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Shocks and Business Cycles

David M. Frankel* Krzysztof Burdzy†

*Iowa State University, dfrankel@econ.iastate.edu

†University of Washington, burdzy@math.washington.edu

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Abstract

A popular theory of business cycles is that they are driven by animal spirits: shifts in expectations brought on by sunspots. A prominent example is Howitt and McAfee (AER, 1992). We show that this model has a unique equilibrium if there are payoff shocks of any size. This equilibrium still has the desirable property that recessions and expansions can occur without any large exogenous shocks. We give an algorithm for computing the equilibrium and study its comparative statics properties. This work generalizes Burdzy, Frankel, and Pauzner (2000) to the case of endogenous frictions and seasonal and mean-reverting shocks.

KEYWORDS: Business Fluctuations and Cycles, Stochastic and Dynamic Games

*Frankel (corresponding author): Department of Economics, Iowa State University, Heady Hall, Ames, Iowa 50011. Burdzy: Department of Mathematics, University of Washington, Seattle, WA 98195. We are grateful for the helpful comments of Jeroen Swinkels (the editor), two anonymous referees, Yossi Feinberg, David Kreps, Jonathan Mattingly, Paul Milgrom, Ady Pauzner, Matthew Rabin, Steve Tadelis, and seminar participants at Berkeley, Caltech, Chicago, Cornell, DELTA, LSE, MIT, Northwestern, NYU, Stanford, Tel Aviv, UCLA, and the 8th World Congress of the Econometric Society.

1 Introduction

Recessions and expansions often occur without any large precipitating shocks. A popular explanation for this is that with externalities, there can be multiple equilibria, so the path followed by the economy can depend on agents' expectations. These expectations can shift unexpectedly when payoff-irrelevant variables ("sunspots") are observed, causing a recession or expansion.

This theory is not entirely satisfactory. First, it is unclear how a large number of firms or workers end up interpreting the *same* sunspots *in the same way*. There is an almost infinite number of sunspot variables that could be used - the daily high temperature in Mongolia, the batting average of the New York Mets, etc.; and each has many potential interpretations. Second and more important, there is no direct evidence that substantial numbers of firms or workers really do condition their economic plans on such variables.[1]

This paper gives a new explanation for business cycles that does not rely on sunspots. The theory still has the property that recessions and expansions can occur suddenly and without any obvious external cause. We begin with Howitt and McAfee's [18] model of sunspot-driven business cycles. We add shocks of any size, in the form of a stochastic parameter that affects the payoff from producing. The shocks lead to a unique equilibrium.[2] There is no role for sunspots. In this equilibrium, small shocks to a *payoff-relevant* variable can lead to recessions and expansions, which take the form of large and sometimes hard-to-reverse changes in economic activity.

The equilibrium falls into one of two categories. If the externality is small enough, there is a unique steady-state employment level for each value of the payoff parameter. The actual employment level, whatever it is, converges gradually to this steady state. A small shock can cause steady-state employment to change disproportionately, causing a recession (if the shock is negative) or expansion (if

[1] An example of what might qualify as direct evidence for sunspots comes from the stock market. Some investors rely on technical buy and sell signals, which predict whether a stock will rise or fall on the basis of its past price behavior. In Shiller's survey following the crash of 1987 [32, p. 394], about 1/3 of investors cited being influenced by the price dropping through a 200-day moving average or other long-term trend line, which is regarded as a sign of weakness (as indeed it was). Froot, Scharfstein, and Stein [14] argue that the popularity of technical indicators may be due to their role as a coordinating device. Such direct evidence for sunspots is lacking in the case of business cycles.

[2] The actual path of employment depends on the realization of the exogenous shocks. Thus, agents' expectations are probabilistic: they consist of a probability distribution over future employment paths. The point is that this probability distribution is uniquely determined by the current state of the economy.

positive). However, the economy does not become "stuck": since the steady state level is unique, an equal and opposite shock will restore the steady-state to its former level, and actual employment will gradually follow.

Other authors have expressed the view that the steady-state unemployment rate changes from time to time. Blanchard and Summers write:

> Most of the time, equilibrium unemployment is stable, and unaffected by movements in the actual rate. But once in a while, a sequence of shocks pushes the equilibrium rate up or down, where it remains until another sequence dislodges it. Such infrequent changes appear to fit quite well with the empirical evidence of unemployment: unemployment seems indeed to be subject to infrequent changes in its mean level. [5, pp. 291-2]

Recent studies have found empirical support for changing mean unemployment, using variants of Hamilton's [16] Markov switching regression.[3]

If the externality is sufficiently large, there can be multiple steady-state employment levels for a range of values of the payoff parameter. The actual employment level, whatever it is, converges gradually to the steady state whose basin of attraction it is currently in. A small shock can cause the economy's current steady-state to disappear. Actual employment will then drift to an adjacent steady state. Once this occurs, an equal an opposite shock will cause the original steady state to reappear. However, employment will not return to this level. This is because the economy is now in the new steady state's basin of attraction. Employment will remain in the new steady state until a sufficiently large shock causes this steady state itself to disappear.

Why do externalities generate multiple steady states? Consider a small negative shock. Each agent in isolation now wants to produce less. This direct effect is present with or without externalities, and can sometimes be quite large. But without externalities, an agent's incentive to produce does not depend on the aggregate employment level. An equal and opposite positive shock will restore her original incentives to produce: the recession will end. The economy does not get stuck.

With externalities, this is not always so. The negative shock causes a decline in employment like before. But now there is an additional multiplier effect: with a search externality, the decline in employment makes it harder to find buyers for one's goods. This further weakens an agent's incentive to produce. This multiplier effect can snowball: a small shock can cause a small decline in employment, which

[3]See Bianchi and Zoega [3], Chauvet, Juhn, and Potter [10], Clemente, Lanaspa, and Montañés [11], and Skalim and Teräsvirta [34].

in turn causes an even larger decline, and so on. Once this happens, an equal and opposite positive shock may not cause a boom: agents' incentive to produce remains weak since employment is now much lower than before. The economy has become mired in recession.

Pissarides [29] analyzes a dynamic model with a unique equilibrium that also has multiple steady states. A continuum of firms decide how many workers to hire in each period. Workers who are unemployed in one period lose skills, so they are less productive in the next period. This creates strategic complementarities across periods: having more jobs in one period makes firms want to create more jobs in the next period since workers will be more productive.

Pissarides analyzes a world with no frictions: all jobs last only one period. This guarantees that the equilibrium is unique. In the more realistic case in which jobs end asynchronously, Pissarides's model would have multiple equilibria. Our model shows that adding productivity shocks solves this problem. This implies that Pissarides's predictions about the form of the equilibrium hold in a much richer and more realistic dynamic setting.

This paper is related to prior research that studies how payoff shocks can eliminate multiplicity of equilibria in dynamic models when there are frictions in changing actions (Burdzy, Frankel, and Pauzner [6], henceforth BFP; Frankel and Pauzner [13], henceforth FP). This paper generalizes these results in two ways. First, BFP and FP assume fixed frictions: each agent receives chances to switch actions at a fixed rate. This assumption is too restrictive for the model of Howitt and McAfee [18], where a firm can fill a vacancy faster by advertising more heavily. In this paper, we let an agent switch faster, possibly at a cost. Formally, each agent chooses a switching rate from a closed interval. In applications, the minimum switching rate can be strictly positive; for example, job attrition in Howitt and McAfee's model implies that jobs become vacant at a strictly positive rate.

The second generalization concerns the shocks. BFP and FP assume that the shocks are i.i.d. From a theoretical point of view, this leaves the impression that the uniqueness result may be a very special case. For applied work, the assumption is undesirable since it rules out such phenomena as mean-reversion and seasonality. By "mean reversion" we mean that a variable's drift depends on its current value: when the variable is high, it is more likely to fall. "Seasonality" means that the drift or variance of the process depends on calendar time: e.g., the daily high temperature might tend to fall and be more volatile in the autumn months. BFP permit mean reversion and seasonality only in various limiting cases (small shocks or small frictions), relying on the fact that taking limits makes the shocks approximately i.i.d. FP rule out such phenomena by assumption.

We show that a unique equilibrium is obtained for a general class of mean-reverting and seasonal shocks. We make essentially no restrictions on the degree of seasonality: both the drift and variance of the stochastic process can depend arbitrarily on time. In addition, the drift can depend linearly on the current value of the process. This generates mean reversion if the drift is a decreasing function of the current value of the process.

We approach mean reversion in two ways. First, consider how fast the payoff parameter drifts towards its mean value when it is at some fixed distance from this mean. We prove uniqueness on the assumption that this rate shrinks to zero over time: if any mean reversion dies out in the long run. This result reveals an essential difference between equilibrium selection in dynamic games (BFP, FP) and equilibrium selection in one-shot “global games” (Carlsson and van Damme [7], henceforth CvD; Morris, Rob, and Shin [24], henceforth MRS). In global games, each player observes a noisy signal of the game’s true payoffs and then chooses an action. A unique equilibrium emerges in the limit as this noise becomes small. In this limit, the noise has a "stationarity" property (in a certain sense that we explain in section 7) that is analogous to the stationarity of the Brownian shocks. For this reason among others, the results of BFP and FP have been viewed as directly analogous to those of CvD and MRS. Our result shows that this analogy is misleading: while stationarity is indispensable in static global games, it can be substantially relaxed in the dynamic setting we study. For any degree of nonstationarity in the static case, there are multiple equilibria if strategic complementarities are strong enough. In contrast, our results yield a large class of nonstationary shocks (those with finite-lived mean reversion) for which the dynamic game has a unique equilibrium for *any* degree of strategic complementarities. This contrast is discussed further in section 7.

Our uniqueness proof does not extend to the case of mean reversion that lasts forever. For that case, we provide an algorithm for computing tight upper and lower bounds on the set of equilibria. We have found very few cases of multiple equilibria using this algorithm, even under strong mean reversion. This suggests that in practical terms, multiple equilibria may not be an important phenomena even with nonvanishing mean reversion. But they do remain a theoretical possibility.

The rest of the paper is organized as follows. We present HM’s model in section 2 and a generalization of it in section 3. Section 4 presents the uniqueness result. Properties of the equilibrium are explored in section 5. An intuition for uniqueness is given in section 6. Section 7 concludes. Appendix A provides a concise guide to the notation used in this paper. An algorithm for computing the equilibrium appears in Appendix B. Appendix C contains proofs of the paper’s theoretical results.

2 The Howitt-McAfee Model

HM assume a large number of identical firms that advertise to hire workers, who quit at a fixed rate. The appeal of hiring is greater when employment is high since it is easier to find customers during a boom. This externality gives rise to multiple equilibria: if a boom is expected, firms will be eager to hire since they expect selling to become easier. There are also sunspot equilibria in which changes in some payoff-irrelevant variable give rise to expectations-driven business cycles.

HM's model has a continuum of identical firms of measure 1. Time t is continuous. Each firm has a single job that is either filled or vacant. Workers are identical and are either employed or unemployed. Unemployed workers search costlessly until they find a job. There is exogenous attrition: each employed worker becomes unemployed according to a Poisson process with fixed arrival rate δ. The number of firms equals the number of workers in the economy, so the proportion of jobs that are filled, X, equals the employment rate (the proportion of workers who have jobs). (Time subscripts are suppressed.)

A filled job yields a surplus $f(X) > 0$ which is divided between worker and firm: the worker receives $(1-\omega)f(X)$ and the firm gets $\omega f(X)$, where ω is a fixed proportion between zero and one. HM assume that the surplus f is increasing in the employment rate. This reflects the idea that marketing is less costly when employment is high.

A firm with a vacancy chooses how intensively to advertise for workers. The firm can vary this intensity over time. A firm that chooses the intensity $\theta \in [0, \overline{\theta}]$ attracts job applicants according to a Poisson process with arrival rate θ. Since the workers and firms are identical, each match results in a hire, so we can think of θ as the hiring rate. A firm that hires at the rate θ while the employment rate is X incurs hiring costs of $c^A(\theta, X)$ per unit of time. These hiring costs are a weakly increasing and left continuous function of the hiring rate θ. They are also weakly increasing in the employment rate X: an increase in employment makes it harder to attract applicants.

We introduce shocks by letting the surplus from a match depend also on a common payoff parameter, W. The surplus from a filled job is now $f(W, X)$, where f is strictly increasing and Lipschitz in both arguments. The surplus can be thought of as the firm's revenue minus all non-wage costs. Thus, changes in W may represent shocks to productivity or to the prices of nonlabor inputs such as energy. We assume that W comes from a parametric family that is described by axiom **A2** in section 3. This family includes seasonal and mean-reverting processes. To avoid complicating the model with layoffs, we assume that the surplus can never be negative.

