_2. By (55) and (56), the type 2 seller uses a hurdle class strategy (DFJ section 3.4) in the MTAS game: either $\widehat{q}_2^2$ is zero (whence the hurdle class c is 1) or $\widehat{q}_1^2$ is one (whence $c = 2$). In the first case, the corresponding security in the MTSD game is

$$\widehat{q}^2 F(Y) = \widehat{q}_1^2 y_1 1_{Y\in\{y_1,y_2\}}$$

which is just standard debt with face value $\widehat{q}_1^2 y_1$. In the second case, the security sold by the high type in the MTSD game is

$$\widehat{q}^2 F(Y) = y_1 1_{Y\in\{y_1,y_2\}} + \widehat{q}_2^2 (y_2 - y_1) 1_{Y=y_2}$$

which is just standard debt with face value $y_1 + \widehat{q}_2^2 (y_2 - y_1)$. Indeed, this is just the result of DFJ section 3.4 as the RSW and RLP allocations coincide (Claim 5).

We now turn to efficient equilibrium allocations other than RSW.[28] By Claim 15, in each such allocation the low type sells her whole portfolio which, as shown above, is equivalent to standard debt with face value y_2. As for the high type, Claim 15 implies that she uses a hurdle class strategy which, as shown in DFJ section 3.4, is also equivalent to standard debt. Since the low type chooses the highest possible face value, the qualitative results of DFJ section 3.4 carry over to the MTSD game: the seller sells standard debt with a face value that is nonincreasing in her type.[29]

How do these non-RSW allocations differ from the RSW allocation? By Claim 14, the low type is paid some $x > 0$ more than the fair value of her security. (In the RSW allocation, she is paid exactly this fair value.) This loosens her IC constraint which, in turn, enables the high type to raise the face value of the security she sells relative to the outcome of DFJ section 3.4. In the case in which Π^2 exceeds the constant ζ defined in (51), the high type can raise her face value all the way to the face value y_2 chosen by the low type: the two types can pool, selling the same standard debt security (By Claim 16). As in section 1.3, neither the pooling outcomes nor the non-RSW separating outcomes survive the intuitive criterion.

[28]By Proposition 2, these do not satisfy the intuitive criterion.

[29]More precisely, it is decreasing in her type in the separating equilibria and constant in her type in the pooling outcomes.

1.5 MT Approach: Proofs

PROOF OF CLAIM 2. Let m_Σ denote the DRM $o_\Sigma^{\cdot}$ defined in (15).[30] Further, consider the profile $\widehat{\Sigma}$ in which the principal always chooses the mechanism m_Σ, the agent accepts, and the principal chooses the outcome o_Σ^t corresponding to her type; if any player ever deviates, they revert to the original equilibrium Σ.[31] If $\widehat{\Sigma}$ is an equilibrium, then it is a DRM equilibrium that implements $o_\Sigma^{\cdot}$. We next show that no player ever wants to deviate in $\widehat{\Sigma}$, working backwards.

Stage 2. Suppose there has been no prior deviation: the principal offered m_Σ in stage 1. If the agent chooses "reject", she gets zero. Let $M_\Sigma \subset M$ denote the set of mechanisms that some type t chooses in Σ with probability $p_m^t > 0$. If the agent accepts, she gets

$$\begin{aligned}
\sum_{t=0}^{T} \Pi^t U^t\left(o_\Sigma^t\right) &= \sum_{t=0}^{T} \Pi^t \sum_{m \in M} p_m^t \sum_{s_\cdot^P \in (S_m^P)^{T+1}} \sum_{s^A \in S_m^A} \pi_m\left(s_\cdot^P, s^A\right) U^t\left(o_m^{s_t^P, s^A}\right) \text{ by (3) and (15)} \\
&= \sum_{m \in M_\Sigma} \sum_{s=0}^{T} \Pi^s p_m^s \sum_{t=0}^{T} \frac{\Pi^t p_m^t}{\sum_{s=0}^{T} \Pi^s p_m^s} \sum_{s_\cdot^P \in (S_m^P)^{T+1}} \sum_{s^A \in S_m^A} \pi_m\left(s_\cdot^P, s^A\right) U^t\left(o_m^{s_t^P, s^A}\right) \\
&= \sum_{m \in M_\Sigma} \sum_{s=0}^{T} \Pi^s p_m^s \sum_{s_\cdot^P \in (S_m^P)^{T+1}} \sum_{s^A \in S_m^A} \pi_m\left(s_\cdot^P, s^A\right) \sum_{t=0}^{T} \mathring{\Pi}^t(m) U^t\left(o_m^{s_t^P, s^A}\right) \\
&= \sum_{m \in M_\Sigma} \sum_{s=0}^{T} \Pi^s p_m^s \sum_{s_\cdot^P \in (S_m^P)^{T+1}} \sum_{s^A \in S_m^A} \pi_m\left(s_\cdot^P, s^A\right) u_m^A\left(s_\cdot^P, s^A\right) \text{ by (9)} \\
&= \sum_{m \in M_\Sigma} \sum_{s=0}^{T} \Pi^s p_m^s \mathring{U}_m(\pi_m) \text{ by (12)}
\end{aligned}$$

which is nonnegative by Claim 1: she is willing to accept m_Σ. As for a type-t principal, in Σ she has the option of imitating any other type t' yet chooses not to. Thus, her payoff

[30] That is, if the agent accepts and the principal reports "t", the mechanism m_Σ implements the outcome o_Σ^t.

[31] Formally, $\widehat{\Sigma} = \left(\widehat{p}_\cdot, \widehat{\Pi}^\cdot(\cdot), \widehat{\pi}_\cdot\right)$ where, for all t, $\widehat{p}_{m_\Sigma}^t = 1$ and, for $m \neq m_\Sigma$, $\widehat{p}_m^t = 0$. If $m = m_\Sigma$, the agent's beliefs are just the prior beliefs($\widehat{\Pi}^\cdot(m_\Sigma) = \Pi^\cdot$) and the stage-2 distribution $\widehat{\pi}_{m_\Sigma}$ puts unit weight on the action profile $\left(s_\cdot^P, s^A\right) = ((0, ..., T), \text{"accept"})$. (By $s_\cdot^P = (0, ..., T)$ we mean the pure strategy in stage 2 in which the principal of each type t truthfully reports her type.) If $m \neq m_\Sigma$, beliefs and stage-2 play are as in Σ: $\widehat{\Pi}^\cdot(m) = \mathring{\Pi}^\cdot(m)$ and $\widehat{\pi}_m = \pi_m$.

from doing so, $V^t\left(o_\Sigma^{t'}\right)$, must not exceed the payoff she gets in equilibrium in Σ, which is $V^t\left(o_\Sigma^t\right)$. Since these are also the principal's payoffs from choosing t' and t, respectively, in $\widehat{\Sigma}$, she will not deviate in $\widehat{\Sigma}$ either.

Stage 1. If the principal deviates in stage 1 by proposing some $m \neq m_\Sigma$, then she and the agent will conform to Σ thereafter. Thus, her payoffs in $\widehat{\Sigma}$ and Σ from proposing m are the same. But this payoff in Σ cannot exceed her payoff from sticking to her equilibrium strategy in Σ, which is $V^t\left(o_\Sigma^t\right)$: the same payoff that she gets in $\widehat{\Sigma}$ in equilibrium. Accordingly, the principal is willing not to deviate in stage 1.

Finally, beliefs in an equilibrium are arbitrary after deviations, so the agent's beliefs following an offer $m \neq m_\Sigma$ may equal $\mathring{\Pi}^{\cdot}(m)$ or anything else. We conclude that $\widehat{\Sigma}$ is an equilibrium as claimed. **Q.E.D.**$_{\text{Claim 2}}$

PROOF OF CLAIM 3. The payoff functions V^t and U^t are linear in the components q^t and ρ^t of $o^{\cdot}$. Hence any convex combination of any two allocations $(o^{\cdot}, \widehat{o}^{\cdot})$ that each satisfies (18) and (19) must also satisfy these conditions. Thus, the set of allocations that satisfy (18) and (19) is convex. The result then follows from Proposition 16.E.2 in Mas-Collel, Whinston, and Greene [18]. **Q.E.D.**$_{\text{Claim 3}}$

PROOF OF CLAIM 5.

Part 1. Define the following set.

Definition 18. A^{Up} is the set of allocations $o^{\cdot} \in O$ that satisfy upwards incentive-compatibility for the principal:

$$V^s\left(o^s\right) \geq V^s\left(o^t\right) \text{ for all types } 0 \leq s < t \leq T; \tag{92}$$

and, for all types, are individually rational for the agent:

$$U^t\left(o^t\right) \geq 0 \text{ for all types } 0 \leq t \leq T. \tag{93}$$

The proof of Part 1 consists of showing (a) any RSW allocation is Pareto optimal in A^{Up} and (b) any solution to RLP is Pareto optimal in A^{Up} and vice-versa. We begin with two useful properties of A^{Up}.

Claim 18. A^{Up} contains a Pareto optimal allocation.

Proof. For any allocation $o^{\cdot}$, let $SW(o^{\cdot})$ denote the sum $\sum_{t=0}^{T} V^t(o^t)$ of payoffs of the different types of principal. The function $SW(\cdot)$ is linear in the arguments q^t and ρ^t of $o^{\cdot}$ and thus continuous in $o^{\cdot}$. Define A_0^{Up} to be the subset of A^{Up} for which each type of principal gets a nonnegative payoff:

$$V^t\left(o^t\right) \geq 0 \text{ for all types } t. \tag{94}$$

The set A_0^{Up} is nonempty (it includes $o_0^{\cdot}$), closed (it is defined by weak inequalities of linear functions), and bounded (as the sum of the payoffs $U^t(o^t) + V^t(o^t) = (1-\delta)\, q^{o^t} f(t)$ is bounded and, by (93) and (94), each payoff is nonnegative). Hence A_0^{Up} is compact. As $SW(\cdot)$ is linear and thus continuous, it must have a maximand $\widetilde{o}^{\cdot}$ on the nonempty, compact set A_0^{Up} by the extreme value theorem. Clearly, $\widetilde{o}^{\cdot}$ is Pareto optimal in A_0^{Up}. If there is an allocation $o^{\cdot}$ in $A^{\mathrm{Up}} - A_0^{\mathrm{Up}}$ that Pareto dominates $\widetilde{o}^{\cdot}$ then $V^t(o^t) \geq V^t(\widetilde{o}^t) \geq 0$ for each type t whence$o^{\cdot}$ is in A_0^{Up} - a contradiction.[32] □

Claim 19. If $\overline{o}^{\cdot} \in A^{\mathrm{Up}}$ is not Pareto optimal in A^{Up}, then there exists an allocation $\widetilde{o}^{\cdot}$ in A^{Up} that Pareto dominates $\overline{o}^{\cdot}$ and is Pareto optimal in A^{Up}.

Proof. By definition 5, there is an allocation $\widehat{o}^{\cdot} \in A^{\mathrm{Up}}$ that Pareto dominates $\overline{o}^{\cdot}$. Let

$$A_{\widehat{o}^{\cdot}}^{\mathrm{Up}} = \left\{ o^{\cdot} \in A^{\mathrm{Up}} : \; V^t(o^{\cdot}) \geq V^t(\widehat{o}^{\cdot}) \text{ for each type } t \right\} \tag{95}$$

be the set of allocations in A^{Up} that are at least as good as $\widehat{o}^{\cdot}$ is for each type t of principal. The set $A_{\widehat{o}^{\cdot}}^{\mathrm{Up}}$ is nonempty (as it contains $\widehat{o}^{\cdot}$); closed (as it is defined by weak inequalities of linear functions); and bounded (as the sum of the payoffs $U^t(o^t) + V^t(o^t) = (1-\delta)\, q^{o^t} f(t)$ is bounded, $U^t(o^t) \geq 0$, and $V^t(o^t) \geq V^t(\widehat{o}^{\cdot}) > -\infty$). Thus, the function $SW(o^{\cdot}) = \sum_{t=0}^{T} V^t(o^t)$ on $A_{\widehat{o}^{\cdot}}^{\mathrm{Up}}$ has a maximizer $\widetilde{o}^{\cdot}$ in $A_{\widehat{o}^{\cdot}}^{\mathrm{Up}}$, which is Pareto optimal in $A_{\widehat{o}^{\cdot}}^{\mathrm{Up}}$ and thus also in A^{Up}. (An allocation $o^{\cdot}$ that Pareto dominated $\widetilde{o}^{\cdot}$ in A^{Up} would, by transitivity, Pareto dominate $\widehat{o}^{\cdot}$ and thus lie in $A_{\widehat{o}^{\cdot}}^{\mathrm{Up}}$, contradicting the Pareto optimality of $\widetilde{o}^{\cdot}$ in $A_{\widehat{o}^{\cdot}}^{\mathrm{Up}}$.) Finally, as $\widetilde{o}^{\cdot}$ is in $A_{\widehat{o}^{\cdot}}^{\mathrm{Up}}$, it Pareto dominates $\overline{o}^{\cdot}$ since $\widehat{o}^{\cdot}$ does. □

[32] See Mas-Collel, Whinston, and Greene [18, Prop. 16.E.2].

We next show that the solutions to RLP coincide with the Pareto optimal allocations in A^{Up}. More precisely, for each Pareto optimal allocation $o^{\cdot}$ in A^{Up} and each type t, the payoff $V^t(o^t)$ of the principal equals the seller's unique payoff $u^*(t)$ in RLP, while the agent's payoff $U^t(o^t)$ is zero as in RLP. Moreover, the issuance function defined by $q^*(t) = q^{o^t}$ solves RLP. Finally, the expected transfer ρ^{o^t} coincides with the unique transfer function $\rho^*(t)$ in RLP and equals the conditional expected value $q^*(t) f(t)$ of the issuance in RLP. The Claim also proves the converse: any solution to RLP corresponds to a Pareto-optimal element of A^{Up}.

Claim 20.

1. Suppose $o^{\cdot}$ is Pareto optimal in the set A^{Up}. Then for each type t, the agent's payoff $U^t(o^t)$ is identically zero and the functions given by $u^*(t) = V^t(o^t)$, $\rho^*(t) = \rho^{o^t}$, and $q^*(t) = q^{o^t}$ solve RLP.

2. Let (u^*, ρ^*, q^*) solve RLP. Then any allocation $o^{\cdot}$ that satisfies $q^{o^t} = q^*(t)$ and $\rho^{o^t} = \rho^*(t)$ for all types t is Pareto optimal in the set A^{Up} and, for each type t, yields the payoffs $V^t(o^t) = u^*(t)$ and $U^t(o^t) = 0$.