The continuation payoff of a firm at time t is the expected integral of future profits less advertising costs, discounted at the constant rate $r > 0$:

$$E_t \int_{s=t}^{\infty} e^{-r(s-t)} \left[\omega f(W_s, X_s)\phi_s - c^A(\theta_s, X_s) \right] ds \tag{1}$$

where ϕ_s equals one if the firm's job is filled at time s and zero otherwise and θ_s is the firm's hiring rate at time s.

We also assume dominance regions: for extreme values of the payoff parameter, firms' behavior is pinned down uniquely. This assumption has two parts:

1. As the payoff parameter rises, the surplus from a match eventually rises to the point that a firm will hire at the maximum feasible rate $\overline{\theta}$, regardless of how it expects other firms to behave.[4]

2. For low enough values of the payoff parameter, firms will not hire.[5]

3 A General Model

We will first prove uniqueness in a more general model and then show that HM's model is a special case. This approach makes it easier to apply the uniqueness result to other models, which may not have HM's special features. It also clarifies that the special assumptions of HM's model can be weakened considerably. These assumptions include the restriction that firms cannot fire workers.

The general model is as follows. There is a continuum of players of measure 1. At any time $t \in [0, \infty)$, each player is locked into one of two modes, 1 or 2. Let the proportion of players who are locked into mode 1 be X_t. In HM, the players are firms, mode 1 (2) is a filled (vacant) position, and X_t is the employment rate.

Players change modes from time to time, according to independent Poisson processes. The arrival rate of this process is a player's *switching rate*. Let k^1 (k^2) be the switching rate of a player who is locked into mode 1 (2). We assume these switching rates are bounded above and below:

A1. Bounded Switching Rates *The switching rate of a player who is in mode* $m = 1, 2$ *comes from a fixed closed interval:* $k^m \in \left[\underline{K}^m, \overline{K}^m\right]$, *where*

[4] A sufficient condition is that for high enough values of the payoff parameter, the present value of profits from a filled vacancy exceeds the marginal benefit of lowering the hiring rate when this rate is at its maximum: if $\frac{\omega \lim_{w\uparrow+\infty} f(w,0)}{r+\delta+\overline{\theta}} > c^A_\theta(\overline{\theta}, x)$ for all $x \in [0, 1]$.

[5] A sufficient condition is that productivity is negative for low enough values of the stochastic parameter: $\lim_{w\downarrow-\infty} f(w, 1) < 0$.

$0 \leq \underline{K}^m \leq \overline{K}^m \leq \infty$. *In particular, no player can choose a switching rate above* $K = \max\left\{\overline{K}^1, \overline{K}^2\right\}$.

If a player who is locked into mode $m = 1, 2$ chooses the switching rate k^m during the infinitesimal period $[t, t + dt]$, she incurs a cost of $c^m(k^m, X_t)dt$ and switches modes with probability $k^m dt$. We assume only that c^m is Lipschitz in X_t and weakly increasing and left-continuous in k^m.[6]

In HM, a firm with a vacant position chooses the switching rate θ (its hiring rate) and its switching cost function is $c^A(\theta, X)$. A firm with a filled job "chooses" the switching rate δ (its attrition rate)[7] and its cost function is identically zero.

In this framework, a player chooses the rate at which switching opportunities arrive. When one arrives, she *must* take it. In the earlier models of BFP and FP, a player costlessly receives switching opportunities at some fixed rate d. She does *not* have to switch. Nevertheless, the switching mechanism of BFP and FP can be captured in the current model by constraining k^1 and k^2 to be in $[0, d]$ and letting c^1 and c^2 be identically zero in this range. Switching with probability p when an opportunity arises in BFP and FP is equivalent to choosing a switching rate of pd in our model.

In addition to paying switching costs, a player may also receive a flow of payoffs from being in a given mode. In HM, this equals a firm's profits if its position is filled and is zero otherwise. We write this direct payoff flow as $u(m, W, X)$, where $m = 1, 2$ is the player's current mode, X is the current proportion of mode-1 players, and W is current value of the stochastic payoff parameter.

A player's time-t continuation payoff is the present discounted value of her direct payoff flow minus her switching costs:

$$E \int_{v=t}^{\infty} e^{-r(v-t)} \left[u(m_v, W_v, X_v) - c^{m_v}(k_v, X_v)\right] dv \tag{2}$$

where $m_v \in \{1, 2\}$ is the mode the player is in at time v, k_v is her switching rate, and $r > 0$ is the common pure rate of time preference.

We now discuss our assumptions on W. For tractability we assume it has continuous paths. Consider the increment $W_{t+dt} - W_t$: the change in the stochastic

[6]Without left continuity an optimum might not exist. Suppose, e.g., that the cost is zero for switching rates below k_0 and $c_0 > 0$ for rates k_0 and above. If the benefit of switching is only $c_0/2$, the agent has no optimal switching rate: any rate below k_0 is too low while any rate greater than or equal to k_0 is too high.

[7]This is captured by setting the upper and lower bounds on her switching rate both equal to δ.

parameter from time t to time $t + dt$. In the general case of an Ito diffusion, this increment is normal with mean $\mu(t, W_t)dt$ (the drift) and variance $\sigma^2(t, W_t)dt$. FP assume that W_t follows a Brownian motion: that $\mu()$ and $\sigma^2()$ are both constants. This rules out any sort of mean-reversion or seasonality. BFP permit the drift to depend on t and W_t; however, they prove uniqueness only in various limiting cases in which the drift becomes negligible. In this paper we will prove uniqueness without taking limits.

We permit both the drift and variance to vary seasonally in an arbitrary way. The drift can also be a linear function of the state W_t; this can capture mean reversion.[8] Formally:

$$dW_t = (\nu_t W_t + \mu_t)\, dt + \sigma_t dB_t \tag{3}$$

where B is a Brownian motion with zero drift and unit variance. If $\nu_t < 0$, for example, W is mean-reverting.

A2. Payoff Shocks: General Case *The drift in W is a (possibly time-varying) linear function of the state:*

$$\mu(t, W_t) = \nu_t W_t + \mu_t \tag{4}$$

where $\nu_t, \mu_t \in \Re$ are deterministic functions of time. The variance of W can change over time but it is independent of the state:

$$\sigma^2(t, W_t) = \sigma_t^2 \tag{5}$$

where $\sigma_t^2 \in \Re^+$ is a deterministic function of time. Moreover, there are constants $0 < N_1 < N_2$ such that, for all t:

1. *The drift terms are bounded:*

$$|\nu_t|\,, |\mu_t| < N_2 \tag{6}$$

2. *Any mean reversion dies out asymptotically:*

$$\int_{s=0}^{\infty} |\nu_s|\, ds < N_2 \tag{7}$$

3. *The variance is nonzero and bounded:*

$$\sigma_t \in [N_1, N_2] \tag{8}$$

[8]For technical reasons, our approach does not admit more general forms of state dependence.

4. *Changes in variance are Lipschitz:*

$$|\sigma_{t'} - \sigma_t| \leq N_2 \,|t' - t| \tag{9}$$

Of the assumed bounds (6)-(9), the most restrictive is (7). It implies that any mean reversion must eventually die out ($\lim_{t\to\infty} \nu_t = 0$). On the other hand, there can be arbitrarily strong mean reversion for an arbitrarily long initial period.

We will also discuss the effects of replacing **A2** with the stronger **A2′**, which states that W is a Brownian motion:

A2′. Payoff Shocks: Brownian Case *W is a Brownian motion with drift μ and variance σ^2: $\mu(t, W_t) = \mu$ and $\sigma^2(t, W_t) = \sigma^2$.*

This assumption makes the environment stationary. We will show that under **A2′**, the equilibrium is also stationary.

Define the *relative payoff flow in mode 1* to be the difference in the payoff flows (the integrand in (2)) in mode 1 versus mode 2:

$$D(W, X, k^1, k^2) = \left[u(1, W, X) - c^1(k^1, X)\right] - \left[u(2, W, X) - c^2(k^2, X)\right] \tag{10}$$

The relative payoff flow in mode 1 is assumed Lipschitz in W and X: there are constants $\overline{\alpha}$ and β such that for all w, w', x, x', k^1, and k^2,

$$D(w, x, k^1, k^2) - D(w, x', k^1, k^2) \leq \beta \,|x - x'| \tag{11}$$

$$D(w, x, k^1, k^2) - D(w', x, k^1, k^2) \leq \overline{\alpha} \,|w - w'| \tag{12}$$

Assumption **A3** states that there are strategic complementarities.

A3. Strategic Complementarities *The relative payoff flow in mode 1 is weakly increasing in the proportion of agents who are in mode 1: for all $x > x'$ and any feasible k^1 and k^2,*

$$D(w, x, k^1, k^2) - D(w, x', k^1, k^2) \geq 0 \tag{13}$$

Assumption **A4** states that a positive shock weakly raises the relative payoff flow in mode 1, and strictly raises it over some interval.

A4. Payoff Monotonicity *The relative payoff flow in mode 1 is weakly increasing in the payoff parameter: for all x, $w > w'$, and any feasible k^1 and k^2,*

$$D(w, x, k^1, k^2) - D(w', x, k^1, k^2) \geq 0 \tag{14}$$

Moreover, there is a nonempty interval (w_1, w_2) such that the inequality holds strictly if w and w' both lie in this interval.

We assume dominance regions: for extreme values of the payoff parameter, players have a strictly dominant switching rate.

A5. Dominance Regions *There are constants* $\overline{w} > \underline{w}$ *such that:*

1. *if* $W_t > \overline{w}$*, it is strictly dominant for players in mode 1 (2) to switch at their minimum (maximum) switching rate.*
2. *if* $W_t < \underline{w}$*, it is strictly dominant for players in mode 1 (2) to switch at their maximum (minimum) switching rate.*

We also assume that the marginal cost of raising one's switching rate depends in a bounded way on the current population distribution:

A6. Bounded Effect of X **on Marginal Cost** *There is a constant* $\eta > 0$ *such that for all* x, x', k, k' *and for each mode* $m = 1, 2$*,*

$$|c^m(k', x') - c^m(k', x) - (c^m(k, x') - c^m(k, x))| \leq \eta \, |k - k'| \, |x - x'| \quad (15)$$

Axiom **A6** implies that the derivative of the marginal switching cost with respect to the proportion in mode 1, if it exists, is bounded: $\partial c_k^m / \partial x \leq \eta$.

A player's information set at time t comprises the public history $(W_v, X_v)_{v \in [0,t)}$ and her private history (the actions she has played and the switching rates she has selected through time t). A (possibly mixed) strategy for a player specifies, at any information set, the distribution of switching rates that she will choose. While we allow players to randomize, in equilibrium they generally do not do so.

4 Theoretical Results

4.1 General Model

Rather than looking for equilibria, we use a more primitive solution concept: the iterative deletion of conditionally dominated strategies (Fudenberg and Tirole [15]). This is an extension of backwards induction to infinite horizon games. At each iteration, we delete all strategies that prescribe playing a strictly dominated action after any history. The process is iterated until no further strategies can be eliminated.

Theorem 1 states that there is a unique outcome that survives the iterative procedure. BFP show that in dynamic models with players who are infinitesimal and anonymous, every Nash equilibrium outcome survives this procedure. Thus, Theorem 1 also implies that the model has a unique Nash equilibrium outcome.

THEOREM 1 (Uniqueness) *The model of section 3 has a unique outcome that survives the iterative deletion of conditionally dominated strategies: for any initial state* (W_0, X_0) *and almost any path of* W*, the path of* X *is uniquely determined.*

Proof: Appendix C.

Theorem 2 states that the probability distribution over what will happen depends only on the current state and time.

THEOREM 2 (The Markov Property) *The distribution of future paths of the state,* (W, X)*, depends only on the current state and time. Under axiom* ***A2′****, this distribution is independent of time.*

Proof: Appendix C.

By the Markov Property, the distribution of future paths of X depends only on the current state and time. But a player's continuation payoff in a given mode depends only on the future paths of X and W. Hence, this continuation payoff is uniquely determined by the current state and time. Theorem 3 also says that small changes in the current state or time lead to small changes in this continuation payoff.

THEOREM 3 (Payoff Continuity) *Let* $V_t^1 = V^1(W, X, t)$ *and* $V_t^2 = V^2(W, X, t)$ *be the continuation payoffs of a player locked into mode 1 and 2, respectively, at state* (W, X) *and time* t*. These are continuous functions of the state and time. Under axiom* ***A2′****, they are independent of time.*

Proof: Appendix C.

We call $V^1 - V^2$ the *relative value of being in mode 1*. Theorem 4 states that the relative value of being in mode 1 is weakly increasing in the proportion of players in mode 1 and in the payoff parameter. It also gives a formula for this relative value: it equals the expected integral of the discounted relative payoff flow in mode 1. The discount rate in this integral is the sum of the rate of time preference and the two switching rates. The reason the switching rates are added is that being in mode 1 rather than mode 2 at time t has an effect on a player's payoff flow until the first time at which either (a) she moves to mode 2 (which occurs at the rate k^1) or (b) if she were in mode 2, she would have moved to mode 1 (which would have occurred at the rate k^2). The hazard rate for one or the other of these events to occur is just the sum of the two events' hazard rates, $k^1 + k^2$, since the probability of their occurring simultaneously in continuous time is zero.

THEOREM 4 (Relative Payoff Monotonicity) *The relative value of being in mode 1, $V^1 - V^2$, equals the expected integral of the discounted relative payoff flow in mode 1:*

$$V_t^1 - V_t^2 = E\left[\int\limits_{v=t}^{\infty} \exp\left(-\int\limits_{s=t}^{v} \left[r + k_s^1 + k_s^2\right] ds\right) D(W_v, X_v, k_v^1, k_v^2) dv\right]$$

This relative value is increasing in the current payoff parameter, W_t, at a rate that is bounded below by a strictly positive constant. It is also weakly increasing in X_t (strictly if the relative payoff flow in mode 1 is strictly increasing in X).

Proof: Appendix C.

Theorem 5 gives a formula for the switching rates in terms of the relative value of being in mode 1 and the switching cost functions. If these functions are differentiable, it implies that a player chooses a switching rate at which the marginal cost of switching faster equals the marginal benefit, which is the relative value of being in the other mode.

THEOREM 5 (Switching Rate Rule) *Agents locked into mode 1 choose a switching rate k that maximizes $k \cdot (V_t^2 - V_t^1) - c^1(k, X_t)$. Agents locked into mode 2 choose a switching rate k that maximizes $k \cdot (V_t^1 - V_t^2) - c^2(k, X_t)$.*

Proof: Appendix C.

Payoff Continuity and the Switching Rate Rule together imply that the rate of change of the proportion of players in mode 1 is almost always uniquely determined by the current state and time. Why? There are $1 - X$ players in mode 2, who enter mode 1 at the common rate k^2, and X players in mode 1, who leave mode 1 at the common rate k^1. Thus:

$$\dot{X} = k^2 \cdot (1 - X) - k^1 \cdot X \tag{16}$$

In addition, Payoff Continuity implies that the relative value of being in mode 1 depends uniquely on the current state and time. By Switching Rate Rule, a player's optimal choice of switching rate depends only on the relative value of being in mode 1 and the population distribution X. This rule almost always gives a unique optimal switching rate for agents in each mode.[9] Together, these imply that at any time t, the switching rates (and thus $\dot{X}$) are uniquely determined at all but a measure-zero set of states.

By Relative Payoff Monotonicity, an increase in the payoff parameter, holding the proportion of agents in mode 1 fixed, raises the relative value of being in mode 1. By the Switching Rate Rule, this implies Theorem 6: an increase in the payoff parameter leads to a weak increase (decrease) in the switching rate of players in mode 2 (1). "Weak" is essential: agents may prefer not to change their switching rates if they are at a corner solution or at a kink of their switching cost function.

THEOREM 6 (Switching Rate Monotonicity) *An increase in the payoff parameter leads to a weakly higher switching rate of agents who are locked into mode 2 and a weakly lower switching rate of agents who are locked into mode 1, holding time and the proportion of agents in mode 1 fixed. More precisely, if $w' > w$, then any switching rate that is optimal for mode-1 (mode-2) agents at state (w, x) at time t is weakly higher (weakly lower) than any switching rate that is optimal for mode-1 (mode-2) agents at state (w', x) at time t.*

Proof: Appendix C.

Referring to equation (16), this implies Theorem 7: an increase in the payoff parameter weakly raises the rate of growth of the proportion of players in mode 1.

THEOREM 7 (Growth Rate Monotonicity) *An increase in the payoff parameter leads to a weakly higher rate of change $\dot{X}$ of the proportion of players locked into mode 1, holding time and the proportion of agents in mode 1 fixed. More precisely, if $w' > w$, then any rate of increase of X that can occur in equilibrium at state (w, x) at time t is weakly lower than any rate of increase of X that can occur in equilibrium at state (w', x) at time t.*

Proof: Appendix C.

[9]There may exist a set of states of measure zero at which agents are indifferent between two or more switching rates. For example, if the marginal cost of switching faster is a constant (e.g., $c^m(k, X) = ck$), then when the relative value of being in the other mode equals this constant, an agent will not care which switching rate she chooses. But the relative value of being in mode 1 is strictly increasing in the payoff parameter by Relative Payoff Monotonicity. Hence, the set of states at which this relative value equals a given constant has measure zero. This holds for any switching cost functions: the set of states at which agents are indifferent between multiple switching rates (and thus where $\dot{X}$ can take multiple values) always has measure zero. Thus, this sort of indeterminacy has no effect on the evolution of X or on players' payoffs.

In contrast, agents' switching rates may not be monotonic in X. Although an increase in X raises the relative payoff flow in mode 1, it can also make it more costly to switch into mode 1.

A key question is how the equilibrium behaves dynamically. We will simplify these dynamics by studying the properties of the contour lines. The contour line corresponding to any real number C is simply the set of states at which $\dot{X} = C$. A particularly important case is $\dot{X} = 0$: the boundary between the regions where X is rising and falling. We study this contour line and its implications in section 5.

X falls at its maximum rate if all agents are in mode 1 and they leave at the maximum possible rate, $\overline{K}^1$. In this case, $\dot{X} = -\overline{K}^1$. Similarly, X rises at its maximum rate if all agents are in mode 2 and they move to mode 1 at the maximum rate, $\overline{K}^2$. In this case, $\dot{X} = \overline{K}^2$. Thus, the rate of change of X must lie in the interval $\left[-\overline{K}^1, \overline{K}^2\right]$. Let C be any fixed real number in this interval. For any such C, Theorem 8 states that there is a certain nonempty interval of values of X in which X can rise at the rate C.[10] For all X below this interval, X must rise at a higher rate; for all X above the interval, X must rise at a lower rate. This interval is independent of W and time. Note that the theorem is not a statement about the equilibrium, but only about what is feasible given the constraints on switching rates in axiom **A1**.

THEOREM 8 *Fix a real number $C \in \left[-\overline{K}^1, \overline{K}^2\right]$. The constraints on the switching rates (**A1**) imply that it is feasible for X to rise at the rate C if and only if X lies in the closed interval $[\underline{X}, \overline{X}] \cap [0,1]$, where*

$$\underline{X} = \frac{\underline{K}^2 - C}{\underline{K}^2 + \overline{K}^1} \text{ and } \overline{X} = \frac{\overline{K}^2 - C}{\overline{K}^2 + \underline{K}^1} \tag{17}$$

***A1** also implies the following:*

1. $\dot{X} < C$ *if* $X > \overline{X}$;
2. $\dot{X} > C$ *if* $X < \underline{X}$;
3. $\underline{X} \leq \overline{X}$, *with equality only if* $\underline{K}^1 = \overline{K}^1$ *and* $\underline{K}^2 = \overline{K}^2$.
4. $\underline{X} \leq 1$ *and* $\overline{X} \geq 0$.

---

[10] Note that C can be negative, in which case X "rises" at a negative rate (i.e., it falls).

Proof: Appendix C.

We now consider the set of states (W, X) for which X lies strictly between $\underline{X}$ and $\overline{X}$. Theorem 9 states that this set can be divided into three regions that vary according to how fast X grows (or shrinks) in equilibrium: a leftmost region where $\dot{X} < C$; a middle region (which may be empty) where $\dot{X} = C$; and a rightmost region where $\dot{X} > C$.

THEOREM 9 *Fix a real number* $C \in \left(-\overline{K}^1, \overline{K}^2\right)$ *and a time* t.[11] *Consider the set of states* (W, X) *for which* $X \in [0, 1]$ *lies strictly between* $\underline{X}$ *and* $\overline{X}$.[12] *In this region, let the* lower Isorate curve *be the left boundary of the set of states at which, in equilibrium at time* t, X *rises at a rate of* C *or higher. Let the* upper Isorate curve *be the right boundary of the set of states where, in equilibrium at time* t, X *rises at a rate of* C *or lower. The following properties hold:*

1. *The lower Isorate curve lies weakly to the left of the upper Isorate curve.*
2. *To the left of the lower Isorate curve,* $\dot{X} < C$.
3. *Between the two curves (if they do not coincide),* $\dot{X} = C$.
4. *To the right of the upper Isorate curve,* $\dot{X} > C$.

Proof: Appendix C.

Theorem 10 states that the boundaries between the three regions—the Isorate curves—are continuous functions from X to W. These functions may also depend on time: the Isorate curves may shift, though they cannot jump discontinuously. If W follows a Brownian motion, the Isorate curves are constant over time.

THEOREM 10 *Fix a real number* $C \in \left(-\overline{K}^1, \overline{K}^2\right)$. *Both Isorate curves corresponding to* C *are continuous functions from* (X, t) *to* W. *If the payoff parameter follows a Brownian motion (axiom* ***A2′****), they are independent of* t: *the Isorate curves do not move.*

---

[11] If $C = -\overline{K}^1$ (respectively, if $C = \overline{K}^2$) one can verify that $\underline{X} = 1$ (respectively, $\overline{X} = 0$). In both cases, there clearly is no $X \in [0, 1]$ that lies in $\left(\underline{X}, \overline{X}\right)$.

[12] If the set is empty (if $\underline{X} = \overline{X}$), then Theorems 9-11 are vacuous. By Theorem 8, this holds only if agents never have any choice about switching rates—if the upper and lower bounds on switching rates coincide.

Proof: Appendix C.

Theorem 11 gives sufficient conditions for the upper and lower Isorate curves to coincide. By Theorem 9, the region where $\dot{X}$ equals C has measure zero in this case. Condition 1 states that the Isorate curves coincide if the rate of increase, C, and the minimum switching rates are not all zero. If these three quantities are all zero, condition 2 states that the Isorate curves still coincide if, in both modes, the marginal cost of raising the switching rate is zero when the switching rate is zero.

THEOREM 11 *Fix a real number $C \in \left(-\overline{K}^1, \overline{K}^2\right)$. The upper and lower Isorate curves corresponding to C must coincide if at least one of the following two conditions holds:*

1. *C, $\underline{K}^1$, and $\underline{K}^2$ are not all zero, or*
2. *$C = \underline{K}^1 = \underline{K}^2 = 0$ and for both modes m, all proportions x of players in mode 1, and any positive quantity ε, there is a feasible switching rate $k > 0$ such that the cost of raising the switching rate from zero to k, divided by the increase k, is less than ε: $\frac{c^m(k,x)-c^m(0,x)}{k} < \varepsilon$.*

Proof: Appendix C.

Suppose conditions 1 and 2 fail: $C = \underline{K}^1 = \underline{K}^2 = 0$ and, in at least one of the modes, the marginal switching cost is strictly positive when the switching rate is zero. Consider the region where, for players in this mode, the relative value of being in the other mode is positive but is less than this marginal switching cost. Throughout this region, players in both modes will choose switching rates of zero, so X will not change. Since the relative value of being in one mode vs. the other is a continuous function of the state, this region has positive area. Hence, the Isorate curves for $C = 0$ (which enclose this region) must be distinct.

Figures 1 and 2 display two computed examples. By Theorem 11, the Isorate curves can be distinct only if $C = \underline{K}^1 = \underline{K}^2 = 0$, in which case $\left[\underline{X}, \overline{X}\right]$ equals the unit interval by (17). Figure 1 illustrates an example in which this is the case. The parameters are $\sigma = 0.2$, $r = 0.1$, $u(1, w, x) - u(2, w, x) = 0.5\,(w + 2x - 1)$, $c^1(k, x) = c^2(k, x) = k$, $\underline{K}^1 = \underline{K}^2 = 0$, and $\overline{K}^1 = \overline{K}^2 = 1$. The marginal cost of switching faster is constant, so an agent will either switch at her maximum rate of one or stay in her current mode. Since the minimum switching rates are zero, $\dot{X}$ can feasibly equal zero anywhere in the state space: $\left[\underline{X}, \overline{X}\right]$ equals the unit interval. There are three regions in equilibrium. In the leftmost region, the relative value of being in mode 2 exceeds the marginal cost of switching faster, so agents in mode 1 switch at their maximum rate while agents in mode 2 remain in mode 2.