Proof. Part 1. Suppose $o^{\cdot}$ is Pareto optimal in A^{Up}. Let t be the lowest type for which $U^t(o^t)$ is not identically zero:

$$0 < U^t\left(o^t\right) = q^{o^t} f(t) - \rho^{o^t}. \tag{96}$$

If the IC constraint (92) does not bind for any $s < t$, then we can raise the transfer ρ^{o^t} that type t gets by $\iota = U^t(o^t) > 0$ without violating (92) or (93), which contradicts the Pareto optimality of$o^{\cdot}$ in A^{Up}. Now suppose the IC constraint (92) does bind for some type $s < t$. Let

$$w = \min\left\{s \le t : V^s(o^s) = V^s\left(o^t\right) \text{ and } U^s\left(o^t\right) > 0\right\} \tag{97}$$

be the lowest type $s \le t$ for which (92) binds and $U^s(o^t) > 0$. (If there is no such type $s < t$, then w equals t.) By construction, $U^w(o^t)$ is positive. Let

$$\varepsilon \in \left(0, \min\left\{1, \frac{U^w(o^t)}{2(1-\delta)\, q^{o^t} f(w)}\right\}\right) \tag{98}$$

and define[33]

$$\iota(\varepsilon) = \min\left\{\frac{U^w(o^t)}{2}, \min_{s<w}\left(V^s(o^s) - V^s\left(o^t\right) + \varepsilon\delta q^{o^t}\left[f(w) - f(s)\right]\right)\right\} > 0. \tag{99}$$

Now consider the alternative allocation $\widehat{o}^{\cdot}$ given by $\widehat{o}^s = o^s$ for $s \neq w$ and $\widehat{o}^w = (q, \rho)$ where $q = (1-\varepsilon) q^{o^t}$ and $\rho = \rho^{o^t} - \varepsilon\delta q^{o^t} f(w) + \iota(\varepsilon)$. For $s < w$,

$$\begin{aligned} V^s(\widehat{o}^s) - V^s(\widehat{o}^w) &= V^s(o^s) - V^s(q, \rho) \\ &= V^s(o^s) - \left[\rho^{o^t} - \varepsilon\delta q^{o^t} f(w) + \iota(\varepsilon) - \delta(1-\varepsilon) q^{o^t} f(s)\right] \\ &= V^s(o^s) - V^s\left(o^t\right) + \varepsilon\delta q^{o^t}\left[f(w) - f(s)\right] - \iota(\varepsilon) \end{aligned}$$

which is nonnegative by (99): no type $s < w$ prefers $\widehat{o}^w$ to $\widehat{o}^s$. Thus, the IC constraint (92) is satisfied in $\widehat{o}^{\cdot}$. Moreover,

$$\begin{aligned} U^w(\widehat{o}^w) &= (1-\varepsilon) q^{o^t} f(w) - \left[\rho^{o^t} - \varepsilon\delta q^{o^t} f(w) + \iota(\varepsilon)\right] \\ &= U^w\left(o^t\right) - \varepsilon(1-\delta) q^{o^t} f(w) - \iota(\varepsilon) \end{aligned}$$

which is nonnegative by (98) and (99): the IR constraint (93) for type w is also satisfied in $\widehat{o}^{\cdot}$. Hence, $\widehat{o}^{\cdot}$ lies in A^{Up}. But the payoff of the type-w principal in $\widehat{o}^{\cdot}$ is

$$\begin{aligned} V^w(\widehat{o}^w) &= \left[\rho^{o^t} - \varepsilon\delta q^{o^t} f(w) + \iota(\varepsilon)\right] - \delta(1-\varepsilon) q^{o^t} f(w) \\ &= V^w\left(o^t\right) + \iota(\varepsilon) > V^w\left(o^t\right) \end{aligned}$$

which contradicts the Pareto optimality of $o^{\cdot}$ in A^{Up}. We conclude that, if $o^{\cdot}$ is Pareto optimal, then $U^t(o^t) = 0$ for all t.

Now let t be the lowest type for which the functions $u^*(t) = V^t(o^t)$, $\rho^*(t) = \rho^{o^t}$, and $q^*(t) = q^{o^t}$ do not solve RLP. As $U^t(o^t) = 0$, it follows that $\rho^{o^t} = q^{o^t} f(t)$, whence $V^t(o^t) = (1-\delta) q^{o^t} f(t)$. Hence, if $q^*(t) = q^{o^t}$ solves RLP, so do $u^*(t) = V^t(o^t)$ and

[33] If $w = 0$, then ι is simply $U^w(o^t)/2 > 0$. If $w > 0$, then the inner min in (99) is positive since for each $s < w$, either $V^s(o^s) > V^s(o^t)$ or $U^s(o^t) \leq 0$; the latter and $U^w(o^t) > 0$ jointly imply $0 < U^w(o^t) - U^s(o^t) = q^{o^t}[f(w) - f(s)]$.

$\rho^*(t) = q^{o^t} f(t)$. Thus, $q^*(t) = q^{o^t}$ must not solve RLP. By assumption, t is the lowest type for which the conclusion fails: $u^*(s) = V^s(o^s)$, $\rho^*(s) = \rho^{o^s}$, and $q^*(s) = q^{o^s}$ solve RLP for each $s < t$. As $o^\cdot$ is Pareto optimal in the set A^{Up}, we have, for all $0 \leq s < t$, $u^*(s) = V^s(o^s) \geq V^s(o^t) = q^{o^t}[f(t) - \delta f(s)]$ whence q^{o^t} is in Q_t. Thus, there must be a q' in Q_t such that $(1-\delta) q' f(t) > (1-\delta) q^{o^t} f(t)$. Let us now replace the allocation $o^\cdot$ with the allocation $\widehat{o}^\cdot$ given by $\widehat{o}^t = (q', q' f(t))$and, for all $s \neq t$, $\widehat{o}^s = o^s$. As $U^t(\widehat{o}^t) = 0$, $\widehat{o}^\cdot$ satisfies (93) for $\bar{t} = T$. Since q' is in Q_t, for each $s < t$ we have $V^s(\widehat{o}^s) = u^*(s) \geq q'[f(t) - \delta f(s)] = V^s(\widehat{o}^t)$: $\widehat{o}^\cdot$ satisfies (92) for $\bar{t} = T$. Thus, $\widehat{o}^\cdot$ is in A^{Up}, each type $s \neq t$ of principal is indifferent between $o^\cdot$ and $\widehat{o}^\cdot$, and the type t principal is better off under $\widehat{o}^\cdot$ than $o^\cdot$. This contradicts the Pareto optimality of $o^\cdot$ in A^{Up}.

Part 2. Let (u^*, ρ^*, q^*) solve RLP. Let $o^\cdot$ be an allocation that satisfies $q^{o^t} = q^*(t)$ and $\rho^{o^t} = \rho^*(t)$ for all types t. This implies that

$$V^t\left(o^t\right) = \rho^{o^t} - \delta q^{o^t} f(t) = \rho^*(t) - \delta q^*(t) f(t) = u^*(t) \tag{100}$$

(which is positive by PROPOSITION 1) and

$$U^t\left(o^t\right) = q^{o^t} f(t) - \rho^{o^t} = q^*(t) f(t) - \rho^*(t) = 0$$

as claimed. Moreover, for $s < t$,

$$\begin{aligned} V^s(o^s) &= u^*(s) \text{ by (100)} \\ &\geq \rho^*(t) - \delta q^*(t) f(s) \text{ as } q^*(t) \in Q_t \text{ and } \rho^*(t) = q^*(t) f(t) \\ &= \rho^{o^t} - \delta q^{o^t} f(s) = V^s\left(o^t\right) \text{ by hypothesis on } o^\cdot. \end{aligned}$$

It follows that $o^\cdot$ lies in A^{Up}. By Claim 18, A^{Up} contains a Pareto optimal element $\widehat{o}^\cdot$. By part 1, for each type t, the agent's payoff $U^t(\widehat{o}^t)$ is zero and the functions given by $u^*(t) = V^t(\widehat{o}^t)$, $\rho^*(t) = \rho^{\widehat{o}^t}$, and $q^*(t) = q^{\widehat{o}^t}$ solve RLP. But by PROPOSITION 1, the solutions $u^*(\cdot)$ and $\rho^*(\cdot)$ to RLP are unique. Accordingly, $V^t(\widehat{o}^t) = V^t(o^t)$ for each t, whence $o^\cdot$ is Pareto optimal in A^{Up}. □

A^{Up} has the following relation to WIE:

Claim 21. 1. If $\overline{o}^{\cdot}$ is Pareto optimal in A^{Up}, then it is WIE.

(a) If $\overline{o}^{\cdot}$ is WIE and $U^t(\overline{o}^t) \geq 0$ for each type t, then $\overline{o}^{\cdot}$ is in A^{Up}.

(b) If $\overline{o}^{\cdot}$ is WIE and $U^t(\overline{o}^t) = 0$ for each type t, then $\overline{o}^{\cdot}$ is Pareto optimal in A^{Up}.

Proof. Part 1. By part 1 of Claim 20, the functions $u^*(t) = V^t(o^t)$, $\rho^*(t) = \rho^{o^t}$, and $q^*(t) = q^{o^t}$ solve RLP. Hence, by PROPOSITION 1, for any types s and t,[34]

$$V^t\left(\overline{o}^t\right) = u^*(t) \geq \rho^*(s) - \delta q^*(s) f(t) = \rho^{\overline{o}^s} - \delta q^{\overline{o}^s} f(t) = V^s\left(\overline{o}^t\right). \tag{101}$$

Thus, $\overline{o}^{\cdot}$ satisfies (16). Now suppose that $\overline{o}^{\cdot}$ is not WIE. Then there is an allocation $o^{\cdot}$ that satisfies (17), (18), and (19). By (18), $o^{\cdot}$ satisfies (92). Since $\overline{o}^{\cdot}$ is in A^{Up}, $V^t(o^t) \geq V^t(\overline{o}^t)$ by (17) and $U^t(o^t) \geq U^t(\overline{o}^t) = 0$ by (19): $o^{\cdot}$ satisfies (93). Thus, $o^{\cdot}$ is in A^{Up} and, by (17), Pareto dominates $\overline{o}^{\cdot}$, which contradicts the assumption that $\overline{o}^{\cdot}$ is Pareto optimal in A^{Up}.

Part 2(a). Since $U^t(\overline{o}^t) \geq 0$ for each type t, $\overline{o}^{\cdot}$ satisfies (93). Since $\overline{o}^{\cdot}$ is WIE, it also satisfies (92) by (16): it is in A^{Up}.

Part 2(b). By part 2(a), $\overline{o}^{\cdot}$ is in A^{Up}. If $\overline{o}^{\cdot}$ is not Pareto optimal in A^{Up}, then by Claim 19 there exists an allocation $\widetilde{o}^{\cdot}$ in A^{Up} that Pareto dominates $\overline{o}^{\cdot}$ and is Pareto optimal in A^{Up}. Hence, by part 1 of Claim 20, the functions given by $u^*(t) = V^t(\widetilde{o}^t)$, $\rho^*(t) = \rho^{\widetilde{o}^t}$, and $q^*(t) = q^{\widetilde{o}^t}$ solve RLP. Thus, by part 1 of PROPOSITION 1, the allocation $\widetilde{o}^{\cdot}$ is incentive compatible: it satisfies (18). As $\widetilde{o}^{\cdot}$ is in A^{Up}, it also satisfies (19) by (93). Finally, by assumption, $\widetilde{o}^{\cdot}$ Pareto dominates $\overline{o}^{\cdot}$: $o^{\cdot} = \widetilde{o}^{\cdot}$ satisfies (17). But this contradicts the assumption that $\overline{o}^{\cdot}$ is WIE. □

Part 1 of Claim 5 now follows from Claims 20 and the following result.

Claim 22. Any RSW allocation $\widehat{o}^{\cdot}$ is Pareto optimal in A^{Up}.

[34]PROPOSITION 1 characterizes a profile e^*, in which a type t seller sells the quantities $q^*(t)$ in return for the revenue $\rho^*(t)$, and gets the payoff $u^*(t)$. Part 3 shows that e^* is an equilibrium, in part by showing the inequality in (101) holds for all types s and t.

Proof. Let $\widehat{o}^{\cdot}$ be an RSW allocation. By Claim 18, there is an allocation $o^{\cdot}$ that is Pareto optimal in A^{Up} and, by part 1 of Claim 20, the functions given by $u^*(t) = V^t(o^t)$, $\rho^*(t) = \rho^{o^t}$, and $q^*(t) = q^{o^t}$ solve RLP. Thus, by part 1 of PROPOSITION 1, the agent's payoff $U^t(o^t)$ is identically zero: $o^{\cdot}$ satisfies (22). By part 3 of PROPOSITION 1, $o^{\cdot}$ also satisfies (21). Thus, by definition 14, $V^t(\widehat{o}^t)$ is not less than $V^t(o^t)$ which, we have shown, equals $u^*(t)$. Since, moreover, $\widehat{o}^{\cdot}$ satisfies (92) by (16) and (93) by (22), $\widehat{o}^{\cdot}$ is in A^{Up}.Now suppose $\widehat{o}^{\cdot}$ is not Pareto optimal in A^{Up}. By Claim 19, there exists an allocation $o^{\cdot}$ in A^{Up} that Pareto dominates $\widehat{o}^{\cdot}$ and is Pareto optimal in A^{Up}. Hence, $o^{\cdot}$ satisfies (21) by part 1 of Claim 21 and (22) by part 1 of Claim 20, so $V^t(\widehat{o}^t) \geq V^t(o^t)$ for all t by Definition 14: $o^{\cdot}$ does not Pareto dominate $\widehat{o}^{\cdot}$ - a contradiction. □