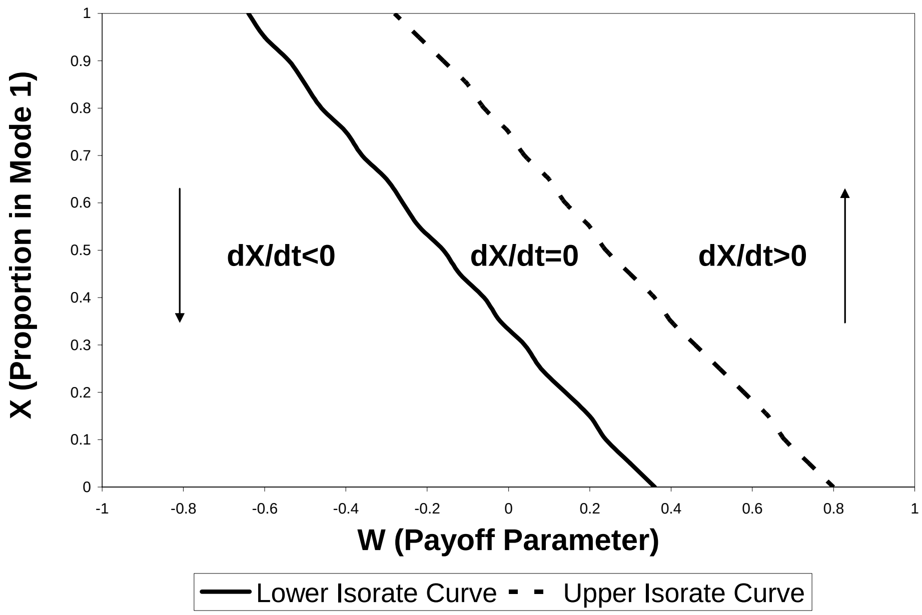

Figure 1:

Hence, X falls. The analogous is true in the rightmost region, where X rises. In the intermediate region, the payoff from switching is less than the marginal cost of switching faster, so all agents remain in their current modes: $\dot{X} = 0$.

Figure 2 shows another example. The parameters are the same as in Figure 1, except that the minimum switching rates, $\underline{K}^1$ and $\underline{K}^2$, now equal 0.05 instead of zero. This guarantees that X must rise (fall) if it is close enough to zero (one) since then nearly all agents are in mode 2 (1) and they must leave at a strictly positive rate. Hence we draw two horizontal dashed lines, one at $\underline{X} = 0.05$ (below which X must rise) and the other at $\overline{X} = 0.95$ (above which X must fall). Between these lines, $\dot{X}$ can feasibly equal zero. This area divides into just two regions. They are separated by the upper and lower Isorate curves, which coincide as Theorem 11 predicts. X falls in the left region and rises in the right one.

The Isorate curves in Figure 2 look quite different from those in Figure 1. Yet the change in parameters is slight: the minimum switching rates are now small and positive rather than zero. To understand why this makes a big difference, suppose we start with Figure 1 and then raise the minimum switching rate in both modes to some small $\kappa > 0$. Since the change is small, the benefit of changing

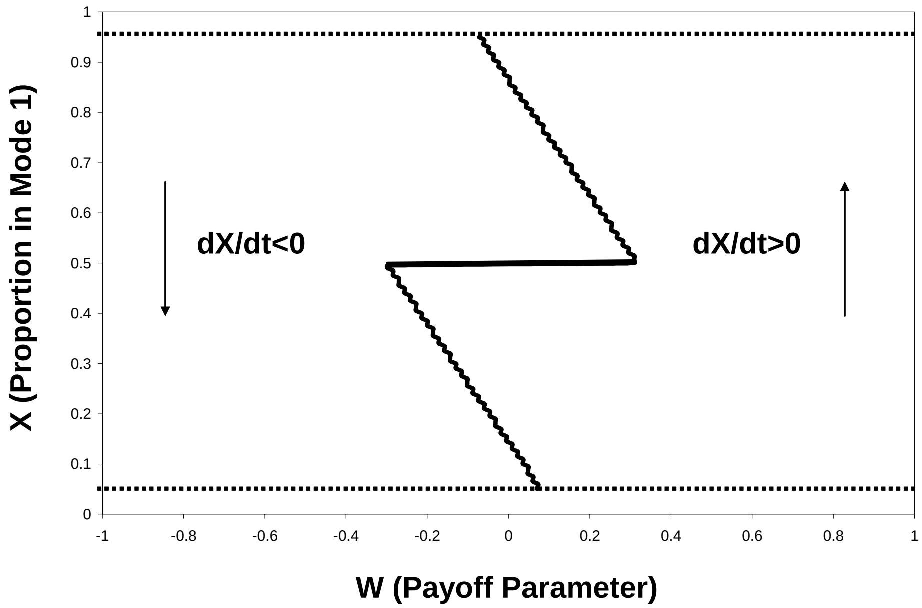

Figure 2:

modes is about the same. Thus, agents are still switching at their minimum rate at nearly all states in the region between the two Isorate curves in Figure 1. However, this minimum rate is now κ rather than zero. This means that $\dot{X}$, which equals $\kappa(1-X)-\kappa X=\kappa(1-2X)$, is now negative (positive) if X exceeds (is less than) one half. The region between the Isorate curves now divides into two; the part in which X exceeds (is less than) one half joins the region where X is falling (rising). There are now only two regions, separated by a single boundary. This explains the difference between the two figures.

4.2 Howitt-McAfee Model

HM show that their model has multiple equilibria in the absence of productivity shocks. Theorem 12 shows that HM's model with shocks is a special case of our general model. Thus, it has a unique equilibrium outcome.

THEOREM 12 *HM's model with shocks (section 2) is a special case of the general model of section 3, if one makes the following assocations:*

1. *the players in the general model correspond to the firms in HM;*
2. *mode 1 corresponds to a filled job;*
3. *mode 2 corresponds to an unfilled job;*
4. *the utility flow in mode 1 corresponds to the profit flow from a filled job,* $\omega f(W_t, X_t)$*;*
5. *the utility flow in mode 2 corresponds to the profit flow from an unfilled job, which is zero;*
6. *The switching rate in mode 1 corresponds to the attrition rate, with lower and upper bounds both equal to the constant* δ*;*
7. *The switching rate in mode 2 corresponds to the hiring rate* θ*, with a lower bound of zero and an upper bound of* $\overline{\theta}$*;*
8. *switching costs in mode 2 correspond to hiring costs,* $c^A(\theta_t, X_t)$*;*
9. *switching costs in mode 1 correspond to attrition costs, which are zero.*

Proof: Appendix C.

Theorems 2-11 imply that the unique outcome has the following properties. The Markov Property implies that the distribution of paths of future employment depends only on the current state and time. If the payoff parameter follows a Brownian motion, this distribution is independent of time. Let V_t^F (V_t^U) be the continuation payoff of a firm with a filled (vacant) position at time t, as given by equation (1). By Payoff Continuity, these are continuous functions of the current state and time; with Brownian motion, they are independent of time.

Relative Payoff Monotonicity states that the relative value of a filled position, $V_t^F - V_t^U$, equals the expected integral of the firm's discounted production profits, plus the search costs it would have incurred if the position were vacant:

$$V_t^F - V_t^U = E\left[\int_{v=t}^{\infty} \exp\left(-\int_{s=t}^{v}[r+\delta+\theta_s]\,ds\right)\begin{pmatrix}\omega f(W_v, X_v)\\ +c^A(\theta_v, X_v)\end{pmatrix} dv\right] \qquad (18)$$

where θ_s is the (common) hiring rate of firms with vacancies at time s. This relative value is strictly increasing in the current payoff parameter and employment rate.

The Switching Rate Rule implies that firms with vacancies at a given state choose the common hiring rate at which the marginal cost of raising the hiring rate,

c_θ^A, equals the marginal benefit, $V^F - V^U$. From Switching Rate Monotonicity we see that that an increase in the payoff parameter leads to a weakly higher hiring rate, controlling for the employment rate and time. By Growth Rate Monotonicity, an increase in the payoff parameter weakly raises the rate of change of the employment rate, which by (16) is

$$\dot{X} = \theta(1 - X) - \delta X \tag{19}$$

If the economy is currently in steady state, the employment rate is neither growing nor shrinking. Hence, by Growth Rate Monotonicity, a positive shock cannot lead the employment rate to fall. Likewise, a negative shock cannot cause it to rise. An important implication of this is that recessions can result only from negative shocks and booms from positive ones. This is typically not the case in models with multiple equilibria.

There is positive attrition in HM's model, so condition 1 of Theorem 11 holds: the Isorate curves for any C coincide. Hence, the regions of rising and falling employment are separated by a curve that is a continuous function from X to W. By Theorem 10, if the payoff parameter follows a Brownian motion, then this curve does not shift over time.

5 Equilibrium Dynamics

We will discuss dynamics in the context of HM's model; this discussion also applies to the general model when the Isorate curves corresponding to $\dot{X} = 0$ coincide (see Theorem 11). For simplicity, we also assume that the payoff parameter follows a Brownian motion. This implies that the Isorate curve does not shift over time.

By Theorems 8-11, the regions of rising and falling employment are separated by a curve (the Isorate Curve corresponding to $\dot{X} = 0$) that is a continuous function from X to W. A computed example appears in Figure 3; the algorithm is set out in Appendix B. Arrows indicate the dynamics of the employment rate. To the right of the curve, employment is rising; to the left and above the curve, it is falling.

A steady state is a state (w, x) at which the local dynamics of X are stable. This means that $\dot{X}$ is negative at states slightly above (w, x) and positive at states slightly below (w, x). The steady states in Figure 3 are shown in bold. They consist of the upwards sloping portion of the Isorate Curve, together with the indicated part of the lower horizontal axis.

The unstable, downward sloping part of the Isorate Curve is shown in dashes. For values of the payoff parameter that correspond to this part of the curve, there are multiple steady states. Suppose the employment rate begins at the higher level and there is a sequence of negative shocks to W. When W reaches about -0.3, the upwards sloping segment of the Isorate Curve folds back on itself. At this

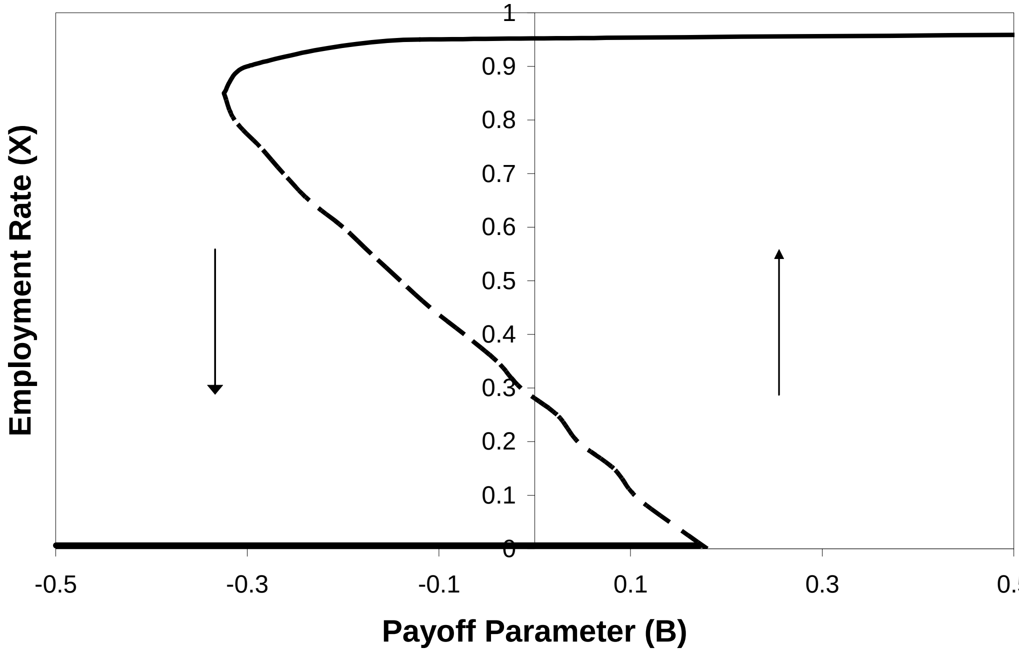

Figure 3:

threshold, an arbitrarily small negative shock to W causes the high steady state to disappear. Since the only remaining steady state is the low one, employment will gradually fall to this new, lower level. A small shock suffices to cause a recession.