Part 2. An RSW allocation $\widehat{o}^{\cdot}$ exists by Claim 4 and, by part 1, satisfies

$$\widehat{V}^t = V^t\left(\widehat{o}^t\right) = u^*(t). \tag{102}$$

Now consider the alternative allocation $o^{\cdot} = (q^{\cdot}, \rho^{\cdot})$ defined by $q^t = q^*(t)$ and $\rho^t = \rho^*(t)$. We claim that it is an RSW allocation as well. We first verify that it satisfies (21) and (22) for all t. Equation (22) holds since, for each s, $U^s(o^s)$ equals $q^*(s) f(s) - \rho^*(s)$ which, by (25), is zero: the agent's reservation payoff. Moreover,

$$V^s(o^s) = \rho^*(s) - \delta q^*(s) f(s) = u^*(s) \tag{103}$$

for all s by (26) and

$$V^s(o^r) = \rho^*(r) - \delta q^*(r) f(s) = q^*(r)\left[f(r) - \delta f(s)\right]$$

by (25). Hence, to show (21), it suffices to show that $u^*(s) \geq q^*(r)\left[f(r) - \delta f(s)\right]$ for all types r, s. But this has already been shown in the proof of part 3 of PROPOSITION 1. We conclude that $o^{\cdot}$ solves (21) and (22) for all t. Finally, for each t, $V^t(o^t)$ attains the maximum payoff $\widehat{V}^t$ possible in Program II^t by (102) and (103). Hence, o^t solves Program II^t for all t: $o^{\cdot}$ is an RSW allocation. **Q.E.D.**$_{\text{Claim 5}}$

PROOF OF CLAIM 6. The payoff functions V^t and U^t are linear in q^t and ρ^t. Hence any convex combination of any two allocations $(o^{\cdot}, \widehat{o}^{\cdot})$ that each satisfies (29) and (30) must also satisfy these conditions. Thus, the set of allocations that satisfy (29) and (30) is convex. The result then follows from Proposition 16.E.2 in Mas-Collel, Whinston, and Greene [18]. **Q.E.D.**$_{\text{Claim 6}}$

PROOF OF CLAIM 7. Let $C = \{o^{\cdot} \in A : V^t(o^t) \geq V^t(o^s) \text{ for all } s,t\}$ denote the set of incentive-compatible allocations. Since the function $V^t(o)$ is linear in each element of $o = (q^o, \rho^o)$, C is a convex subset of A.[35] Let $\overline{o}^{\cdot}$ be WIE. By Claim 3 and Bertsekas [3, Proposition 5.3.1],[36]

$$\overline{o}^{\cdot} \in \arg\max_{o^{\cdot} \in C} \left\{ \sum_{t=0}^{T} w^t V^t\left(o^t\right) + \sum_{t=0}^{T} \nu^t \left[U^t\left(o^t\right) - U^t\left(\overline{o}^t\right)\right] \right\}$$

for some positive weights $\{w^t\}_{t=0}^{T}$ and nonnegative multipliers $\{\nu^t\}_{t=0}^{T}$. Let $\nu = \sum_{t=0}^{T} \nu^t$. For each t, let Π^t equal ν^t/ν if $\nu > 0$ and $1/(T+1)$ otherwise. Then

$$\overline{o}^{\cdot} \in \arg\max_{o^{\cdot} \in C} \left\{ \sum_{t=0}^{T} w^t V^t\left(o^t\right) + \nu \sum_{t=0}^{T} \Pi^t \left[U^t\left(o^t\right) - U^t\left(\overline{o}^t\right)\right] \right\}.$$

Hence, by Claim 6 and Bertsekas [3, Proposition 5.3.1], $\overline{o}^{\cdot}$ is IE with respect to the beliefs $\Pi^{\cdot}$: $\Pi(\overline{o}^{\cdot})$ is nonempty. The proof of convexity of the set $\Pi(\overline{o}^{\cdot})$ is identical to that of MT [20, p. 17]. **Q.E.D.**$_{\text{Claim 7}}$

PROOF OF CLAIM 8. Let $\widehat{o}^{\cdot}$ be an RSW allocation. By Claim 4, $\widehat{o}^{\cdot}$ is Pareto optimal in A^{Up} whence, by part 1 of Claim 20, the agent's payoff $U^t(\widehat{o}^t)$ is identically zero and the functions given by $u^*(t) = V^t(\widehat{o}^t)$, $\rho^*(t) = \rho^{\widehat{o}^t}$, and $q^*(t) = q^{\widehat{o}^t}$ solve RLP. Now fix a small $\varepsilon > 0$ and consider the allocation $\widetilde{o}^{\cdot}$ given by, for each type t, $\widetilde{q}^t = q^{\widehat{o}^t}$ and $\widetilde{\rho}^t = \rho^{\widehat{o}^t} - (t+1)\varepsilon$. Then for any types s and t,

$$V^s\left(\widetilde{o}^t\right) = V^s\left(\widehat{o}^t\right) - (t+1)\varepsilon. \tag{104}$$

[35]Defined in (5), A is the set of all possible allocations.

[36]To satisfy condition (1) in Bertsekas [3, Proposition 5.3.1], begin with $\overline{o}^{\cdot}$ and then lower the transfer ρ^t that the agent pays the principal by some common (type-independent) amount $\varepsilon > 0$. The resulting allocation is in C and, moreover, strictly satisfies (19): condition (1) holds.

If type s proposes $\widetilde{o}^{\cdot}$, the agent will accept. Why? First, the agent gets a positive payoff if type s chooses $\widetilde{o}^{s}$ as

$$U^{s}\left(\widetilde{o}^{s}\right)=(s+1)\,\varepsilon>0. \tag{105}$$

Moreover, type s will not imitate any type $t>s$ since $V^{s}\left(\widetilde{o}^{s}\right)-V^{s}\left(\widetilde{o}^{t}\right)=V^{s}\left(\widehat{o}^{s}\right)-V^{s}\left(\widehat{o}^{t}\right)+(t-s)\,\varepsilon$ by (104) which is positive as $\widehat{o}^{\cdot}$ is in A^{Up}. Finally, if type t imitates type $s<t$ then, by Assumption 1, the agent gets no less than $U^{s}\left(\widetilde{o}^{s}\right)$ which again is positive by (105). As the agent knows her payoff will be positive,she will accept. As for the principal, if her type is s then by proposing $\widetilde{o}^{\cdot}$ and then choosing $\widetilde{o}^{s}$ she receives $V^{s}\left(\widehat{o}^{s}\right)-(s+1)\,\varepsilon$ by (104). Thus, she will deviate from any putative equilibrium that offers her less than this. As this is true for all $\varepsilon>0$,her payoff in any equilibrium cannot be less than $V^{s}\left(\widehat{o}^{s}\right)$. **Q.E.D.**$_{\text{Claim 8}}$

PROOF OF PROPOSITION 1.

By Claim 10, the RSW allocation $\widehat{o}^{\cdot}$ is IE relative to some nonnegative beliefs $\widehat{\Pi}^{\cdot}\in\Re_{+}^{T+1}$

Part 1: only if. Condition (33) is necessary since, otherwise, the principal could strictly gain by imitating a different type. Condition (34) is necessary since, otherwise, the agent could do better by always playing ”reject”. And (35) is necessary by Claim 8. We conclude that (33), (34), and (35) are necessary conditions for $o^{\cdot}$ to be the expected allocation of an equilibrium.

Part 2: if. Consider any allocation $o^{\cdot}=\overline{o}^{\cdot}$ that satisfies (33), (34), and (35). We will prove the existence of a DRM equilibrium $\overline{o}^{\cdot}$, which is intuitive if $\overline{o}^{\cdot}$ is an RSW allocation $\widehat{o}^{\cdot}$.

In such an equilibrium, the principal always chooses the DRM $\overline{o}^{\cdot}$. Thus, if the principal conforms to the equilibrium in stage 1, the agent's interim beliefs are simply her prior beliefs $\Pi^{\cdot}$: by (34), she will accept in stage 2. Moreover, the principal is willing to truthfully report her type in the DRM $\overline{o}^{\cdot}$ by (33). It remains only to show that there exist interim beliefs that deter the principal from deviating to a mechanism other than $\overline{o}^{\cdot}$, and that these beliefs are intuitive if $\overline{o}^{\cdot}$ is an RSW allocation.

For any alternative mechanism $m \in M$, define

$$\Delta_m^\Sigma = \begin{cases} \left\{ \mathring{\Pi}^\cdot \in \Delta^T : \ \mathring{\Pi}^t = 0 \text{ for all } t \in J_m \right\} & \text{if} \quad J_m \neq \Upsilon \\ \Delta^T & \text{if} \quad J_m = \Upsilon \end{cases} \tag{106}$$

where J_m, defined in (31), is the set of types t who will never choose m over the RSW allocation. If some types are not in J_m, then Δ_m^Σ is the set of interim beliefs that assign probability zero to all types in J_m; else it is the full set of all possible beliefs. As m is unexpected, any interim beliefs $\mathring{\Pi}^\cdot(m)$ are consistent with Bayes's Rule. If, in addition, the interim beliefs $\mathring{\Pi}^\cdot(m)$ lie in Δ_m^Σ, then they satisfy Int_m^Σ:

Claim 23. Δ_m^Σ is compact and the beliefs in Δ_m^Σ satisfy Int_m^Σ.

Proof. Compactness is immediate from (106). As for Int_m^Σ, if $J_m \neq \Upsilon$ then the beliefs in Δ_m^Σ put zero weight on any type in J_m by (106). And if $J_m = \Upsilon$, then no type t in $\Upsilon - J_m$ satisfies (32) as $\Upsilon - J_m$ is empty: Int_m^Σ holds for any beliefs. In both cases, Int_m^Σ holds for all beliefs in Δ_m^Σ. □

We will show the following condition holds for each m.

Deter(m)**.** For each mechanism m, there exist interim beliefs $\mathring{\Pi}^\cdot(m)$ in Δ_m^Σ and associated equilibrium payoffs $\left(\mathring{V}^\cdot(m), \mathring{U}(m)\right)$ in $\Psi_m\left(\mathring{\Pi}^\cdot(m)\right)$ such that $\mathring{V}^t(m) \leq \widehat{V}^t$ for all t: the type-t principal does not prefer m to $\overline{o}^t$.

Suppose Deter(m) holds for each out-of-equilibrium mechanism m. Consider the strategy profile in which any unexpected mechanism m leads to the beliefs $\mathring{\Pi}^\cdot(m)$ in Δ_m^Σ and associated equilibrium payoffs $\left(\mathring{V}^\cdot(m), \mathring{U}(m)\right)$ described in Deter(m). Then as long as the agent will surely accept the DRM $\overline{o}^\cdot$, a type-t principal is willing to choose $\overline{o}^\cdot$ and then, by (33), to choose the outcome $\overline{o}^t$. And since each type of principal chooses $\overline{o}^\cdot$, the agent's interim beliefs in response to $\overline{o}^\cdot$ are her prior beliefs $\Pi^\cdot$, so she is indeed willing to accept by (34). Hence, the DRM $\overline{o}^\cdot$ is supported by an equilibrium. If, further, $\overline{o}^\cdot = \widehat{o}^\cdot$, then Int_m^Σ holds since $\mathring{\Pi}^\cdot(m)$ lies in Δ_m^Σ: the given equilibrium is intuitive, so the RSW allocation $\widehat{o}^\cdot$ is the expected allocation of an intuitive equilibrium as claimed.

It remains only to prove Deter(m) for each mechanism m other than the DRM $\overline{o}^{\cdot}$. Suppose, to the contrary, that there is a mechanism m for which Deter(m) fails to hold: for any interim beliefs $\mathring{\Pi}^{\cdot}(m)$ in Δ_m^{Σ} and equilibrium payoffs $\left(\mathring{V}^{\cdot}(m), \mathring{U}(m)\right)$ in $\Psi_m\left(\mathring{\Pi}^{\cdot}(m)\right)$, there is a type t for whom $\mathring{V}^t(m) > \widehat{V}^t$. We will derive a contradiction. A key step is the use of Kakutani's [16] fixed point theorem. To verify the assumptions of Kakutani's theorem, we must first derive a series of technical results.

Claim 24. Let S be any compact subset of Δ^T. The correspondences $\phi_m : S \rightrightarrows \Phi_m$ and $\psi_m : S \rightrightarrows \Re^{T+1} \times \Re$ defined in (11) and (14) are upper hemicontinuous and, for any interim beliefs $\mathring{\Pi}^{\cdot} \in S$, the sets $\phi_m\left(\mathring{\Pi}^{\cdot}\right)$ and $\psi_m\left(\mathring{\Pi}^{\cdot}\right)$ are nonempty and convex.

Proof. We first show an equivalence between $CEIG_m^{\mathring{\Pi}^{\cdot}}$ and Aumann's [2] original notion of correlated equilibrium in a symmetric information setting.

Lemma 1. *A distribution $\pi_m \in \Phi_m$ is a $CEIG_m^{\mathring{\Pi}^{\cdot}}$ if and only if it is a correlated equilibrium (Aumann [2]) of the complete information game with pure strategy sets $\left(S_m^P\right)^{T+1}$ and S_m^A and payoff functions*

$$u_m^P\left(s_{\cdot}^P, s^A\right) = \sum_{t=0}^{T} V^t\left(o_m^{s_t^P, s^A}\right). \tag{107}$$

and $u^A_{m,\mathring{\Pi}^{\cdot}}$ for the principal and agent, respectively. These payoff functions are continuous in the interim beliefs $\mathring{\Pi}^{\cdot}$, and the pure strategy sets $\left(S^P\right)^{T+1}$ and S^A are finite.

Proof. Equation (7) holds if and only if

$$\sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) u_m^P\left(s_{\cdot}^P, s^A\right) \geq \sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) u_m^P\left(\widehat{s}_{\cdot}^P, s^A\right) \text{ for any } s_{\cdot}^P, \widehat{s}_{\cdot}^P \in \left(S_m^P\right)^{T+1} \tag{108}$$

Why? If (7) holds then, summing over types t, one obtains (108). Conversely, if (7) does not hold then, for some type t and actions $s_{\cdot}^P, \widetilde{s}_{\cdot}^P$, $\sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) V^t\left(o_m^{s_t^P, s^A}\right) < \sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) V^t\left(o_m^{\widetilde{s}_t^P, s^A}\right)$. Defining $\widehat{s}_{\cdot}^P$ by $\widehat{s}_t^P = \widetilde{s}_t^P$ and, for each $t' \neq t$, $\widehat{s}_{t'}^P = s_{t'}^P$, we have

$$\sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) u_m^P\left(s_{\cdot}^P, s^A\right) < \sum_{s^A \in S_m^A} \pi_m\left(s_{\cdot}^P, s^A\right) u_m^P\left(\widehat{s}_{\cdot}^P, s^A\right),$$

whence (108) fails, as claimed. Hence, π_m is a $CEIG_m^{\mathring{\Pi}^{\cdot}}$ if and only the conditions (8) and (108) hold. By Fudenberg and Tirole [11, p. 57], these two conditions are necessary and sufficient for π_m to be a correlated equilibrium in the sense of Aumann [2]. Finally, continuity and finiteness are trivial. □

By Lemma 1 and Cotter's [7] result for the degenerate symmetric information case, the correspondence ϕ_m is upper hemicontinuous on Δ^T and, for any interim beliefs $\mathring{\Pi}^{\cdot} \in \Delta^T$, the set $\phi_m\left(\mathring{\Pi}^{\cdot}\right)$ is nonempty and convex. By (14), $\psi_m\left(\mathring{\Pi}^{\cdot}\right)$ is nonempty since $\phi_m\left(\mathring{\Pi}^{\cdot}\right)$ is. By (12) and (13), for any λ in $[0,1]$,

$$\begin{aligned}&\lambda\left(\mathring{V}_m^{\cdot}(\pi_m),\mathring{U}_m(\pi_m)\right)+(1-\lambda)\left(\mathring{V}_m^{\cdot}(\pi_m'),\mathring{U}_m(\pi_m')\right)\\&=\left(\mathring{V}_m^{\cdot}\left(\lambda\pi_m+(1-\lambda)\pi_m'\right),\mathring{U}_m\left(\lambda\pi_m+(1-\lambda)\pi_m'\right)\right)\end{aligned}$$

whence $\psi_m\left(\mathring{\Pi}^{\cdot}\right)$ is convex as well.

It remains to show that ϕ_m and ψ_m are upper hemicontinuous on any compact set $S \subset \Delta^T$. Since ϕ_m is upper hemicontinuous on Δ^T, if $\left(\mathring{\Pi}_n^{\cdot},\pi_m^n\right) \to \left(\mathring{\Pi}^{\cdot},\pi_m\right)$with $\mathring{\Pi}_n^{\cdot} \in \Delta^T$ and $\pi_m^n \in \phi_m\left(\mathring{\Pi}_n^{\cdot}\right)$ for all n, then $\pi_m \in \phi_m\left(\mathring{\Pi}^{\cdot}\right)$, so it suffices to show that if $\mathring{\Pi}_n^{\cdot} \in S$ for all n, then $\mathring{\Pi}^{\cdot} = \lim_{n\to\infty}\mathring{\Pi}_n^{\cdot}$ is also in S, but this must hold as S is compact: ϕ_m is upper hemicontinuous on S. As for upper hemicontinuity of ψ_m on S, suppose $\left(\mathring{\Pi}_n^{\cdot},\left(\mathring{V}_n^{\cdot},\mathring{U}_n\right)\right) \to \left(\mathring{\Pi}^{\cdot},\left(\mathring{V}^{\cdot},\mathring{U}\right)\right)$with $\mathring{\Pi}_n^{\cdot} \in S$ and $\left(\mathring{V}_n^{\cdot},\mathring{U}_n\right) \in \psi_m\left(\mathring{\Pi}_n^{\cdot}\right)$for all n. As just shown, $\mathring{\Pi}^{\cdot} \in S$, so we must merely show that $\left(\mathring{V},\mathring{U}\right) \in \psi_m\left(\mathring{\Pi}^{\cdot}\right)$. By (14), for each $n$there is an $\pi_m^n \in \phi_m\left(\mathring{\Pi}^{\cdot}\right)$ such that $\mathring{V}_n^{\cdot} = \mathring{V}_m^{\cdot}(\pi_m^n)$ and $\mathring{U}_n = \mathring{U}_m(\pi_m)$. By taking subsequences if needed, we may assume that $\pi_m = \lim_{n\to\infty}\pi_m^n$exists. Hence, $\left(\mathring{\Pi}_n^{\cdot},\pi_m^n\right) \to \left(\mathring{\Pi}^{\cdot},\pi_m\right)$with $\pi_m^n \in \phi_m\left(\mathring{\Pi}_n^{\cdot}\right)$for all n. Thus, by the upper hemicontinuity of ϕ_m on S, $\pi_m \in \phi_m\left(\mathring{\Pi}^{\cdot}\right)$ whence, by (14), $\left(\mathring{V}_m^{\cdot}(\pi_m),\mathring{U}_m(\pi_m)\right) \in \psi_m\left(\mathring{\Pi}^{\cdot}\right)$. Finally, since π_m^n converges to π_m, (12) and (13) imply that $\left(\mathring{V}_m^{\cdot}(\pi_m^n),\mathring{U}_m(\pi_m^n)\right) = \left(\mathring{V}_n^{\cdot},\mathring{U}_n\right)$ converges to $\left(\mathring{V}_m^{\cdot}(\pi_m),\mathring{U}_m(\pi_m)\right)$ but by assumption $\lim_{n\to\infty}\left(\mathring{V}_n^{\cdot},\mathring{U}_n\right) = \left(\mathring{V}^{\cdot},\mathring{U}\right)$ whence $\left(\mathring{V}^{\cdot},\mathring{U}\right) = \left(\mathring{V}_m^{\cdot}(\pi_m),\mathring{U}_m(\pi_m)\right) \in \psi_m\left(\mathring{\Pi}^{\cdot}\right)$ as claimed: ψ_m is upper hemicontinuous on S. This concludes the proof of Claim 24. □

Let

$$\Psi_m = \bigcup_{\mathring{\Pi}^{\cdot} \in \Delta^T} \psi_m\left(\mathring{\Pi}^{\cdot}\right) \tag{109}$$

be the set of all correlated equilibrium payoff vectors in the mechanism m, for *any* interim beliefs $\mathring{\Pi}^{\cdot}$ in Δ^T.

Claim 25. Ψ_m is compact.

Proof. We first show that it is bounded. Since we assume $\rho \in [-\overline{\rho}, \overline{\rho}]$ for some arbitrarily large but finite $\overline{\rho}$, the worst that could happen to the principal is to pay the agent $\overline{\rho}$ and also give the agent her entire portfolio; her payoff would then be at least $-\overline{\rho} - \delta af(T)$. The best that could occur is for him to retain her whole portfolio and receive a transfer of $\overline{\rho}$ from the agent, giving him a payoff of $\overline{\rho}$. Hence, the principal's payoff lies in $[-\overline{\rho} - \delta af(T), \overline{\rho}]$ for each type. Likewise, the agent's payoff lies in $[-\overline{\rho}, \overline{\rho} + af(T)]$. Hence, Ψ_m is bounded.

We now show that it is closed. Let $\left(\mathring{V}_n^{\cdot}, \mathring{U}_n\right) \to \left(\mathring{V}^{\cdot}, \mathring{U}\right)$with $\left(\mathring{V}_n^{\cdot}, \mathring{U}_n\right) \in \Psi_m$for all n. Then by definition of Ψ_m, for each n there is a $\mathring{\Pi}_n^{\cdot}$ in Δ^T such that $\left(\mathring{V}_n^{\cdot}, \mathring{U}_n\right) \in \psi_m\left(\mathring{\Pi}_n^{\cdot}\right)$. By taking subsequences if needed, we can assume that $\mathring{\Pi}_n^{\cdot}$ converges to a limit $\mathring{\Pi}^{\cdot}$. That is, $\left(\mathring{\Pi}_n^{\cdot}, \left(\mathring{V}_n^{\cdot}, \mathring{U}_n\right)\right) \to \left(\mathring{\Pi}^{\cdot}, \left(\mathring{V}^{\cdot}, \mathring{U}\right)\right)$with $\mathring{\Pi}_n^{\cdot} \in \Delta^T$ and $\left(\mathring{V}_n^{\cdot}, \mathring{U}_n\right) \in \psi_m\left(\mathring{\Pi}_n^{\cdot}\right)$for all n. As shown in the proof of Claim 24, this implies that $\mathring{\Pi}^{\cdot} \in \Delta^T$ and $\left(\mathring{V}, \mathring{U}\right)$ lies in $\psi_m\left(\mathring{\Pi}^{\cdot}\right)$ which is a subset of Ψ_m: Ψ_m is closed. □

By Claim 10, the RSW allocation is IE with respect to some nonnegative prior beliefs $\widehat{\Pi}^{\cdot}$. Let us now modify the game in two ways: (a) the agent's prior beliefs are $\widehat{\Pi}^{\cdot}$ rather than $\Pi^{\cdot}$ and (b) the principal must choose either the mechanism m or the RSW DRM $\widehat{o}^{\cdot}$. In the modified game $\hat{G}$, let $(V^{\cdot}, U) \in \Psi_m$ denote the correlated equilibrium payoff vector that results when the principal chooses m. In this situation,

$$P^t(V^{\cdot}, U) = \arg\max_{p \in [0,1]} \left[pV^t + (1-p)\widehat{V}^t\right] \tag{110}$$

is the type-t principal's set of optimal probabilities of choosing m. Let $P^{\cdot}(V^{\cdot}, U) = \prod_{t=0}^{T} P^t(V^{\cdot}, U)$

be the corresponding set of optimal probability vectors[37] and let

$$P(\Psi_m) = \{P^{\cdot}(V^{\cdot}, U) : (V^{\cdot}, U) \in \Psi_m\}$$

be the image of Ψ_m under $P^{\cdot}$. Clearly, $P(\Psi_m)$ is a subset of

$$P_m^{\Sigma} = \left\{ p^{\cdot} \in [0,1]^{T+1} : \; p^t = 0 \text{ for all } t \in J_m \right\}. \tag{111}$$

Claim 26. The correspondence $P^{\cdot} : \Psi_m \rightrightarrows P_m^{\Sigma}$ is upper hemicontinuous and, for any payoff vector $(V^{\cdot}, U) \in \Psi_m$, the set $P^{\cdot}(V^{\cdot}, U)$ is nonempty and convex. Moreover, P_m^{Σ} is compact.

Proof. Compactness of P_m^{Σ} is immediate from (111). For any $V^{\cdot}$ in $\Re^{T+1}$, $P^t(V^{\cdot})$ is clearly nonempty and convex whence so is $P^{\cdot}(V^{\cdot})$. For upper hemicontinuity, suppose that $(V_n^{\cdot}, p_n^{\cdot}) \to (V^{\cdot}, p^{\cdot})$with $V_n^{\cdot} \in S$ and $p_n^{\cdot} \in P^{\cdot}(V_n^{\cdot})$for all n. Then $V^{\cdot} \in S$ as S is compact. It remains to show that $p^{\cdot} \in P^{\cdot}(V^{\cdot})$ - or, equivalently, that $p^t \in P^t(V^{\cdot})$ for each type t. This is trivial if $V^t = \widehat{V}^t$ since then $P^t(V^{\cdot}) = [0,1]$ which must contain p^t. If instead $V^t < (>) \widehat{V}^t$, then $P^t(V^{\cdot})$ equals $\{0\}$ (resp., $\{1\}$) but also, for high enough n, $V_n^t < (>) \widehat{V}^t$ whence p_n^t equals 0 (resp., 1) and thus $\lim_{n\to\infty} p_n^t$ is also 0 (resp., 1) which lies in $P^t(V^{\cdot})$, a contradiction. Since this is so for each type t, $p^{\cdot} = \lim_n p_n^{\cdot}$ lies in $P^{\cdot}(V^{\cdot})$ as claimed. □

In the modified game $\hat{G}$, let m lead to the correlated equilibrium payoff vector $(V^{\cdot}, U)$. Let the principal's strategy be $p^{\cdot}$ where p^t is the type t principal's chance of choosing m. If the principal chooses m, what is the set of possible interim beliefs $\mathring{\Pi}^{\cdot}$ of the agent if her prior is $\widehat{\Pi}^{\cdot}$?First, if the principal's ex-ante probability $\widehat{P}(p^{\cdot}) = \sum_{t=0}^{T} p^t \widehat{\Pi}^t$ of choosing m is zero, the choice of m is unexpected: the agent's beliefs are arbitrary. In this case, we will restrict beliefs to lie in the set Δ_m^{Σ} defined in (106). On the other hand, if $\widehat{P}(p^{\cdot})$ is positive, the agent's beliefs are uniquely given by Bayes's Rule: $\mathring{\Pi}^t = p^t \widehat{\Pi}^t / \widehat{P}(p^{\cdot})$ for all t. Hence, $\mathring{\Pi}^t$ is positive only if $p^t > 0$ which, by (111), implies $t \notin J_m$ since $p^{\cdot} \in P_m^{\Sigma}$. Thus, $\mathring{\Pi}^t$ lies in Δ_m^{Σ} in this case as well. Combining the two cases, our assumptions imply that for each $p^{\cdot}$

[37] The symbol $\prod$ here denotes the Cartesian product.

in P_m^Σ, the agent's interim beliefs $\mathring{\Pi}^\cdot$ on seeing m will lie in the set

$$\beta\left(p^\cdot\right)=\begin{cases}\left\{\left(\frac{p^0\widehat{\Pi}^0}{\widehat{P}(p^\cdot)},\ldots,\frac{p^T\widehat{\Pi}^T}{\widehat{P}(p^\cdot)}\right)\right\}\subset\Delta_m^\Sigma & \text{if} \quad \widehat{P}\left(p^\cdot\right)>0\\ \Delta_m^\Sigma & \text{if} \quad \widehat{P}\left(p^\cdot\right)=0\end{cases} \tag{112}$$

which, in each case, is a subset of Δ_m^Σ. Let $\beta\left(P_m^\Sigma\right)=\left\{\beta\left(p^\cdot\right):p^\cdot\in P_m^\Sigma\right\}\subset\Delta_m^\Sigma$ be the image of P_m^Σ under β.

Claim 27. The correspondence $\beta:P_m^\Sigma\rightrightarrows\Delta_m^\Sigma$ is upper hemicontinuous and, for any vector $p^\cdot\in P_m^\Sigma$ of probabilities, the set $\beta\left(p^\cdot\right)$ is nonempty and convex.