Once this happens, an equal and opposite shock to W will cause the high-employment steady state to reappear. However, the low steady state is also dynamically stable. Thus, the economy will remain in recession until a sufficiently large shock to W causes the low steady state itself to disappear. With multiple steady states, small shocks can have cumulatively large and hard-to-reverse effects on the employment rate.[13]

We will see in section 5.2 that small shocks can have large effects also with a unique steady state. However, in this case the economy does not get stuck: an equal and opposite shock will always restore the steady state to its initial level.

We have assumed that the payoff parameter follows a Brownian motion. This simplifies the exposition since, with a stationary environment, the Isorate Curve is constant over time. With seasonal and mean-reverting shocks, the environment

[13]This is an example of a cusp catastrophe (see Zeeman [35]).

players face at any given state (W, X) changes over time. Hence, their optimal behavior may also change. This yields an Isorate Curve that can fluctuate over time as the environment changes. However, Figure 3 still accurately describes the dynamics of employment at any *given* time. Employment still rises to the right of the current Isorate Curve (wherever it is) and falls to the left. In addition, our discussion of the system's dynamics remains accurate as long as the shifts in the Isorate curve occur sufficiently slowly, relative to changes in the employment rate.

5.1 When are there Multiple Steady States?

There are multiple steady states in regions where the Isorate Curve has downward sloping segments. The Isorate Curve is upward (downward) sloping if an increase in employment, holding the payoff parameter fixed, lowers (raises) the employment growth rate. To see why, suppose we are initially at $(W, X) = (w, x)$ and consider what happens if there is a one-shot increase of dx in the employment rate. If this increase lowers employment growth below zero, then by Growth Rate Monotonicity the payoff parameter must rise in order to restore it to zero: the Isorate Curve must be upward sloping. On the other hand, if the increase in the employment rate raises the employment growth rate above zero, a decrease in the payoff parameter is needed to return growth to zero: the Isorate Curve is downward sloping.

We thus need to determine how an increase in employment affects the employment growth rate. In HM's model, this is the sum of three effects. The first is the change in employment growth that results if employment rises holding firm behavior (hiring rates) constant. This effect on employment growth is negative. By (19), the decline in the employment growth rate in HM is exactly $-(\delta + \theta)dx$: the larger is the sum of the attrition and hiring rates, the more negative is the first effect. This is because there are dx fewer firms with vacancies, who are hiring at the rate θ, and dx more firms with filled jobs, whose workers are leaving at the rate δ.

The second effect comes from changes in agents' behavior. Because there are strategic complementarities in production, the increase in employment raises the incentive to produce. This second effect tends to raise the employment growth rate by raising the hiring rate in HM.

There may also be a third effect. If an increase in employment alters hiring costs, this can affect a firm's incentive to fill a vacancy. For example, there may be congestion: marginal hiring costs may be higher when the vacancy rate is low, since fewer workers are searching for jobs. HM assume this.[14] With congestion, an

[14]One might also argue for *anticongestion*: it may be easier to hire when employment is high since there is less competition from other firms. With anticongestion, the third effect of an increase in employment on employment growth is positive. On the other hand, anticongestion may weaken

increase in employment tends to lower employment growth by lowing hiring rates. But there can also be a reverse effect: an anticipated increase in employment may raise the incentive to hire now, before hiring costs rise. Via this effect, an increase in employment can raise employment growth. Hence, the overall effect of congestion on the shape of the Isorate Curve is ambiguous.

To conclude, the effect of the increase in employment on employment growth is the sum of the negative effect when agents' behavior is held constant and the positive effect via strategic complementarities. There is also a congestion effect of ambiguous sign. If the sum of these effects is positive, the payoff parameter must fall to restore employment growth to zero: the Isorate Curve is downward sloping. If the sum is negative, the Isorate Curve is upward sloping. In HM, for instance, the more rapid is job attrition, the stronger the production spillovers must be in order for the equilibrium to have multiple steady states.

5.2 Comparative Statics

In this section we use simulations to study the effects of strategic complementarities and congestion on the shape of the curve that separates the regions of rising and falling employment in HM's model. We also depict the case analyzed by HM, in which shocks are absent. Figures 4-7 depict four cases that differ in the presence or absence of strategic complementarities and congestion in hiring. They illustrate that even when the steady state employment level is always unique, there are regions where small shocks can have disproportionately large effects on this steady state.

We contrast the surplus functions $f = W + X$ (strategic complementarities) and $f = W + 0.5$ (no complementarities).[15] Firms keep half the surplus. We also contrast hiring cost functions $c^A = \frac{\theta^2}{1-X}$ (congestion) and $c^A = \theta^2$ (no congestion). To simplify the exposition, we assume W follows a Brownian motion. The Brownian motion has zero drift and variance $\sigma^2 = 0.04$. The maximum advertising rate is $\bar{\theta} = 1$ and the attrition rate is $\delta = 0.02$. The discount rate is $r = 0.1$. The Isorate curves coincide in this case by Theorem 11.

Figure 4 depicts the case in which there are both complementarities and congestion. This is the case studied by HM. The two dashed curves and the narrow line

strategic complementarities: if other firms hire, a given firm's incentive to hire is weakened since it will become less costly to hire later. Since this weakens the second effect, the net effect of anticongestion on the shape of the Isorate Curve is ambiguous. This assumes that any anticongestion is weak enough that the critical assumption of Strategic Complementarities (**A3**) still holds; if **A3** is violated, our results do not apply.

[15] The constant 0.5 just ensures that the Isorate curves appear in about the same place in the two cases. It does not affect the shape of the curves.

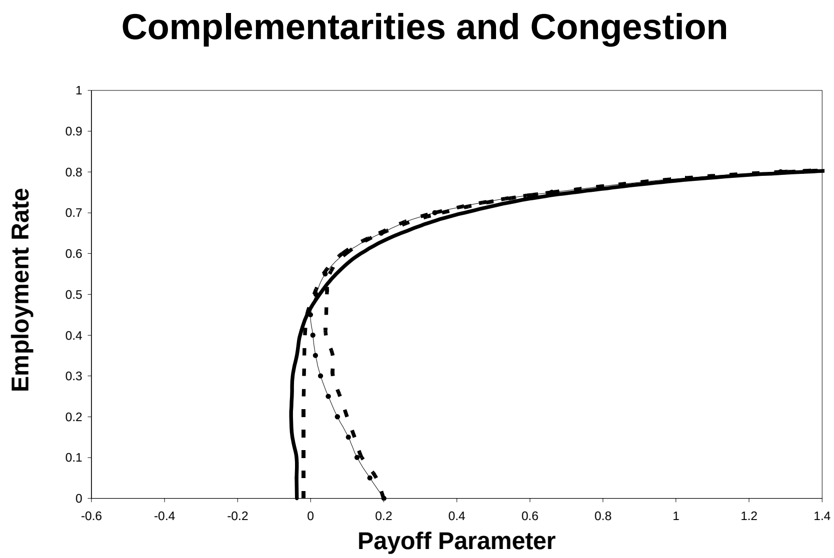

Figure 4:

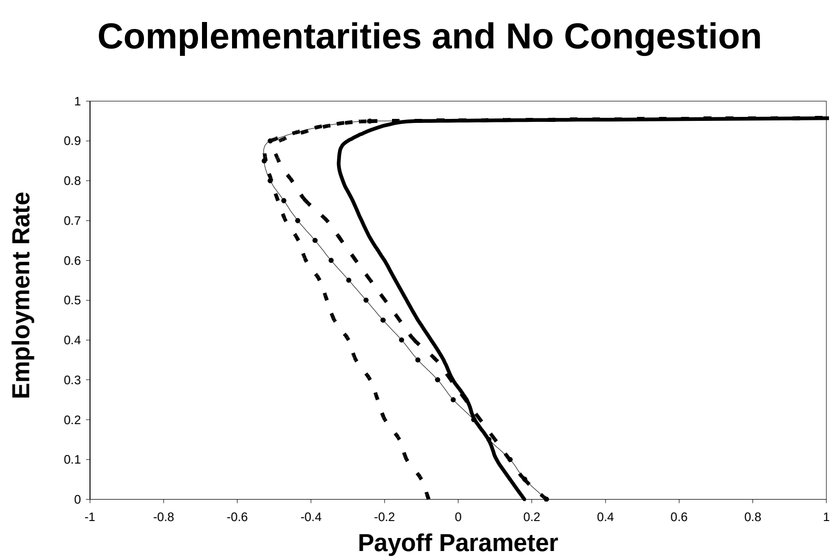

Figure 5:

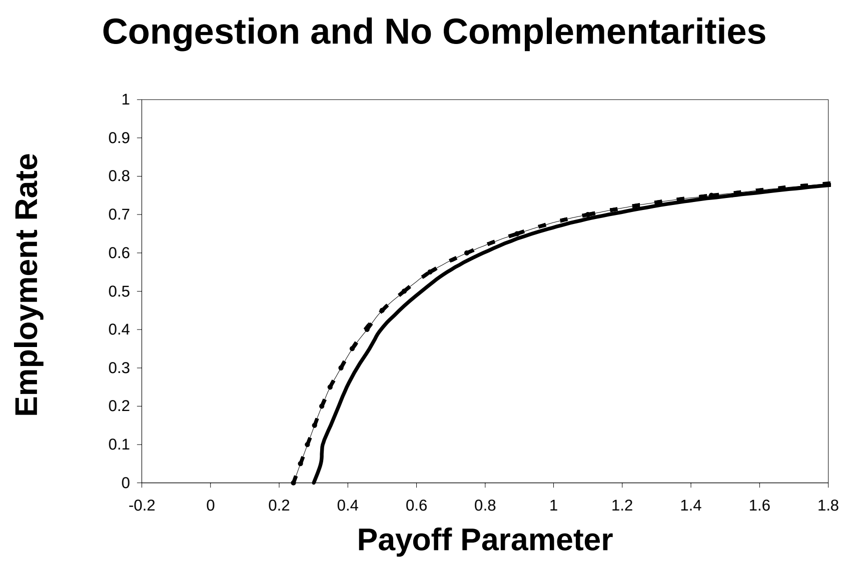

Figure 6:

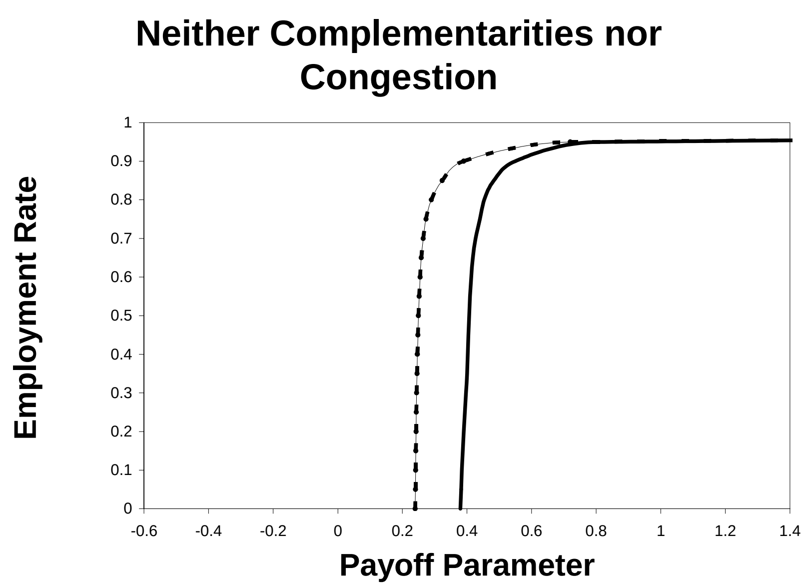

Figure 7:

marked with dots pertain to a world without shocks, in which W is constant. The two dashed curves divide the state space into three regions. In the leftmost region, employment must be falling in any equilibrium without shocks. In the rightmost region, it must be rising without shocks. In the region between the dashed curves, employment can either rise or fall, depending on agents' expectations. The narrow line marked with small dots is the set of steady states in a world without shocks: the states at which, if firms expect employment to remain at its current level forever, they will choose hiring and layoff rates that keep employment constant.

The solid curve pertains to a world with shocks, where the equilibrium is unique. To the right of this curve, employment is rising; to the left, it is falling. There is a negligible region of multiple steady states in this case. The effects of attrition and congestion, which make the curve upwards sloping, offset the strategic complementarities, which tend to make the curve slope down.

This is not the case in Figure 5, which depicts the case in which there are complementarities in the production function but no congestion in hiring. The curves have the same interpretation in this figure. The solid curve has a downwards sloping segment: there are multiple steady states.