Proof. By (112), $\beta\left(p^\cdot\right)$ is nonempty and convex. For upper hemicontinuity, suppose $\left(p_n^\cdot,x_n^\cdot\right)\to\left(p^\cdot,x^\cdot\right)$with $p_n^\cdot\in P_m^\Sigma$ and $x_n^\cdot\in\beta\left(p_n^\cdot\right)$for all n. Then $p^\cdot\in P_m^\Sigma$ since P_m^Σ is compact (Claim 26). It remains to show that $x^\cdot\in\beta\left(p^\cdot\right)$. Since $x_n^\cdot\in\beta\left(p_n^\cdot\right)\subset\Delta_m^\Sigma$ for all n and Δ_m^Σ is closed, $x^\cdot=\lim_{n\to\infty}x_n^\cdot$ is also in Δ_m^Σ; thus, if $\widehat{P}\left(p^\cdot\right)$ is zero we are done. If $\widehat{P}\left(p^\cdot\right)$ is positive, then, since the function $\widehat{P}\left(p_n^\cdot\right)$ is continuous in $p_n^\cdot$, there is a n^* such that, for all $n>n^*$, $\widehat{P}\left(p_n^\cdot\right)$ is positive whence $x_n^\cdot\in\beta\left(p_n^\cdot\right)$ must equal $\left(\frac{p_n^0\widehat{\Pi}^0}{\widehat{P}(p_n^\cdot)},\ldots,\frac{p_n^T\widehat{\Pi}^T}{\widehat{P}(p_n^\cdot)}\right)$ and thus[38]

$$x^\cdot=\lim_{n\to\infty}x_n^\cdot=\lim_{n\to\infty}\left(\frac{p_n^0\widehat{\Pi}^0}{\widehat{P}\left(p_n^\cdot\right)},\ldots,\frac{p_n^T\widehat{\Pi}^T}{\widehat{P}\left(p_n^\cdot\right)}\right)=\left(\frac{p^0\widehat{\Pi}^0}{\widehat{P}\left(p^\cdot\right)},\ldots,\frac{p^T\widehat{\Pi}^T}{\widehat{P}\left(p^\cdot\right)}\right)\in\beta\left(p^\cdot\right).$$

Hence β is upper hemicontinuous. □

Let D_m denote the Cartesian product $\Psi_m\times P_m^\Sigma\times\Delta_m^\Sigma$, which is compact by Claims 25, 26, and 23, resp. Consider the correspondence $r:D_m\rightrightarrows D_m$ that maps each $d=\left(\left(V^\cdot,U\right),p^\cdot,\mathring{\Pi}^\cdot\right)$ in D_m to $r(d)=\psi_m\left(\mathring{\Pi}^\cdot\right)\times P^\cdot\left(V^\cdot\right)\times\beta\left(p^\cdot\right)$. For any d in D_m, $r(d)$ is nonempty and convex since $\psi_m\left(\mathring{\Pi}^\cdot\right)$, $P^\cdot\left(V^\cdot\right)$, and $\beta\left(p^\cdot\right)$ are by Claims 24, 26, and 27, resp. Moreover, r is upper hemicontinuous since ψ_m, $P^\cdot$, and β have this property by the same Claims. Thus, by Kakutani's [16] fixed point theorem, r has a fixed point

$$\left(\left(V_*^\cdot,U_*\right),p_*^\cdot,\mathring{\Pi}_*^\cdot\right)\in D_m \tag{113}$$

[38]This uses the fact that $f(x,y)=g(x)/h(y)$ is continuous at (x_0,y_0) if g and h are continuous and $h(y_0)\neq 0$.

satisfying

$$\left(V_*^{\cdot}, U_*\right) \in \psi_m\left(\mathring{\Pi}_*^{\cdot}\right), \tag{114}$$

$$p_*^{\cdot} \in P^{\cdot}\left(V_*^{\cdot}\right), \text{ and} \tag{115}$$

$$\mathring{\Pi}_*^{\cdot} \in \beta\left(p_*^{\cdot}\right). \tag{116}$$

By (114), there is a correlated equilibrium $\pi_m^* \in \phi_m\left(\mathring{\Pi}^{\cdot}\right)$ such that

$$\left(V_*^{\cdot}, U_*\right) = \left(\mathring{V}_m^{\cdot}\left(\pi_m^*\right), \mathring{U}_m\left(\pi_m^*\right)\right). \tag{117}$$

Moreover, the beliefs $\mathring{\Pi}_*^{\cdot}$ are in Δ_m^{Σ} by (113) and thus satisfy Int_m^{Σ} by Claim 23.

For the mechanism m, we have now constructed beliefs $\mathring{\Pi}^t(m) = \mathring{\Pi}_*^{\cdot}$ (which satisfy Int_m^{Σ}) and associated correlated equilibrium payoffs $\left(\mathring{V}^{\cdot}(m), \mathring{U}(m)\right) = \left(\mathring{V}_*^{\cdot}, \mathring{U}_*\right)$ of the stage-2 game. By hypothesis, there is a type t whose payoff $\mathring{V}^t(m)$ exceeds $\widehat{V}^t$.

What is the expected type-contingent outcome in the stage-2 equilibrium of m? For each pure action profile $\left(s_{\cdot}^P, s^A\right) \in \left(S_m^P\right)^{T+1} \times S_m^A$, the probability is $\pi_m^*\left(s_{\cdot}^P, s^A\right)$ that $\left(s_{\cdot}^P, s^A\right)$ is played. If, moreover, the principal's type is t, the resulting outcome is $o_m^{s_t^P, s^A}$ for sure. Thus, the expected outcome conditional on the principal choosing m and her type being t is

$$o_*^t = \sum_{s^A \in S_m^A} \sum_{s_{\cdot}^P \in \left(S_m^P\right)^{T+1}} \pi_m^*\left(s_{\cdot}^P, s^A\right) o_m^{s_t^P, s^A}. \tag{118}$$

Now consider the following scenario of the modified game $\hat{G}$.

Stage 1: the principal of type t proposes m with probability p_*^t and an RSW allocation $\widehat{o}^{\cdot}$ with complementary probability.

Stage 2: if the principal chose $\widehat{o}^{\cdot}$, the agent plays "accept" and the principal chooses the outcome $\widehat{o}^t$ that corresponds to her type t. If the principal chose m, the players play the correlated equilibrium π_m^* whence the expected outcome, for a principal of type t, is o_*^t.

We claim that the agent's *ex ante* expected payoff in Σ is nonnegative. This payoff is

$$\sum_{t=0}^{T} \widehat{\Pi}^t\left[p_*^t U^t\left(o_*^t\right) + \left(1 - p_*^t\right) U^t\left(\widehat{o}^t\right)\right] \geq \sum_{t=0}^{T} \widehat{\Pi}^t p_*^t U^t\left(o_*^t\right) \text{ by (22).} \tag{119}$$

If $\widehat{\Pi}^t p_*^t$ is zero for all t, we are done as the right hand side is zero. Else $\sum_{s=0}^{T} \widehat{\Pi}^s p_*^s$ is positive, whence the right hand side of (119) has the same sign as

$$\begin{aligned}
\sum_{t=0}^{T} \frac{\widehat{\Pi}^t p_*^t}{\sum_{s=0}^{T} \widehat{\Pi}^s p_*^s} U^t\left(o_*^t\right) &= \sum_{t=0}^{T} \mathring{\Pi}^t(m)\, U^t\left(o_*^t\right) \text{ by (10) with } \Pi^{\cdot} = \widehat{\Pi}^{\cdot} \\
&= \sum_{s^A \in S_m^A} \sum_{s_{\cdot}^P \in \left(S_m^P\right)^{T+1}} \pi_m^*\left(s_{\cdot}^P, s^A\right) \sum_{t=0}^{T} \mathring{\Pi}^t(m)\, U^t\left(o_m^{s_t^P, s^A}\right) \text{ by (3) and (118)} \\
&= \sum_{s^A \in S_m^A} \sum_{s_{\cdot}^P \in \left(S_m^P\right)^{T+1}} \pi_m^*\left(s_{\cdot}^P, s^A\right) u_m^A\left(s_{\cdot}^P, s^A\right) \text{ by (9)} \\
&= \mathring{U}_m\left(\pi_m^*\right) \text{ by (12)}
\end{aligned}$$

which is nonnegative by Claim 1.

Now let us give the principal a third option at the initial stage in $\hat{G}$: she can propose the DRM $\widetilde{o}^{\cdot}$ defined, for each t, by $\widetilde{o}^t = p_*^t o_*^t + (1 - p_*^t)\,\widehat{o}^t$. If the agent accepts and the principal chooses $\widetilde{o}^t$, the players get the same type-contingent payoffs as in Σ above. Thus, as Σ gives the agent a nonnegative payoff, she will accept the DRM. And as the principal does not want to deviate in Σ, she will choose $\widetilde{o}^t$: $\widetilde{o}^{\cdot}$is incentive-compatible. But recall that m is a mechanism for which, for any beliefs $\mathring{\Pi}^{\cdot} \in \Delta^T$ and any associated equilibrium payoffs $\left(\mathring{V}^{\cdot}, \mathring{U}\right)$, there is a type t who does better than in the RSW allocation: $\mathring{V}^t > \widehat{V}^t$. Letting $\mathring{\Pi}^{\cdot} = \mathring{\Pi}^{\cdot}(m)$, it follows that there is a type t for which $V_*^t > \widehat{V}^t$ and thus, by (114), $p_*^t = 1$, so $V^t\left(\widetilde{o}^t\right) > \widehat{V}^t$. Since, by (114), $V^t\left(\widetilde{o}^t\right) = \max\left\{V_*^t, \widehat{V}^t\right\} \geq \widehat{V}^t$, we have produced an allocation, $\widetilde{o}^{\cdot}$, that is incentive-compatible, Pareto dominates $\widehat{o}^{\cdot}$, and gives the agent a nonnegative payoff under the beliefs $\widehat{\Pi}^{\cdot}$. Thus, by definition 15, $\widehat{o}^{\cdot}$ is not IE under the beliefs $\widehat{\Pi}^{\cdot}$ - a contradiction. **Q.E.D.**$_{\text{Proposition 1}}$

PROOF OF PROPOSITION 2. Let $\Sigma = \left(p_{\cdot}^{\cdot}, \mathring{\Pi}^{\cdot}(\cdot), \pi_{\cdot}\right)$ be an intuitive equilibrium whose expected allocation is $o^{\cdot} = o_{\Sigma}^{\cdot} = (q^{\cdot}, \rho^{\cdot})$. Assume w.l.o.g. that Σ is a DRM equilibrium.[39]

[39] If not, use Σ to construct the equivalent DRM equilibrium $\widehat{\Sigma}$ as in footnote 31. It has the same equilibrium payoffs of principal and agent for each type of principal. Moreover, beliefs following any deviation m of the principal are the same in $\widehat{\Sigma}$ as in Σ. Hence, since π_m is intuitive, so is the belief function $\widehat{\pi}_m$ of Σ: We have thus constructed an intuitive DRM equilibrium that implements the expected allocation $o_{\Sigma}^{\cdot}$ of Σ.

When considering deviations by the principal, it will suffice to focus on simple mechanisms in which the agent has two actions, accept and reject, and the principal has one action: "do nothing". If the agent accepts, a given outcome (q,ρ) is implemented; else the reservation allocation (in which both get zero) is implemented. We will refer to such a mechanism, abusing notation slightly, as $m=(q,\rho)$.

We first show that Σ is fairly priced: the amount ρ^t that the agent pays to a principal of each type t equals the conditional expected value $q^t f(t)$ of the portfolio that she receives. For suppose not. First suppose that the portfolio of some type t is underpriced: $\rho^t < q^t f(t)$. Let s be the largest type such that $q^t f(s) < q^t f(t)$. (If there is no such type s, then the principal can instead select the mechanism $m=(q^t, q^t f(t)-\varepsilon)$ for any $\varepsilon \in (0, q^t f(t)-\rho^t)$ and the agent will accept so Σ is not an equilibrium.) Choose any $\lambda \in [0,1)$ satisfying

$$\frac{\rho^t - \delta q^t f(t)}{q^t [f(t) - \delta f(t)]} < \lambda < \frac{\rho^t - \delta q^t f(s)}{q^t [f(t) - \delta f(s)]}. \tag{120}$$

Such a λ must exist as the ratio on the right is increasing in $-q^t f(s)$ since $\rho^t < q^t f(t)$. Also choose any

$$\varepsilon \in \left(0, \lambda q^t [f(t) - \delta f(t)] - \rho^t - \delta q^t f(t)\right) \tag{121}$$

where the interval is nonempty by the first inequality in (120). Now suppose type t deviates to the mechanism $m=(\lambda q^t, \lambda q^t f(t)-\varepsilon)$. By Assumption 1, for all $s' < s$, $q^t f(s') \leq q^t f(s) < q^t f(t)$ and so, by (120), $\lambda q^t [f(t) - \delta f(s')]$ is less than $\rho^t - \delta q^t f(s')$ which, in turn, is no greater than $V^{s'}\left(o^{s'}\right)$ since s' is willing not to imitate t in Σ. Thus, since $\varepsilon > 0$ by (121), in any stage-2 correlated equilibrium following a deviation to m, each type $s' \leq s$ gets less than her equilibrium payoff $V^{s'}\left(o^{s'}\right)$: each type $s' \leq s$ is in the set J_m defined in (31). Now suppose the principal chooses m and the agent is sure the agent's type is not in J_m. Then by Assumption 1, she is sure that the portfolio λq^t is worth at least $\lambda q^t f(t)$. As accepting thus gives her at least $\varepsilon > 0$, she will accept in any correlated equilibrium π_m.[40] This implies, in turn, that a type-t principal gains by deviating to m for any π_m since,by

[40] Since the principal's choice in stage 2 is trivial, correlated equilibrium implies merely that the agent will act optimally given her beliefs.

(120) and (121), $V^t(o^t) = \rho^t - \delta q^t f(t)$ is less than $\lambda q^t [f(t) - \delta f(t)]$. It follows from Int_m^Σ that on seeing m, the agent's interim beliefs $\mathring{\Pi}^{\cdot}(m)$ must put zero weight on any type in J_m: she will accept. But then t will surely defect: Σ is not an equilibrium.

We have shown that no portfolio is underpriced in Σ: for each t, $\rho^t \leq q^t f(t)$. Can there be overpricing? No: as each agent has positive ex-ante probability Π^t, this (with the absence of underpricing) would imply that the agent's ex-ante expected payoff from accepting the DRM Σ is negative - which contradicts Claim 1.

Since Σ is fairly priced, the price paid by the agent equals the value of the assets she receives for each type t:

$$\rho^t = q^t f(t) \text{ for } t = 0, ..., T. \tag{122}$$

Hence, the payoff $U^t(o^t) = q^t f(t) - \rho^t$ of the agent is identically zero. By Claim 8, the payoff $V^t(o^t)$ of the type-t principal is not less than $\widehat{V}^t$ which, in turn, equals the solution $u^*(t)$ to RLP by Corollary 5. We claim that $V^t(o^t) \leq u^*(t)$ as well and that q^t lies in the set Q_t defined in (23).If not, let t be the lowest type for which $V^t(o^t) > u^*(t)$. By fair pricing,

$$V^t\left(o^t\right) = \rho^t - \delta q^t f(t) \text{ equals } (1-\delta) q^t f(t) \text{ for each } t. \tag{123}$$

But by (24), (25), and (26), $u^*(t)$ equals $\max_{q \in Q_t}[(1-\delta) q f(t)]$. Thus, if $V^t(o^t) > u^*(t)$, then q^t must not lie in Q_t. By (23), this means that there is a type $s < t$ whose equilibrium payoff $V^s(o^s) = u^*(s)$ in Σ is less than $q^t[f(t) - \delta f(s)]$ which in turn, by (122), equals her payoff $\rho^t - \delta q^t f(s)$ from imitating t in Σ. Thus, Σ is not an equilibrium - a contradiction.

We have shown that in any intuitive equilibrium Σ, the payoff $V^t(o^t)$ of the type-t principal equals her RSW payoff $\widehat{V}^t$ which, in turn, equals $u^*(t)$ and the agent's corresponding payoff $U^t(o^t)$ also equals her RSW payoff of zero. Moreover, $q^t \in Q_t$. If q^t does not maximize $q^t f(t)$ in this set then

$$\begin{aligned} V^t\left(o^t\right) &= (1-\delta) q^t f(t) \text{ by (123)} \\ &< (1-\delta) q^*(t) f(t) \text{ by (24)} \\ &= u^*(t) \text{ by (25) and (26)} \\ &= V^t\left(o^t\right) \text{ as shown above.} \end{aligned}$$

This is a contradiction. We thus have confirmed that $q^*(t) = q^t$, $\rho^*(t) = \rho^t$, and $u^*(t) = V^t(o^t)$ are a solution to RLP. Accordingly, by part 2 of Claim 5, the allocation $o^\cdot$ is an RSW allocation as claimed.

We have shown that any DRM $\overline{o}^\cdot$ that satisfies (33), (34), and (35) is supported by an equilibrium as claimed. Moreover, for each mechanism $m \in M$, the property Int_m^Σ holds since the agent's interim beliefs $\mathring{\Pi}^\cdot(m)$ have been constructed to lie in Δ_m^Σ. Thus, if $\overline{o}^\cdot$ is an RSW allocation $\widehat{o}^\cdot$, the equilibrium is intuitive as well: any RSW allocation $\widehat{o}^\cdot$ is the expected allocation of an intuitive equilibrium, as claimed. **Q.E.D.**$_{\text{Proposition 2}}$

PROOF OF CLAIM 12. Equations (52) through (58) are proved in the text, and (68) follows from (44), (45), and (54). Substituting (53) into $V^2(\widehat{o}^1) = \widehat{\rho}^1 - \delta a f(2)$ we obtain the equality in (59). The RSW allocation $\widehat{o}^\cdot$ is an equilibrium allocation by Claim 11 and $\widehat{q}^2 \neq \widehat{q}^1 = a$ by Claim 12, whence the inequality in (59) follows from the following lemma.

Lemma 2. *In any equilibrium allocation $o^\cdot$ in which $q^2 \neq q^1 = a$, $V^2(o^2) > V^2(o^1)$.*

Proof. By type 1's IC constraint,

$$\begin{aligned} V^2(o^2) - V^2(o^1) &\geq V^2(o^2) - V^2(o^1) - \left[V^1(o^2) - V^1(o^1)\right] \\ &= \delta\left(a - q^2\right)\Delta^1 > 0 \end{aligned}$$

where the strict inequality is by (36). □

By (54) and (58), $\widehat{V}^1 - \widehat{V}^2$ equals $\widehat{q}^2[\Delta - (1-\delta)f(2)]$ which in turn, by (45), equals $\delta\widehat{q}^2\Delta^1$, proving (60). As for (62),

$$\widehat{V}^1 - V^2(\widehat{o}^1) = (1-\delta)af(1) - a[f(1) - \delta f(2)] = \delta a \Delta^1$$

as claimed whence, by (60),

$$\widehat{V}^2 - V^2(\widehat{o}^1) = \left[\widehat{V}^1 - V^2(\widehat{o}^1)\right] - \left[\widehat{V}^1 - \widehat{V}^2\right] = \delta\left(a - \widehat{q}^2\right)\Delta^1$$

proving (63). Equation (64) follows from (48) and (49). As for (65) and (66), there are two cases.

1. If $\widehat{V}^1 \leq \Delta_1$, (55) and (56) imply $\widehat{q}_1^2 = \frac{\widehat{V}^1}{\Delta_1}$ and $\widehat{q}_2^2 = 0$. Substituting these into (58), clearing denominators, and multiplying by -1, we obtain $(\delta - 1)\widehat{V}^1 f_1(2) = -\widehat{V}^2\Delta_1$. Adding $\Delta_1\widehat{V}^1$ to both sides then yields $\delta\widehat{V}^1\Delta_1^1 = \Delta_1\left(\widehat{V}^1 - \widehat{V}^2\right)$, which proves (a) the $<$ and = parts of (67), (b) that the second elements of the mins in (55) and (65) are equal, and (c) that the second entry of the max in (65) is zero. This verifies (65) and (66) for this case.

2. If $\widehat{V}^1 > \Delta_1$, (55) and (56) imply $\widehat{q}_1^2 = 1$ and $\widehat{q}_2^2 = \frac{\widehat{V}^1 - \Delta_1}{\Delta_2}$. Substituting these into (58) and clearing denominators yields

$$\widehat{V}^2\Delta_2 = (1-\delta)\left[f_1(2)\Delta_2 + \left(\widehat{V}^1 - \Delta_1\right)f_2(2)\right].$$

Isolating the Δ_2 terms on the right hand side yields

$$\left(\widehat{V}^1 - \Delta_1\right)\left[(\delta - 1) f_2(2)\right] = \Delta_2\left\{-\widehat{V}^2 - (\delta - 1) f_1(2)\right\}.$$

Adding $\left(\widehat{V}^1 - \Delta_1\right)\Delta_2$ to both sides and using (45) to substitute for Δ_2 on the left and for Δ_1 on the right we obtain

$$\left(\widehat{V}^1 - \Delta_1\right)\delta\Delta_2^1 = \Delta_2\left\{\widehat{V}^1 - \widehat{V}^2 - \delta\Delta_1^1\right\},$$

which (a) proves the $>$ part of (67) and (b) implies that the second elements in the maxes in (56) and (66) are equal. Hence, by hypothesis that $\widehat{V}^1 > \Delta_1$, the second element of the max in (66) is positive, whence the second element in the min in (65) exceeds 1. Finally, This verifies (65) and (66) for this case.

As for (61), by (55), (56), and (60), there are two cases.

1. If $\widehat{V}^1 \leq \Delta_1$ then $\widehat{q}^2 = \left(\frac{\widehat{V}^1}{\Delta_1}, 0\right)$ so

$$\widehat{V}^1 - \widehat{V}^2 = \delta\frac{\widehat{V}^1}{\Delta_1}\Delta_1^1. \tag{124}$$

2. If $\widehat{V}^1 > \Delta_1$ then $\widehat{q}^2 = \left(1, \frac{\widehat{V}^1 - \Delta_1}{\Delta_2}\right)$ so

$$\widehat{V}^1 - \widehat{V}^2 = \delta\left\{\Delta_1^1 + \frac{\widehat{V}^1 - \Delta_1}{\Delta_2}\Delta_2^1\right\}. \tag{125}$$

Combining the two cases we obtain (61). Finally, we verify (68). We compute

$$\zeta = \frac{\gamma}{a\Delta^1} > \frac{\delta\Delta_1^1}{\Delta_1} = \eta \iff \Delta_1^1 a\Delta^1 < \frac{\Delta_1\gamma}{\delta} = \Delta_1\left(a - \widehat{q}^2\right)\Delta^1 = \Delta_1 a\Delta^1 - \Delta_1\widehat{q}^2\Delta^1$$

$$\iff 0 < \left(\Delta_1 - \Delta_1^1\right)a\Delta^1 - \Delta_1\widehat{q}^2\Delta_1^1 \tag{126}$$

When $\nu > 0$, by (55), (56), and (67), this becomes

$$\begin{aligned}
&0 < \left(\Delta_1 - \Delta_1^1\right)a\Delta^1 - \Delta_1\frac{\widehat{V}^1}{\Delta_1}\Delta_1^1 = (1-\delta)f_1(1)a\Delta^1 - \widehat{V}^1\Delta_1^1\\
&= (1-\delta)f_1(1)a[f(2)-f(1)] - (1-\delta)af(1)[f_1(2)-f_1(1)]\\
&= (1-\delta)\{f_1(1)a[f(2)-f(1)] - af(1)[f_1(2)-f_1(1)]\}\\
&= (1-\delta)\{f_1(1)[f_1(2)-f_1(1)] + f_1(1)[f_2(2)-f_2(1)] - [f_1(1)+f_2(1)][f_1(2)-f_1(1)]\}\\
&= (1-\delta)\{f_1(1)[f_2(2)-f_2(1)] - f_2(1)[f_1(2)-f_1(1)]\}\\
&= (1-\delta)\{f_1(1)f_2(2) - f_2(1)f_1(2)\}
\end{aligned}$$

which is true by (37). When $\nu \le 0$, by (55), (56), and (67), (126) becomes

$$\begin{aligned}
0 &< \left(\Delta_1 - \Delta_1^1\right)a\Delta^1 - \Delta_1\left(\Delta_1^1 + \frac{\widehat{V}^1 - \Delta_1}{\Delta_2}\Delta_2^1\right)\\
&= -\Delta_1^1 a\Delta^1 + \Delta_1\left(a\Delta^1 - \Delta_1^1 - \frac{\widehat{V}^1 - \Delta_1}{\Delta_2}\Delta_2^1\right)\\
&= -\Delta_1^1 a\Delta^1 + \frac{\Delta_1\Delta_2^1}{\Delta_2}\left(a\Delta - \widehat{V}^1\right)\\
&= -\Delta_1^1 a\Delta^1 + \frac{\Delta_1\Delta_2^1}{\Delta_2}a\Delta^1 \text{ by (68)}\\
&= \Delta_1\left[\frac{\Delta_2^1}{\Delta_2} - \frac{\Delta_1^1}{\Delta_1}\right]a\Delta^1
\end{aligned}$$

which is positive by (36) and (46).

Q.E.D.$_{\text{Claim 12}}$

Proof of Claim 13. By Claim 12, it suffices to show that $o^{\cdot}$ is an RSW allocation. Since $\widehat{V}^1 = (1-\delta)af(1)$ is the maximum gains from trade when the seller's type is 1, $V^1\left(o^1\right) + U^1\left(o^1\right) \le \widehat{V}^1 = V^1\left(o^1\right)$ whence $U^1\left(o^1\right) \le 0$ and thus, by (40), $U^2\left(o^2\right) \ge 0$.

First suppose $U^1\left(o^1\right)=0$. Then $o^{\cdot}$ clearly satisfies (22). As $o^{\cdot}$ is an equilibrium allocation, it also satisfies (21). Since, moreover, it gives each type of seller her RSW payoff, it solves Program II^t in Definition 14 and hence is an RSW allocation as claimed. Now suppose instead that $U^1\left(o^1\right)<0$, whence $U^2\left(o^2\right)>0$; we will prove a contradiction. Consider the alternative allocation $\widetilde{o}^{\cdot}$ defined by $\widetilde{o}^1=\widehat{o}^1$ and $\widetilde{o}^2=o^2$. We have

$$U^1\left(\widetilde{o}^1\right)=U^1\left(\widehat{o}^1\right)=0 \text{ by part 1 of Claim 5;}$$

$$U^2\left(\widetilde{o}^2\right)=U^2\left(o^2\right)>0 \text{ by assumption;}$$

$$V^1\left(\widetilde{o}^2\right)=V^1\left(o^2\right)\leq V^1\left(o^1\right)=V^1\left(\widehat{o}^1\right)=V^1\left(\widetilde{o}^1\right) \text{ by construction, as } o^{\cdot} \text{ is an equilibrium, by hypothesis, and by construction, resp.;}$$

$$V^2\left(\widetilde{o}^2\right)=V^2\left(o^2\right)=V^2\left(\widehat{o}^2\right)\geq V^2\left(\widehat{o}^1\right)=V^2\left(\widetilde{o}^1\right) \text{ by construction, by hypothesis, by Claim 4, and by construction, resp.}$$

Accordingly, $\widetilde{o}^{\cdot}$ satisfies (21) and (22) and gives each type of seller her RSW payoff. Thus,$\widetilde{o}^{\cdot}$ solves Program II^t in Definition 14 and so is an RSW allocation. However, $U^2\left(\widetilde{o}^2\right)>0$, which contradicts part 1 of Claim 5.

Q.E.D.$_{\text{Claim 13}}$

PROOF OF CLAIM 14. By part (c) of Proposition 1, the payoff $V^1\left(o^1\right)$ of the low type seller cannot be less than her RSW payoff $\widehat{V}^1$ which, by parts 1 of Claim 5 as well as Claim 10 of DFJ2, equals $(1-\delta)af(1)$. But this is the maximum gains from trade $V^1\left(o^1\right)+U^1\left(o^1\right)=(1-\delta)q^1f(1)$ when the seller's type is low. Hence $U^1\left(o^1\right)\leq 0$ and thus, by (40), $U^2\left(o^2\right)\geq 0$. Accordingly, if $U^1\left(o^1\right)=0$, then $o^{\cdot}$ satisfies (22). As $o^{\cdot}$ is an equilibrium allocation, it also satisfies (21) and thus, by Definition 14, $V^t\left(o^t\right)\leq\widehat{V}^t$ for each t, which contradicts P0. We conclude that $U^1\left(o^1\right)<0$ whence, by (40), $U^2\left(o^2\right)>0$.

Now suppose that $V^1\left(o^1\right)\leq\widehat{V}^1$; we will derive a contradiction. By (41), $V^1\left(o^1\right)=\widehat{V}^1$, and by P0, $V^2\left(o^2\right)>\widehat{V}^2$. By part 2 of Claim 5, the above unique solution to RLP induces an RSW allocation $\widehat{o}^{\cdot}$ defined by $\widehat{o}^t=(\widehat{q}^t,\widehat{\rho}^t)=(q^*(t),\rho^*(t))$ for $t=1,2$. Consider the allocation $\widetilde{o}^{\cdot}$ defined by $\widetilde{o}^1=\widehat{o}^1$ and $\widetilde{o}^2=o^2$. We have

$$U^1\left(\widetilde{o}^1\right) = U^1\left(\widehat{o}^1\right) = 0 \text{ by part 1 of Claim 5;}$$
$$U^2\left(\widetilde{o}^2\right) = U^2\left(o^2\right) > 0 \text{ as shown above;}$$
$$V^1\left(\widetilde{o}^2\right) = V^1\left(o^2\right) \leq V^1\left(o^1\right) = V^1\left(\widehat{o}^1\right) = V^1\left(\widetilde{o}^1\right) \text{ by construction, as } o^{\cdot} \text{ is an equilibrium, as shown above, and by construction, resp.;}$$
$$V^2\left(\widetilde{o}^2\right) = V^2\left(o^2\right) \geq V^2\left(\widehat{o}^2\right) \geq V^2\left(\widehat{o}^1\right) = V^2\left(\widetilde{o}^1\right) \text{ by construction, as shown above, by Claim 4, and by construction, resp.}$$

Accordingly, $\widehat{o}^{\cdot}$ satisfies (21) and (22) yet gives type 2 seller more than her RSW payoff - contradicting Definition 14.