Figure 6 corresponds to the case in which there is congestion in hiring but no complementarities in production. Here the solid curve is upwards sloping: the steady state employment level is unambiguously unique for each value of the payoff parameter. In Figure 6, the boundaries of the region of multiplicity without shocks (the two dashed curves) coincide since there is a unique equilibrium without shocks. This is also so in Figure 7, which depicts the case in which there are neither complementarities in production nor congestion in hiring. The black curve in Figure 7 is upwards sloping, so the steady state employment level is always unique. However, it is much steeper than in Figure 6 since there is no congestion. Near the steep segment, a small shock to the payoff parameter has a large effect on the steady state employment level. But since the steady state is unique, this large effect can be undone by an equal and opposite shock. This contrasts with the case of multiple steady states.

Figure 8 illustrates how the equilibrium changes as the strategic complementarities in production become stronger. The figure depicts the Isorate curve for $\dot{X} = 0$ in the stochastic case for a range of parameters: $f(w, x) = w + ax - a/2$ where $a =$ 0, 2, 4, 8, and 16. This captures strategic complementarities that range from nonexistent ($a = 0$) to relatively strong ($a = 16$). The rest of the parameters are as in Figures 4-7 for the congestion case ($c^A = \frac{\theta^2}{1-X}$). With no complementarities ($a = 0$), there is a unique steady state for every value of the payoff parameter. With positive but relatively weak complementarities ($a = 2$), there is a small area of multiple steady states. As the complementarities grow, this region expands.

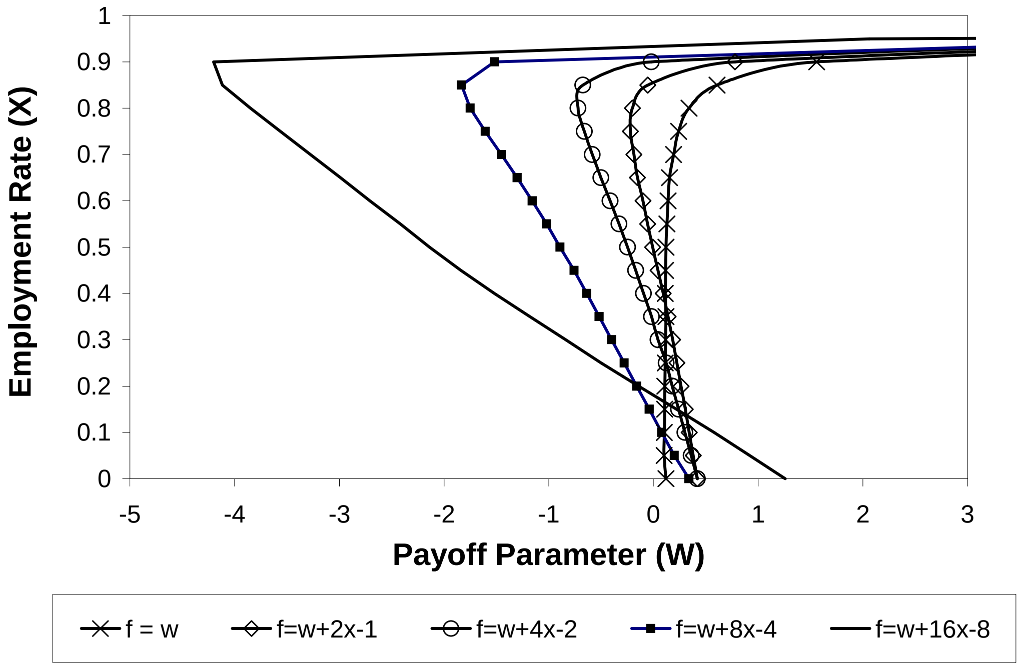

Figure 8:

6 Intuition

This section presents a detailed intuition for the uniqueness result. We focus on HM's model; the intuition for the general model is analogous. We begin with the case of Brownian shocks (axiom **A2′**); section 6.1 explains what changes with seasonal and mean-reverting shocks.

The game has strategic complementarities: firms hire more intensively if other firms are expected to do so since the resulting increase in employment makes it easier to market one's products. By a result in Milgrom and Roberts [22], the hiring rates chosen by firms in any equilibrium are bounded above and below by strategy profiles ($\overline{S}$ and $\underline{S}$, respectively) that are also equilibria of the model. Moreover, these equilibria are monotonic: an increase in the payoff parameter or employment rate, ceteris paribus, leads firms to choose a weakly higher hiring rate.

Our task is to explain why these strategy profiles must coincide. They are depicted in Figure 9. Since we have only two dimensions, we (a) assume the strategies are independent of time t and (b) depict the relationship between the payoff parameter W and the hiring rate θ for some *fixed* employment rate X. In any equilibrium, firms choose hiring rates no lower than those prescribed by $\underline{S}$ and no higher than those given by $\overline{S}$.

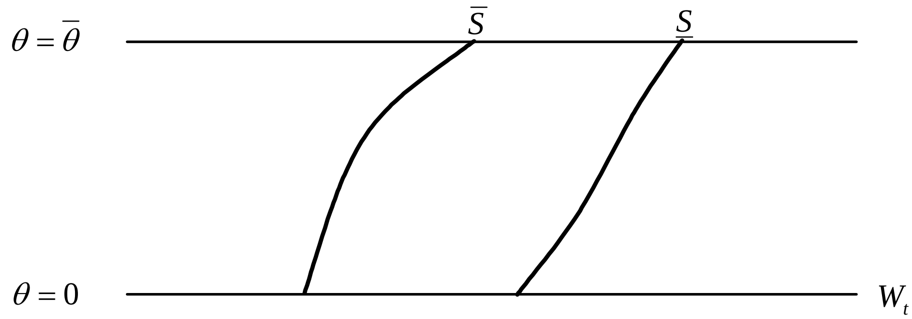

Figure 9:

Since there are dominance regions, the strategy $\overline{S}$ must prescribe hiring at the minimum rate of zero for low enough values of the payoff parameter. Thus, there is a translation $\widetilde{S}$ of $\underline{S}$ that lies entirely to the left of $\overline{S}$ and touches $\overline{S}$ at some w^* (Figure 10). Let Δ be the horizontal distance between $\widetilde{S}$ and $\overline{S}$.[16]

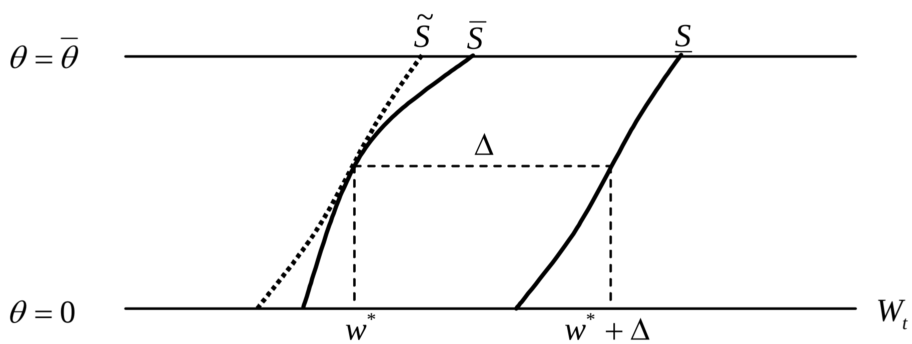

Figure 10:

Since the two curves touch at w^*, we know that[17]

$$\overline{S}\,(w^*) = \underline{S}\,(w^* + \Delta)\,. \tag{20}$$

[16] $\widetilde{S}$ is a translation of the entire strategy, not just the part depicted in the figures. That is, the hiring rate prescribed by $\widetilde{S}$ when the employment rate is X_t and the state is W_t is set equal to the hiring rate prescribed by $\underline{S}$ when the employment rate is X_t and the state is $W_t + \Delta$, *for any* X_t.

[17] We suppress the dependence of the strategies on time and X_t for expositional clarity.

Let θ^* be the optimal hiring rate of a firm when the payoff parameter is w^*, if the firm believes that all other firms will use the strategy $\widetilde{S}$. Since $\widetilde{S}$ is everywhere above $\overline{S}$, a firm expects to see higher employment rates in the future if it believes that all other firms will use the strategy $\widetilde{S}$ than if it expects them to use $\overline{S}$. Thus, since there are strategic complementarities,

$$\theta^* \geq \overline{S}\,(w^*)\,. \tag{21}$$

We will now show that θ^* must be strictly less than $\underline{S}\,(w^* + \Delta)$ unless $\widetilde{S}$ and $\underline{S}$ *coincide*. Since this inequality would contradict (20) and (21), the two curves must indeed coincide. As $\overline{S}$ lies entirely between them, it must also coincide with $\underline{S}$; this will show that the equilibrium is unique.

Consider two firms: firm i is at state (w^*, x) and expects other firms to use strategy $\widetilde{S}$; firm j finds itself at state $(w^*+\Delta, x)$ and expects $\underline{S}$ to be used. Since the payoff parameter follows a Brownian motion, which has stationary and independent increments, the changes in the payoff parameter must have the same distribution at the two points. Thus, since one strategy is an exact translation of the other, agents i and j must also expect the same *joint* distribution of changes in the state, (W, X).[18] The only difference is that j expects a payoff parameter that is always Δ *less* than the payoff parameter that i expects. Since the profit flow from a filled position is increasing in the payoff parameter, the gains from hiring are strictly lower for j than for i. So j must choose a lower hiring rate than i: j's optimal rate, θ^*, is lower than i's optimal rate, $\underline{S}(w^* + \Delta)$.[19] This completes the argument: $\overline{S}$ must coincide with $\underline{S}$, so there is a unique equilibrium.

The argument relies on the fact that i and j expect the same joint distribution of X and changes in W. More precisely, $(W_v - W_t, X_v)_{v \geq t}$ has the same distribution at the two states. Since a Brownian motion has i.i.d. increments, $(W_v - W_t)_{v \geq t}$ has the same distribution. But since $\widetilde{S}$ is a translation of $\underline{S}$ and since X_t is the same at the two states, firms i and j expect any given path $(W_v - W_t)_{v \geq t}$ to generate the same path $(X_v)_{v \geq t}$ of employment rates. Thus, $(W_v - W_t, X_v)_{v \geq t}$ has the same distribution at the two states.

The above argument presumes that each continuation path of W generates a unique path of X. By equation (19), X is the solution to the differential equation

[18] More precisely, $(W_v - W_t, X_v)_{v \geq t}$ has the same distribution at the two states. Since a Brownian motion has i.i.d. increments, $(W_v - W_t)_{v \geq t}$ has the same distribution. But since $\widetilde{S}$ is a translation of $\underline{S}$ and since X_t is the same at the two states, i and j expect any given path $(W_v - W_t)_{v \geq t}$ to generate the same path $(X_v)_{v \geq t}$ of proportions of players in mode-1. Thus, $(W_v - W_t, X_v)_{v \geq t}$ has the same distribution at the two states.

[19] Strictly speaking, it is only weakly lower. This technical point is addressed in the proof.

$\dot{X} = \theta(1 - X) - \delta X$, where the hiring rate θ depends on the state (W, X) and time t. This equation would have a unique solution if θ were Lipschitz continuous in X. However, it may not be: a small increase in X can easily lead to a jump in the optimal hiring rate, making θ discontinuous. Instead, we show uniqueness using a weaker property: that the effect of X on θ is bounded *relative* to the effect of W on θ. This follows from equation (11), which provides an upper bound on the effect of X on workers' productivity, and assumption **A4**, which implies a lower bound of the effect of W on their productivity.[20] In addition, the changes in X are infinitesimal compared to the changes in W over small time intervals.[21] Thus, over small time intervals, changes in W effectively "blur" the effect of X on θ, making θ Lipschitz in a *probabilistic* sense in X.

This completes the intuition for the case of Brownian shocks.

6.1 Seasonality and Mean-Reversion

The same argument might appear to break down when W displays mean reversion and seasonality (axiom **A2**). The problem is in the last step: if $\Delta > 0$, the distribution of changes in W will not, in general, be the same at the two states, so the distribution of changes in X may differ as well. This problem is clearest in the case of mean-reversion. For example, suppose W reverts to a mean value that lies somewhere between w^* and $w^* + \Delta$. Firm i at w^* would expect W to trend upwards while firm j at $w^* + \Delta$ will expect it to drift downwards. Since $\widetilde{S}$ is an exact translation of $\underline{S}$, i will expect, on average, higher values of X than j. On the other hand, W will tend to be lower for firm i than for firm j since it starts lower. These two differences go in opposite directions: one makes hiring more appealing and one less. Thus, the two firms may well *both* want to choose the same hiring rate: the two strategies may not coincide.