Q.E.D.$_{\text{Claim 14}}$

Proof of Claim 15. The set S is compact by Claim 11. Hence, S_x is also compact as it adds an equality to the set of constraints that define S. Thus, if S_x is nonempty, then the continuous function J has a maximum on S_x by the Extreme Value Theorem: $\overline{S}_x$ is nonempty as well.

Now suppose S_x is nonempty and let $o^{\cdot} \in \overline{S}_x$. We will show that $o^{\cdot}$ satisfies P1-P4 in turn. Suppose first that $o^{\cdot}$ violates P1. First suppose $q^1 \neq a$. Let $\widehat{q}^1 = a$ and $\widehat{\rho}^1 = \rho^1 + \delta\left(a - q^1\right) f(1)$, and $\widehat{o}^2 = o^2$. One can see that $V^t\left(\widehat{o}^t\right) = V^t\left(o^t\right)$ for $t = 1, 2$: $\widehat{o}^{\cdot}$ satisfies P0 and (41) since $o^{\cdot}$ does. Moreover, $V^1\left(\widehat{o}^2\right) = V^1\left(o^2\right)$ since $\widehat{o}^2 = o^2$, and

$$V^2\left(\widehat{o}^1\right) - V^2\left(o^1\right) = \widehat{\rho}^1 - \rho^1 - \delta\left(a - q^1\right) f(2) = \delta\left(a - q^1\right)\left[f(1) - f(2)\right] \leq 0$$

whence $\widehat{o}^{\cdot}$ satisfies (38) since $o^{\cdot}$ does. Finally,

$$U^1\left(\widehat{o}^1\right) - U^1\left(o^1\right) = (1-\delta)\left(a - q^1\right) f(1) > 0 \tag{127}$$

by (36) while $U^2\left(\widehat{o}^2\right) = U^2\left(o^2\right)$ whence $\widehat{o}^{\cdot}$ satisfies (40) as $o^{\cdot}$ does: $\widehat{o}^{\cdot}$ is in S_x. But by (127), $J\left(\widehat{o}^{\cdot}\right) > J\left(o^{\cdot}\right)$: $o^{\cdot}$ is not in $\overline{S}_x$, a contradiction. Finally, using $q^1 = a$, we set $V^1\left(o^1\right) = \rho^1 - \delta a f(1)$ equal to the required payoff of $\widehat{V}^1 + x$ and use (54) to obtain $\rho^1 = a f(1) + x$.

Now say $o^{\cdot}$ violates P2. By (38), this implies $V^1\left(o^2\right) < V^1\left(o^1\right)$. Hence $q^2 \neq a$ since otherwise type 2 would deviate to o^1.[41] For any $\varepsilon > 0$, consider the alternative allocation $\widehat{o}$ defined by $\widehat{o}^1 = o^1$, $\widehat{q}^2 = (1-\varepsilon)q^2 + \varepsilon a$, and $\widehat{\rho}^2 = \rho^2 + \delta\left(\widehat{q}^2 - q^2\right)f(2)$. By construction, $V^1\left(\widehat{o}^1\right) = V^1\left(o^1\right) = \widehat{V}^1 + x$, $V^2\left(\widehat{o}^2\right) = V^2\left(o^2\right)$, and $U^2\left(\widehat{o}^2\right) > U^2\left(o^2\right)$, which imply (39), (42), and (40). Finally, since $V^1\left(o^2\right) < V^1\left(o^1\right)$, we can choose $\varepsilon > 0$ small enough that (38) holds so $\widehat{o}$ is in S_x whence (as $U^2\left(\widehat{o}^2\right) > U^2\left(o^2\right)$) the original allocation $o^{\cdot}$ is not in $\overline{S}_x$ - a contradiction.

Suppose now that $o^{\cdot}$ violates P3: $q^2 \neq a$ and $V^2\left(o^2\right) > \widehat{V}^2$. For any $\varepsilon > 0$, consider the alternative allocation $\widehat{o}$ defined by $\widehat{o}^1 = o^1$, $\widehat{q}^2 = (1-\varepsilon)q^2 + \varepsilon a$, and $\widehat{\rho}^2 = \rho^2 + \delta\left(\widehat{q}^2 - q^2\right)f(1)$. By construction, $V^1\left(\widehat{o}^1\right) = V^1\left(o^1\right) = \widehat{V}^1 + x$. By P2, $V^1\left(\widehat{o}^2\right) = V^1\left(o^2\right) = V^1\left(o^1\right) = V^1\left(\widehat{o}^1\right)$ and $U^2\left(\widehat{o}^2\right) > U^2\left(o^2\right)$ which imply (38) and (40), respectively. Moreover, by P1 and P2,

$$\begin{aligned} V^2\left(o^2\right) - V^2\left(o^1\right) &= \rho^2 - \rho^1 + \delta\left(a - q^2\right)f(2) \\ &= V^1\left(o^2\right) - V^1\left(o^1\right) + \delta\left(a - q^2\right)\Delta^1 \\ &> V^1\left(o^2\right) - V^1\left(o^1\right) = 0 \end{aligned}$$

where the inequality relies on Assumption 1 and genericity. Thus, if ε is small enough, (39) and (42) hold whence $\widehat{o}$ is in S_x. But since $U^2\left(\widehat{o}^2\right) - U^2\left(o^2\right) > 0$, $o^{\cdot}$ is not in $\overline{S}_x$ - a contradiction.

Finally, suppose $o^{\cdot}$ violates P4: $q_1^2 < 1$, $q_2^2 > 0$, and hence, by P3, $V^2\left(o^2\right) = \widehat{V}^2$. Let us now raise q_1^2 slightly while lowering q_2^2 and adjusting ρ^2 so as to leave the payoffs of each type of seller from o^2 unchanged. We will show that this makes the agent better off when facing the type-2 seller, so $o^{\cdot}$ is not in $\overline{S}_x$ - a contradiction. For any small constants $\varepsilon, \iota, \omega$, consider the alternative allocation $\widehat{o}$ defined by $\widehat{o}^1 = o^1 = o^1$, $\widehat{q}_1^2 = q_1^2 + \varepsilon$, $\widehat{q}_2^2 = q_2^2 - \iota$, and $\widehat{\rho}^2 = \rho^2 + \omega$. Given $\varepsilon > 0$, we seek ι and ω such that the change does not alter the payoff of either seller from choosing the outcome designated for either seller, whence (38), (39),

[41] If $q^2 = a = q^1$ then $V^2\left(o^1\right) - V^2\left(o^2\right) = \rho^1 - \rho^2 = V^1\left(o^1\right) - V^1\left(o^2\right) > 0$.

(41), and (42) still hold. As $\widehat{o}^1 = o^1$, $V^t\left(\widehat{o}^1\right) = V^t\left(o^1\right)$ for $t = 1, 2$. Now,

$$V^1\left(\widehat{o}^2\right) - V^1\left(o^2\right) = \omega - \delta\varepsilon f_1(1) + \delta\iota f_2(1)$$

and

$$V^2\left(\widehat{o}^2\right) - V^2\left(o^2\right) = \omega - \delta\varepsilon f_1(2) + \delta\iota f_2(2).$$

Subtracting and dividing by δ, the two expressions are equal as long as $\iota = \varepsilon\psi$ where $\psi = \Delta_1^1/\Delta_2^1$ is positive by (36). Finally, the second expression (and thus the first) is zero as long as $\omega = \delta\varepsilon\left[f_1(2) - \psi f_2(2)\right]$. With these choices, the agent's payoff from selling to type 2 changes by

$$U^2\left(\widehat{o}^2\right) - U^2\left(o^2\right) = \varepsilon f_1(2) - \iota f_2(2) - \omega = \varepsilon(1-\delta)\left[f_1(2) - \psi f_2(2)\right]$$

If we multiply the term in square brackets by Δ_2^1, which is positive by (36), it becomes

$$\Delta_2^1 f_1(2) - \Delta_1^1 f_2(2) = f_1(1) f_2(1)\left[\frac{f_2(2)}{f_2(1)} - \frac{f_1(2)}{f_1(1)}\right]$$

which is positive by (37). Accordingly, $U^2\left(\widehat{o}^2\right) > U^2\left(o^2\right)$, so $o^{\cdot}$ is not in $\overline{S}_x$ - a contradiction.

We conclude that $o^{\cdot}$ satisfies P1, P2, P3, and P4, as claimed. **Q.E.D.**$_{\text{Claim 15}}$

PROOF OF CLAIM 16. By Claim 15, any such allocation satisfies P1-P4. Given P1, $q^1 = q^2$ implies $q^2 = a$. Moreover, by P2, $\rho^2 = \rho^1 = af(1) + x$: a pooling menu is offered. As the agent's payoff must be nonnegative, using the notation $U(o^{\cdot})$ of (40),

$$0 \le U(o^{\cdot}) = \sum_{t=1}^{2} \Pi^t\left(q^t f(t) - \rho^t\right) = a\sum_{t=1}^{2} \Pi^t f(t) - \left[af(1) + x\right]$$
$$= \Pi^2 a\Delta^1 - x.$$

Accordingly, we must have $x \le \Pi^2 a\Delta^1$. By Claim 11, for this to be an equilibrium it suffices that the type 2 seller gets at least her RSW payoff: that $V^2\left(o^2\right) = \rho^1 - \delta af(2) = a\left[f(1) - \delta f(2)\right] + x$ is not less than $\widehat{V}^2$ or, equivalently, $x \ge \widehat{V}^2 - a\left[f(1) - \delta f(2)\right] = \widehat{V}^2 - V^2\left(\widehat{o}^1\right)$. Combining these inequalities and using (63), x must lies in the interval (78) as claimed. Moreover, for this interval to be nonempty, we must have $\delta\left(a-\widehat{q}^2\right)\Delta^1 \leq \Pi^2 a\Delta^1$, which is equivalent to (77) as claimed.[42] **Q.E.D.**$_{\text{Claim 16}}$

Proof of Proposition 3. Let $o^{\cdot} \in \overline{S}_x$ satisfy $q^2 \neq q^2$. By P2 we have $V^1\left(o^2\right) = \rho^2 - \delta q^2 f(1) = V^1\left(o^1\right) = \widehat{V}^1 + x$, whence

$$\rho^2 = \delta q^2 f(1) + \widehat{V}^1 + x. \tag{128}$$

Combining (79) and (128) to eliminate ρ^2 yields

$$x = \delta q^2 \Delta^1 - \left(\widehat{V}^1 - \widehat{V}^2\right). \tag{129}$$

By P4, there are now two subcases.

1. Assume (70). Then by (55), (65), and (66), type 2's RSW outcome satisfies (72). There are now two intervals in which x may lie.

   (a) If x lies in (80) then, by (36) and (129), q^2 is given by (82) so, as $x \to 0$, q^2 converges to $\widehat{q}^2$ by (72) and hence, by (79), ρ^2 converges to $\delta\widehat{q}^2 f(2) + \widehat{V}^2$ which equals $\widehat{\rho}^2$ by (57) and (58). It follows that the agent's expected payoff (40) converges to her RSW payoff of zero as $x \to 0$. For $x > 0$, her type-contingent payoff $U^t\left(o^t\right) = q^t f(t) - \rho^t$ equals $-x$ when $t = 1$ and, using (79),

$$\begin{aligned} U^2\left(o^2\right) &= q^2 f(2) - \rho^2 = (1-\delta)\, q^2 f(2) - \widehat{V}^2 \\ &= (1-\delta)\,\frac{\widehat{V}^1 - \widehat{V}^2 + x}{\delta\Delta_1^1} f_1(2) - \widehat{V}^2 \end{aligned}$$

   when $t = 2$. Hence, using $\Pi^1 = 1 - \Pi^2$, her unconditional expected payoff (40), for a given increment x in type 1's payoff, is

$$U\left(o^{\cdot}\right) = U\left(o^{\cdot}|x\right) = -x + \Pi^2\left[(1-\delta)\,\frac{\widehat{V}^1 - \widehat{V}^2 + x}{\delta\Delta_1^1} f_1(2) + x - \widehat{V}^2\right] \tag{130}$$

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[42] The right hand side of (77) is positive by (36) and (58). It is less than one since $\widehat{q}^2 \neq a$ by (55) and (56) and thus $\widehat{V}^2 = (1-\delta)\,\widehat{q}^2 f(2)$ (using (58)) is less than $(1-\delta)\, a f(2) = V^2\left(\widehat{o}^1\right) + a\Delta^1$ by (36).

which at the top endpoint of (80) equals$\widehat{V}^1 - \widehat{V}^2 - \delta\Delta_1^1 + \Pi^2 \left[\Delta_1 - \widehat{V}^1\right]$ which by (71) is nonnegative if and only if[43]

$$\Pi^2 \geq \frac{\delta\Delta_1^1 - \left(\widehat{V}^1 - \widehat{V}^2\right)}{\Delta_1 - \widehat{V}^1} = \eta, \tag{131}$$

which confirms (81). Alternatively, one can differentiate (130) with respect to x:

$$\frac{d}{dx}U\left(o^{\cdot}|x\right) = -1 + \Pi^2 \frac{(1-\delta)\, f_1(2) + \delta\Delta_1^1}{\delta\Delta_1^1} = -1 + \Pi^2 \frac{\Delta_1}{\delta\Delta_1^1} = -1 + \frac{\Pi^2}{\eta}. \tag{132}$$

As $U\left(o^{\cdot}|x\right)$ is zero when $x = 0$, the constraint (40) holds at x if and only if this derivative is nonnegative, which is equivalent to (81).

(b) If

$$x \geq \nu \tag{133}$$

then, by (129), q^2 is given by (85). The requirement that $q_2^2 \leq 1$ then imposes the additional constraint on x that

$$x \leq \delta a\Delta^1 - \left(\widehat{V}^1 - \widehat{V}^2\right) = \gamma \tag{134}$$

by (63) and (48); thus, x must lie in (83) as claimed.[44] It remains to check that the agent's payoff (40) is nonnegative. As in part 1(a), her type-contingent payoff $U^t\left(o^t\right) = q^t f(t) - \rho^t$ satisfies

$$U^1\left(o^1\right) = -x \text{ and} \tag{135}$$

$$U^2\left(o^2\right) = (1-\delta)\, q^2 f(2) - \widehat{V}^2. \tag{136}$$

[43]By (45), (65), (65), and (70),

$$\frac{\delta\Delta_1^1 - \left(\widehat{V}^1 - \widehat{V}^2\right)}{\Delta_1 - \widehat{V}^1} = \frac{\delta\Delta_1^1}{\Delta_1} \frac{1 - \frac{\widehat{V}^1 - \widehat{V}^2}{\delta\Delta_1^1}}{1 - \widehat{V}^1/\Delta_1} = \frac{\delta\Delta_1^1}{\Delta_1} = \eta.$$

[44]Intuitively, for $o^{\cdot}$ to be an equilibrium, type 2's payoff $V^2\left(\widehat{o}^1\right) + x$ from type 1's outcome o^1 cannot exceed her payoff $V^2\left(o^2\right)$ from her own outcome which, in turn, equals $\widehat{V}^2$ by P3.

Thus, using (85) and $\Pi^1 = 1 - \Pi^2$, condition (40) can be rewritten as

$$0 \leq U\left(o^{\cdot}|x\right) = -x + \Pi^2 \left\{ (1-\delta) \left[f_1(2) + f_2(2) \frac{\widehat{V}^1 - \widehat{V}^2 + x - \delta\Delta_1^1}{\delta\Delta_2^1} \right] + x - \widehat{V}^2 \right\}. \tag{137}$$

The left hand side is linear and continuous in x, so $U\left(o^{\cdot}|x\right) = U\left(o^{\cdot}|\nu\right) + (x-\nu)\,\partial U\left(o^{\cdot}|x\right)/\partial x$ as $\partial U\left(o^{\cdot}|x\right)/\partial x$ is constant over $x > \nu$. Thus, using (137) and (49) to compute

$$U\left(o^{\cdot}|\nu\right) = -\nu + \Pi^2 \left[(1-\delta) f_1(2) + \nu - \widehat{V}^2 \right] = -\nu + \Pi^2 \left(\Delta_1 - \widehat{V}^1 \right)$$

and using (137) to compute

$$\frac{\partial U\left(o^{\cdot}|x\right)}{\partial x} = -x + \Pi^2 \left\{ (1-\delta) f_2(2) \frac{1}{\delta\Delta_2^1} + 1 \right\} = -1 + \Pi^2 \frac{\Delta_2}{\delta\Delta_2^1},$$

we find that $U\left(o^{\cdot}|x\right)$ is nonnegative at x in (83) if and only if

$$\begin{aligned} 0 \leq U\left(o^{\cdot}|x\right) &= -\nu + \Pi^2 \left(\Delta_1 - \widehat{V}^1 \right) + (x-\nu) \left[-1 + \Pi^2 \frac{\Delta_2}{\delta\Delta_2^1} \right] \\ &= -x + \Pi^2 \left[\Delta_1 - \widehat{V}^1 + (x-\nu) \frac{\Delta_2}{\delta\Delta_2^1} \right] \end{aligned} \tag{138}$$

or equivalently if $\Pi^2 \geq \phi(x)$ (defined in (84)) as claimed. We next show that $\phi(x)$ is in (0,1) for x in (83). As $x \geq \nu > 0$, it suffices to show that

$$\begin{aligned} & x < \Delta_1 - \widehat{V}^1 + (x-\nu) \frac{\Delta_2}{\delta\Delta_2^1} \\ \Longleftrightarrow & x \left[1 - \frac{\Delta_2}{\delta\Delta_2^1} \right] < \Delta_1 - \widehat{V}^1 - \nu \frac{\Delta_2}{\delta\Delta_2^1} \\ \Longleftrightarrow & -\frac{(1-\delta) f_2(2)}{\delta\Delta_2^1} x < \Delta_1 - \widehat{V}^1 - \nu \frac{\Delta_2}{\delta\Delta_2^1} \\ \Longleftrightarrow & 0 < (1-\delta) f_2(2) x + \left(\Delta_1 - \widehat{V}^1 \right) \delta\Delta_2^1 - \Delta_2 \nu \end{aligned}$$

As this expression is increasing in x, it suffices to check the inequality at $x = \nu$ when it becomes

$$0 < \left(\Delta_1 - \widehat{V}^1 - \nu \right) \delta\Delta_2^1$$

or, equivalently, by (36),

$$
\begin{aligned}
0 < \Delta_1 - \widehat{V}^1 - \nu &= \Delta_1 - \widehat{V}^1 - \delta\Delta_1^1 + \left(\widehat{V}^1 - \widehat{V}^2\right) \\
&= (1-\delta)\, f_1(2) - \widehat{V}^2 = (1-\delta)\left[f_1(2) - \widehat{q}^2 f(2)\right] \text{ by (58)} \\
&= (1-\delta)\left[\Delta_1 - \widehat{V}^1\right]\frac{f_1(2)}{\Delta_1} \text{ by (55), (56), and (71)}
\end{aligned}
$$

which is positive by (36) and (71) as claimed. Next, $\phi(x)$ is of the form $\frac{x}{\alpha+\beta x}$ where $\beta = \frac{\Delta_2}{\delta\Delta_2^1} > 1$ by (47), so

$$
\phi'(x) = \frac{\alpha}{(\alpha+\beta x)^2}
$$

which is clearly finite (so ϕ is continuous). It is positive if and only if

$$
\begin{aligned}
& 0 < \alpha = \Delta_1 - \widehat{V}^1 - \nu\frac{\Delta_2}{\delta\Delta_2^1} \\
\Longleftrightarrow\, & \delta\Delta_2^1\left(\Delta_1 - \widehat{V}^1\right) > \nu\Delta_2 = \left[\delta\Delta_1^1 - \left(\widehat{V}^1 - \widehat{V}^2\right)\right]\Delta_2 = \left[\delta\Delta_1^1 - \delta\frac{\widehat{V}^1}{\Delta_1}\Delta_1^1\right]\Delta_2 \text{ by (72)} \\
\Longleftrightarrow\, & \delta\Delta_2^1\left(\Delta_1 - \widehat{V}^1\right)\Delta_1 > \delta\Delta_1^1\left(\Delta_1 - \widehat{V}^1\right)\Delta_2 \Longleftrightarrow \Delta_2^1\Delta_1 > \Delta_1^1\Delta_2
\end{aligned}
$$

which holds by (46). We next must evaluate $\phi(x)$ at the endpoints of (83). At the lower endpoint we obtain

$$
\begin{aligned}
\phi(\nu) &= \frac{\nu}{\Delta_1 - \widehat{V}^1} = \frac{\delta\Delta_1^1 - \delta\widehat{q}^2\Delta^1}{\Delta_1 - \widehat{V}^1} \text{ by (64) and (60)} \\
&= \frac{\delta\Delta_1^1 - \delta\frac{\widehat{V}^1}{\Delta_1}\Delta_1^1}{\Delta_1 - \widehat{V}^1} = \frac{\delta\Delta_1^1}{\Delta_1} = \eta
\end{aligned}
$$

as claimed, while at the upper endpoint we obtain

$$
\begin{aligned}
\phi(\gamma) &= \frac{\gamma}{\Delta_1 - \widehat{V}^1 + (\gamma-\nu)\frac{\Delta_2}{\delta\Delta_2^1}} = \frac{\gamma}{\Delta_1 - \widehat{V}^1 + \delta\Delta_2^1\frac{\Delta_2}{\delta\Delta_2^1}} \\
&= \frac{\gamma}{\Delta_1 + \Delta_2 - \widehat{V}^1} = \frac{\gamma}{a\Delta - (1-\delta)\,a f(1)} = \frac{\gamma}{a\Delta^1} = \zeta \text{ by (59) and (63)}
\end{aligned}
$$

as claimed. Next, it is trivial from (84) to see that $\lim_{x\to\infty}\phi(x)$ equals $\delta\Delta_2^1/\Delta_2$; this exceeds ζ as ϕ is increasing and $\phi(\gamma) = \zeta$. Finally, since ϕ is continuous and increasing on $[\nu,\gamma]$ and maps this interval to $[\eta,\zeta]$, it has a continuous and increasing inverse function ϕ^{-1} defined on $[\eta,\zeta]$ that maps this interval to $[\nu,\gamma]$, and for $\left(x,\Pi^2\right)\in[\nu,\gamma]\times[\eta,\zeta]$, $\Pi^2\geq\phi(x)$ if and only if $x\leq\phi^{-1}\left(\Pi^2\right)$. Hence, the region R defined by (83) and (84) is the union of two regions. The first, region (i), is the subset of R on which Π^2 is in $[\eta,\zeta]$. Since, at all points in R, x lies in $[\nu,\gamma]$, $\Pi^2\geq\phi(x)$ is equivalent to $x\leq\phi^{-1}\left(\Pi^2\right)$. Hence, region (i) is defined by the two conditions (86) and (87). The second, region (ii), is the subset of R on which (89) holds. Since $\phi(x)\leq\zeta$ when x is in $[\nu,\gamma]$, condition (84) holds automatically if, in addition, (83) holds, region (ii) is defined simply by (89) and (83).

2. Suppose (73) holds. In this case, $x>0$ clearly cannot be less than $\delta\Delta_1^1-\left(\widehat{V}^1-\widehat{V}^2\right)$. Hence, reasoning as in case (a)(ii), x must lie in (90) and q^2 must be given by (85). This has two implications. First, since $q_2^2\leq 1$, x must lie in (90) as claimed. Second, as $x\to 0$, q^2 converges to $\widehat{q}^2$ by (75) and hence, by (79), ρ^2 converges to $\delta\widehat{q}^2 f(2)+\widehat{V}^2$ which equals $\widehat{\rho}^2$ by (57) and (58). So as in case (a)(i), the agent's expected payoff (40) converges to her RSW payoff of zero as $x\to 0$. Since, moreover, the agent's expected payoff is again given by (137), a necessary and sufficient condition for there to be a separating equilibrium with x in (90) is that

$$0\leq\frac{d}{dx}U\left(o^{\cdot}|x\right)=-1+\Pi^2\left\{\frac{(1-\delta)\,f_2(2)}{\delta\Delta_2^1}+1\right\}=-1+\Pi^2\frac{\Delta_2}{\delta\Delta_2^1}$$

or, equivalently, that

$$\Pi^2\geq\frac{\delta\Delta_2^1}{\Delta_2}. \tag{139}$$

Since (73),

$$
\begin{aligned}
\frac{\gamma\Delta_2}{\delta} &= \Delta_2\left(a-\widehat{q}^2\right)\Delta^1 \text{ by (48)}\\
&= \Delta_2\left((1,1)-\left(1,\frac{\widehat{V}^1-\Delta_1}{\Delta_2}\right)\right)\Delta^1 \text{ by (56) and (65)}\\
&= \Delta_2\left(1-\frac{\widehat{V}^1-\Delta_1}{\Delta_2}\right)\Delta_2^1 = \left(\Delta_1+\Delta_2-\widehat{V}^1\right)\Delta_2^1 = \Delta_2^1\left\{a\left[f(2)-\delta f(1)\right]-(1-\delta)\,af(1)\right\}\\
&= \Delta_2^1 a\Delta^1
\end{aligned}
$$

and thus $\frac{\delta\Delta_2^1}{\Delta_2}=\frac{\gamma}{a\Delta^1}$, whence, (139) is equivalent to (91) when (73) holds. **Q.E.D.**$_{\text{Proposition 3}}$

PROOF OF CLAIM 17. Clearly, if $o^{\cdot}$ is in $\overline{S}$, it must also be in $\overline{S}_x$ for that x equal to the (constant) increment $V^1\left(o^1\right)-\widehat{V}^1$ that the type 1 seller gets in $o^{\cdot}$. Thus, to characterize $\overline{S}$, we can restrict attention to allocations that are in $\overline{S}_x$ for some payoff increment x. Any element of $\overline{S}$ must therefore satisfy P1 and P4: type 1 sells her whole portfolio and type 2 sells all of asset 1 before she sells any of asset 2. Moreover, by (76), any element of $\overline{S}$ must maximize (in $\overline{S}$) the value $\sum_{t=1}^{2}\Pi^t q^t f(t)$ of assets sold. Hence, every allocation in $\overline{S}_x^{\text{pool}}$ (for any x) must be in $\overline{S}$ since, in each such allocation, the type 2 seller sells her entire portfolio. Moreover, if $\overline{S}_x^{\text{pool}}$ is nonempty for some x, then no element of any $\overline{S}_x^{\text{sep}}$ can be in $\overline{S}$ as any such allocation involves type 2 retaining some units of asset 2. Finally, by Claim 16, $\overline{S}_x^{\text{pool}}$ is nonempty for some x if and only if $\overline{S}_\gamma^{\text{pool}}$ is nonempty, which is so if $\Pi^2\geq\zeta$. It follows that, if $\Pi^2\geq\zeta$, $\overline{S}$ consists only of pooling allocations in which, for some x in the interval (78), the agent pays $\rho^t=af(1)+x$ to each type of seller for her whole portfolio.

Now suppose $\Pi^2<\gamma$: $\overline{S}_x^{\text{pool}}$ is empty for any $x>0$. Hence, any element of $\overline{S}$ must be in $\overline{S}_x^{\text{sep}}$ for some $x>0$. But by Proposition 3, an increase in x always leads to higher welfare as it leads type 2 to sell more of one asset and no less of the other. Thus, for any $\Pi^2<\gamma$, $\overline{S}$ consists of the (by Proposition 3) singleton element of $\overline{S}_x^{\text{sep}}$ for the highest x for which $\overline{S}_x^{\text{sep}}$ is nonempty. As there is no such x when either (a) $\nu>0$ and $\Pi^2<\eta$ or (b) $\nu\leq 0$ and $\Pi^2<\gamma$, $\overline{S}$ is empty in these cases. But if $\nu>0$ and $\Pi^2\in[\eta,\gamma)$, the highest x for which

$\overline{S}_x^{\mathrm{sep}}$ is nonempty is $x = \phi^{-1}\left(\Pi^2\right)$ by case 1(b) of Proposition 3. This proves the result. **Q.E.D.**$_{\text{Claim 17}}$

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