How do we overcome this problem? We exploit the dynamic structure of the game: we use a translation of $\underline{S}$ that *varies over time*: $(\widetilde{S}_t)_{t\geq 0}$ instead of $\widetilde{S}$. We move the translation in such a way that if firm i believes others will choose hiring rates given by $\widetilde{S}_t$ at each future time t, it *does* expect the same distribution of paths of X as firm j, which expects others to choose hiring rates given by $\underline{S}$. For example, if the payoff parameter is mean-reverting, then the strategy $\widetilde{S}_t$ would drift upwards

[20] More precisely, W is expected to spend a positive amount of time in the future in the interval (w_1, w_2), where it will have a positive effect on workers' productivity. This creates a strictly positive effect of the current value of W on future productivity and thus on firms' current hiring incentives.

[21] Changes in W over a short time interval ε have a large random component: their standard deviation is of order $\sqrt{\varepsilon}$. On the other hand, since firms' hiring and layoff rates are bounded, the changes in X are of order ε, which becomes infinitely smaller than $\sqrt{\varepsilon}$ as $\varepsilon \to 0$.

with W over time, gradually approaching $\underline{S}$. An example appears in Figure 11 for some times t and t', where $t < t'$.

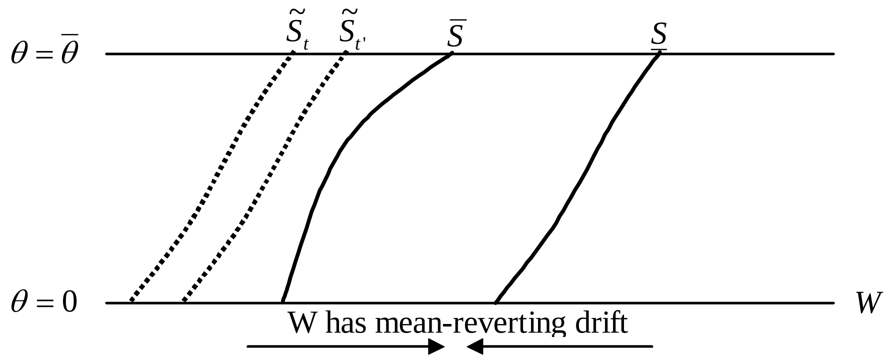

Figure 11:

The argument relies crucially on the condition in assumption **A2** that any mean reversion in W eventually die out. This condition implies that the translations don't have to actually converge to $\underline{S}$ over time.[22] If they did, then regardless of how far to the left we shifted the time 0 translation of $\underline{S}$, subsequent translations would have to converge eventually to $\underline{S}$ and thus would not all lie to the left of $\overline{S}$ as the argument requires.

7 Concluding Remarks

7.1 Relation to Global Games

Our limit uniqueness argument generalizes the arguments of BFP and FP. However, providing a general argument for the case of endogenous frictions and seasonal and mean-reverting shocks requires a significant extension of that logic. The intuition is most closely related to results in Frankel, Morris, and Pauzner [12] (henceforth, FMP). They study a static model with incomplete information. Players choose

[22] More precisely, the translations are given by $\widetilde{S}_t(W_t) = \underline{S}_t(W_t + g(t)\Delta)$, where $g(t) = \exp\left(\int_{s=0}^{t} \nu_s ds\right)$. (We write $\underline{S}_t$ since this lowest surviving profile can also move over time - as can the highest, $\overline{S}_t$.) For example, if there is mean reversion, then $\nu_t < 0$, so $g(t)$ shrinks over time: $\widetilde{S}$ gradually approaches $\underline{S}$. The parameter Δ is the infimum of parameters for which $\widetilde{S}_t$ lies to the left of $\overline{S}_t$ for all t. The condition in **A2** that $\exp\left(\int_{s=0}^{\infty} |\nu_s|\, ds\right) < \infty$ implies that $g(t)$ is bounded above by a strictly positive constant, so by the existence of dominance regions such a Δ must exist.

from a compact set of actions. Each player receives a private signal of a common parameter that affects everyone's payoffs. The prior over this parameter includes "dominance regions" in which the highest and lowest actions are strictly dominant. FMP's model is an example of a global game (Carlsson and van Damme [7], Morris and Shin [25]).

FMP show that a unique equilibrium survives iterative dominance. While the details are different, there is an analogy. In both cases, players play against opponents in different but nearby "states": the value of the Brownian motion at the moment when the opponent picks her action in this paper and a player's payoff signal in FMP. This local interaction gives rise to a contagion effect that begins in the dominance regions and spreads throughout the state space. The whole state space is affected because the interaction structure is stationary: the probability of playing against an agent who sees a state a given distance from one's own state is independent of one's own state.[23] Using this property, a translation argument implies that the lowest and highest strategies surviving iterative dominance must coincide.

The translation argument itself is also similar in some respects but not in others. In both papers, the vertical axis captures a player's action and the horizontal axis represents the payoff parameter. But in this paper, strategies depend also on the population action distribution X, which is not pictured in the graphs of section 6. To display a strategy in its entirety we would need three dimensions. In addition, we have to establish that the system is determinate: that for almost any path of the Brownian motion, the path of X is unique. This difficulty is absent in the static case. These are the essential differences.

As explained in section 6.1, mean-reversion poses a problem for our translation argument. A phenomenon like mean reversion can also occur in FMP's setting (and in other global games). It occurs when the prior over the true payoff parameter is single-peaked and the noise in the signals does not vanish. Then when a player gets a high signal, chances are her signal error was positive, so she believes that her opponent probably got a signal below hers and thus is likely to pick a lower action. Analogously, when W is mean-reverting, if a player sees a high value of W, those who pick actions after her will tend to see lower values of W and thus pick lower actions. In both cases, the strategic effect (expecting others to pick lower actions) can offset the direct effect of seeing a higher signal/parameter on a player's incentive to pick a higher action, and the translation argument may not

[23]In FMP, a player's signal asymptotically has no effect on her posterior belief that her opponent's signal differs from hers by a given amount. In this paper, the stationarity of Brownian motion implies that the payoff parameter a player sees when choosing her action has no effect on the probability that she will meet an opponent who will have chosen his action when the payoff parameter will have shifted by a given amount.

work.

As we show in section 6.1, mean-reversion in dynamic games can be overcome by exploiting the game's dynamic structure (as long as the mean reversion asymptotically disappears). Each successive player uses a translation that is a bit closer (in a sense) to the strategy $\underline{S}$ that is being translated. This approach does not seem to work in static games. Since player i plays against player j, who in turn plays against player i, one cannot place each player's strategy closer than the other's to a given strategy. If j's strategy is closer than i's, then i's strategy must be farther away than j's.

Indeed, static games with noisy payoff signals can have multiple equilibria when the noise is not taken to zero (Morris and Shin [26, 27]). A stronger statement can be made: if the prior is single-peaked and the noise is positive, then there must exist payoffs for which the equilibrium is not unique. In contrast, in our setup, if the mean reversion eventually dies out, then the equilibrium is unique for *any* payoffs—regardless of how strong the strategic complementarities are.

We will show the result for static games using a simple example. Consider the following payoff matrix, where x_i is player i's payoff signal and $c > 0$ is a constant that is common knowledge.[24]

		Player 2 R	Player 2 L
Player 1	R	$1 + cx_1, 1 + cx_2$	$cx_1, 0$
	L	$0, cx_2$	$1, 1$

Suppose $x_i = \theta + \varepsilon_i$ where $\theta \sim N(0, 1)$, $\varepsilon_1, \varepsilon_2 \sim N(0, \sigma^2)$, and θ, ε_1, and ε_2 are independent. This game has dominance regions[25] and strategic complementarities, and an increase in a player's signal raises the relative payoff from playing R. By standard reasoning (e.g., Morris and Shin [25]), there is a unique equilibrium in the limit as the signal noise vanishes.

We consider threshold equilibria in which a player plays R if her signal exceeds some cutoff x^* and L otherwise. For this to be an equilibrium, a player with signal $x_i = x^*$ must be indifferent between R and L. We will now show that as long as the noise is positive, there exist payoffs for which there is more than one threshold equilibrium:

[24] We simplify by supposing that a player's payoff depends on her signal rather than on θ. This is not necessary for there to be multiple equilibria.

[25] If $x_i > 1/c$, it is strictly dominant to play R; if $x_i < -1/c$, L is strictly dominant.

CLAIM 1 *No matter how small is $\sigma^2 > 0$, there are multiple threshold equilibria in the above global game if $c > 0$ is small enough.*

Proof: Appendix C.

The intuition is that raising the cutoff threshold x^* has two opposing effects on the incentives of a player with signal $x_i = x^*$. The first is the direct effect on the payoff matrix: raising x^* makes R more appealing for a player with signal $x_i = x^*$, since her signal is higher. If this were the only effect, there would be a unique equilibrium. But with positive noise there is also a strategic effect: since the prior is single-peaked, raising x^* leads a player with signal x^* to conclude that her opponent is more likely to get a signal that is less than her own. Since players with signals below the cutoff play L, this strategic effect makes L more appealing by strategic complementarities. If the strategic effect is strong enough relative to the direct effect, there can be multiple threshold equilibria.

The direct effect is proportional to c. The strategic effect is increasing in σ^2 since a less precise signal gives the prior more weight. As long as the prior is single-peaked, the strategic effect is positive for any $\sigma^2 > 0$. Thus, if the direct effect (as measured by c) is small enough given σ^2, there can be multiple threshold equilibria.

A Notation Guide

Tables 1 and 2, which appear at the end of this paper, define the principal notation used in the body of this paper.

B Computing the Equilibrium

This section gives an algorithm for computing tight upper and lower bounds on the switching rates that can be chosen in any equilibrium in the general model. For simplicity, we assume the shocks are stationary: the change in W over time dt has mean $-\nu(W_t - W_{\text{mean}})dt$ and variance $\sigma^2 dt$, where $\nu \geq 0$ and $\sigma^2 > 0$ are constants. This means that:

$$dW_t = -\nu(W_t - W_{\text{mean}})dt + \sigma dB_t \tag{22}$$

where ν is a nonnegative constant and B is a Brownian motion with zero drift and unit variance.

There are two cases. If $\nu = 0$, W follows a Brownian motion. The upper and lower bounds must be equal since the equilibrium is unique by Theorem 1. If

$\nu > 0$, W reverts to the mean value W_{mean}. Since condition 2 in axiom **A2** does not hold in this case, there may be multiple equilibria: equality of the upper and lower bounds is not guaranteed.

In both cases, the bounds are tight. In particular, the following two strategy profiles are equilibria, where the functions $\overline{k}^m(W,X)$ and $\underline{k}^m(W,X)$ give the computed upper and lower bounds on the switching rate of an agent in mode $m = 1,2$ at each state.

Highest-X Equilibrium: Mode-1 players play $\underline{k}^1$ while mode-2 players play $\overline{k}^2$.

Lowest-X Equilibrium: Mode-1 players play $\overline{k}^1$ while mode-2 players play $\underline{k}^2$.

Why? We find each strategy profile by iterating the best-reply correspondence until it converges. Since the result is a fixed point of the best-reply correspondence, it must be an equilibrium. The only difference is the starting point of the iterations. In the Highest-X Equilibrium, we begin the iterations with players choosing $k^1 = \underline{K}^1$ and $k^2 = \overline{K}^2$ at each state. This creates the highest feasible path of X, which by Strategic Complementarities creates the strongest incentives to be in mode 1. Likewise, in the Lowest-X Equilibrium, we begin the iterations with players choosing $k^1 = \overline{K}^1$ and $k^2 = \underline{K}^2$ at each state. This creates the lowest feasible path of X, which by Strategic Complementarities creates the strongest incentives to be in mode 2.

We first prove some useful bounds on the relative value of being in mode 1 vs. mode 2 for a payoff-maximizing player.

LEMMA 1 *Suppose W follows the process $dW_t = -\nu(W_t - W_{mean})dt + \sigma dB_t$ where B is a Brownian motion with zero drift and unit variance and $\nu \geq 0$. Define:*[26]

$$
\begin{aligned}
c_0 &= \frac{\overline{\alpha}}{r} \\
c_1 &= \frac{\max_{x\in[0,1]} |\Delta u(0,x)|}{r} + \overline{\alpha}\left[\sigma + \frac{\sigma}{r^2}\right] \\
c_1' &= \frac{\max_{x\in[0,1]} |\Delta u(0,x)|}{r} + \overline{\alpha}\left[|W_{mean}|\left(\frac{1}{r} - \frac{1}{r+\nu}\right) + \frac{\sigma}{r\sqrt{2\nu}}\right]
\end{aligned}
$$

where $\Delta u(w,x) = u(1,w,x) - u(2,w,x)$ is the difference in direct payoff flows in mode 1 vs. mode 2 at state (w,x). Fix a strategy profile for all players but one.

[26]The term $\max_{x\in[0,1]} |u(1,0,x) - u(2,0,x)|$ equals the greatest absolute difference in direct utility between the two modes when $W = 0$ and X can take any value between 0 and 1.

Call her player i. Let V_t^m be i's continuation payoff at time t if she is in mode $m = 1, 2$ and plays optimally from time t onwards. The following bounds hold regardless of the strategy profile of the other players.

1. *Upper Bounds: If $\nu = 0$, then $|V_t^1 - V_t^2| \le c_0 |W_t| + c_1$. If $\nu > 0$, then $|V_t^1 - V_t^2| \le c_0 |W_t| + c_1'$.*

2. *Lower Bounds: Assume there is an $\underline{\alpha} > 0$ such that for all x, $w > w'$, and any feasible k^1 and k^2,*

$$D(w, x, k^1, k^2) - D(w', x, k^1, k^2) > \underline{\alpha}\,(w - w')$$

If $\nu = 0$, then for $W_t \ge 0$, we have $V_t^1 - V_t^2 \ge -c_1 + \frac{\alpha}{r+2K} W_t$; for $W_t \le 0$ we have $V_t^1 - V_t^2 \le c_1 + \frac{\alpha}{r+2K} W_t$. If $\nu > 0$, then for $W_t \ge 0$, we have $V_t^1 - V_t^2 \ge -c_1' + \frac{\alpha}{r+2K+\nu} W_t$; for $W_t \le 0$ we have $V_t^1 - V_t^2 \le c_1' + \frac{\alpha}{r+2K+\nu} W_t$.

Proof: *p. 79.*

We discretize time into periods of length $\Delta \approx 0$: time t equals $0, \Delta, 2\Delta$, and so on. An agent who chooses the switching rate k in a given period switches modes with probability $k\Delta$. W takes values that are always an integer multiple of $\sigma\sqrt{\Delta}$; in each period, it jumps up or down by this amount. The probability of jumping up is $p_\nu(W) = \frac{1}{2} - \frac{\nu\sqrt{\Delta}}{2\sigma}(W - W_{\text{mean}})$. This guarantees that the expected change in W over one period has mean $-\nu(W - W_{\text{mean}})\Delta$ and variance $\sigma^2\Delta$. Equation (22) thus holds in the limit as $\Delta \to 0$.

First we find the equilibrium with the lowest rate of increase of X in each state (W, X). Then we find the equilibrium with the highest rate. In the continuous-time model with $\nu = 0$, these equilibria are identical (Theorem 1). In discrete time, the equilibria may differ in principle; in practice, we find that they usually agree or are very close.

We find the lowest equilibrium in the following way; the highest equilibrium is found analogously. We first compute very low and high values $\underline{w}$, $\overline{w}$, such that players have strictly dominant switching rates for all $W \le \underline{w}$ and for all $W \ge \overline{w}$.[27] We now need to determine how players behave for W between $\underline{w}$ and $\overline{w}$. Clearly, we cannot determine how they behave for the entire continuum of possible values

[27] This is made easy if we assume that the relative payoff flow in mode 1, $D(W_t, X_t, k^1, k^2)$, is increasing in W_t at some minimum rate $\underline{\alpha} > 0$. Using part 2 of Lemma 1, one can compute $\underline{w}$ and $\overline{w}$ using the Switching Rate Rule. Without this extra assumption, one must guess at values of $\underline{w}$ and $\overline{w}$, compute the equilibrium as described below, and then check that at $\underline{w}$, mode 1 (2) agents do pick their highest (lowest) switching rates, and that the reverse holds at $\overline{w}$.

of X between zero and one. Instead, we compute their behavior at values of X that are integer multiples of $1/N$ for some large integer N. We thus have a finite grid of states (w, x). At each such state, we need to compute players' switching rates in modes 1 and 2. We do this as follows.

Let

$$\Delta < \min\left\{\frac{1}{r+2K}, \frac{1}{NK}\right\} \tag{23}$$

where K is the maximum switching rate that any player can ever choose. For any (w, x) in the grid, let $k^1_{-1}(w, x) = k^1_0(w, x) = \overline{K}^1$ and $k^2_{-1}(w, x) = k^2_0(w, x) = \underline{K}^2$. Let $\nabla_{-1}(w, x) = \nabla_0(w, x)$ be the lower bound, given in Lemma 1, on the relative value of being in mode 1 at state (w, x) in any equilibrium. This bound is weakly increasing in w and x. For any $n > 0$, let $k^1_n(w, x)$ and $k^2_n(w, x)$ be the optimal switching rates under the belief that all other agents will switch at rates given by k^1_{n-1} and k^2_{n-1} and that continuation payoffs in the next period will be given by ∇_{n-1}. Let $\nabla_n(w, x)$ be the relative value of being in mode 1 at state (w, x) under this belief. More precisely, optimal switching rates are given by

$$\begin{aligned} k^1_n(W_t, X_t) &= \text{argmax}_k \left[-(k\Delta)E^{n-1}_t \nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) - c^1(k, X_t)\Delta\right] \\ k^2_n(W_t, X_t) &= \text{argmax}_k \left[(k\Delta)E^{n-1}_t \nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) - c^2(k, X_t)\Delta\right] \end{aligned}$$

and the relative value of being in mode 1 is given by

$$\nabla_n(W_t, X_t) = D(W_t, X_t, k^1_n, k^2_n)\Delta + (1 - r\Delta - k^1_n\Delta - k^2_n\Delta)E^{n-1}_t \nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta})$$

(ignoring terms of second order in Δ) where $E^{n-1}_t \nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta})$ denotes the expected relative value of being in mode 1 in the next period if other players switch at the rates $k^1 = k^1_{n-1}(W_t, X_t)$ and $k^2 = k^2_{n-1}(W_t, X_t)$ in the current period:

$$\begin{aligned} &E^{n-1}_t \nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) \\ &= p_\nu(W)\nabla_{n-1}(W_t + \sigma\sqrt{\Delta}, X_{t+\Delta}) + [1 - p_\nu(W)]\,\nabla_{n-1}(W_t - \sigma\sqrt{\Delta}, X_{t+\Delta}) \end{aligned}$$

As implied by this expression, the change in X is deterministic. Since there is a continuum of agents, the proportion of mode 1 agents who switch to mode 2 is $k^1_{n-1}\Delta$ and the proportion of mode 2 agents who switch to mode 1 is $k^2_{n-1}\Delta$. Thus, $X_{t+\Delta} - X_t = -X_t k^1_{n-1}\Delta + (1 - X_t)k^2_{n-1}\Delta$. Even if X_t is an integer multiple of $1/N$, $X_{t+\Delta}$ may not be. Thus, we use linear interpolation to approximate the relative value of being in mode 1 at $X_{t+\Delta}$. Since $\Delta < 1/KN$, X never leaves the unit interval: $|X_{t+\Delta} - X_t| < 1/N$.

For any $n \geq 0$, the following properties hold by induction:

1. At any state (w, x), the one-period increase in X that would result if all agents were to use the switching rates $k_n^1(w, x)$ and $k_n^2(w, x)$ constitutes a lower bound on how much X can ever rise in any equilibrium.[28]

2. ∇_n gives a lower bound on the relative value of being in mode 1 at each state in any equilibrium, and is weakly increasing in w and x;

3. k_n^1 gives an upper bound on the switching rate chosen by mode-1 players at each state in any equilibrium;

4. k_n^2 gives a lower bound on the switching rate chosen by mode-2 players at each state in any equilibrium;

5. ∇_n is weakly greater than ∇_{n-1} at each state;[29]

6. k_n^1 is weakly less than k_{n-1}^1 at each state;

[28] Since there are $1 - X$ mode-1 agents, who would switch with probability $k_n^1\Delta$, and X mode-2 agents, who would switch with probability $k_n^2\Delta$, the one-period increase in X would be $k_n^1\Delta(1 - X) - k_n^2\Delta X$.

[29] At the state (W_t, X_t),

$$\begin{aligned}
\nabla_n - \nabla_{n-1} &= \left[-c^1(k_n^1, X_t) + c^2(k_n^2, X_t)\right]\Delta + (1 - r\Delta - k_n^1\Delta - k_n^2\Delta)E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) \\
&\quad - \left[-c^1(k_{n-1}^1, X_t) + c^2(k_{n-1}^2, X_t)\right]\Delta \\
&\quad + (1 - r\Delta - k_{n-1}^1\Delta - k_{n-1}^2\Delta)E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta}) \\
&= A + B + C + D + E \\
\text{where } A &= -c^1(k_n^1, X_t)\Delta - k_n^1\Delta E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) \\
&\quad - \left[-c^1(k_{n-1}^1, X_t)\Delta - k_{n-1}^1\Delta E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta})\right] \\
B &= -k_{n-1}^1\Delta\left[E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) - E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta})\right] \\
C &= c^2(k_n^2, X_t)\Delta - k_n^2\Delta E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) \\
&\quad - \left[c^2(k_n^2, X_t)\Delta - k_n^2\Delta E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta})\right] \\
D &= c^2(k_n^2, X_t)\Delta - k_n^2\Delta E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta}) \\
&\quad - \left[c^2(k_{n-1}^2, X_t)\Delta - k_{n-1}^2\Delta E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta})\right] \\
E &= (1 - r\Delta)\left[E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) - E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta})\right]
\end{aligned}$$

Note that $A \geq 0$ since k_n^1 solves $\max_k\left[-c^1(k, X_t)\Delta - k\Delta E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta})\right]$; $D \geq 0$ since k_{n-1}^2 solves $\min_k\left[c^2(k, X_t)\Delta - k\Delta E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta})\right]$; and

$$B+C+E = \left[1 - r\Delta - k_{n-1}^1\Delta - k_n^2\Delta\right]\left[E_t^{n-1}\nabla_{n-1}(W_{t+\Delta}, X_{t+\Delta}) - E_t^{n-2}\nabla_{n-2}(W_{t+\Delta}, X_{t+\Delta})\right]$$

which by induction is nonnegative since $\Delta < 1/(r + 2K)$.

7. k_n^2 is weakly greater than k_{n-1}^2 at each state.

In addition, the switching rates are bounded by assumption, and it can be shown easily that ∇_n is bounded as well. Hence, these three sequences (∇_n, k_n^1, and k_n^2) are all monotone and bounded as n grows. They therefore must converge as n goes to infinity. Let them converge to ∇_∞, k_∞^1, and k_∞^2, respectively. By the above properties we have:

1. ∇_∞ gives a lower bound on the relative value of being in mode 1 at each state in any equilibrium;
2. k_∞^1 gives an upper bound on the switching rate chosen by mode-1 players at each state in any equilibrium;
3. k_∞^2 gives a lower bound on the switching rate chosen by mode-2 players at each state in any equilibrium.

Properties 2 and 3 imply that $k_\infty^2(1-X) - k_\infty^1 X$ is a lower bound on the rate of change of X in any equilibrium, as claimed. Analogously, we can obtain an upper bound on the rate of change of X. If these bounds coincide, they must be the unique equilibrium with shocks.

C Proofs

First, some preliminaries. Since $D(w, x, k^1, k^2)$ is strictly increasing in w, there must be a constant $\underline{\alpha} > 0$ such that if $w > w'$,

$$D(w, x, k^1, k^2) - D(w', x, k^1, k^2) \in [\underline{\alpha}(w - w'), \overline{\alpha}\,(w - w')] \tag{24}$$

for all $w, w' \in [\underline{w}, \overline{w}]$, $x \in [0,1]$, $k^1 \in [\underline{K}^1, \overline{K}^1]$, and $k^2 \in [\underline{K}^2, \overline{K}^2]$ (all compact sets).

In the remainder of the proof, we normalize the cost of choosing the lowest possible switching rate to zero, by letting $\widehat{u}(m, w, x) = u(m, w, x) - c^m(\underline{K}^m, x)$ and $\widehat{c}^m(k, x) = c^m(k, x) - c^m(\underline{K}^m, x)$. We then relabel $\widehat{u}$ and $\widehat{c}$ to u and c^m, respectively.

Lemma 2 shows that W can be written in terms of a Brownian motion by simultaneously transforming space and time. This is the key result that lets us prove uniqueness with seasonal and mean-reverting shocks.

LEMMA 2 *Consider the diffusion given by $dW_t = (\nu_t W_t + \mu_t)dt + \sigma_t dB_t$ where B is a Brownian motion with zero drift and unit variance. For the following functions g and h, the process $g(t, B_{h(t)})$ has the same distribution as the process W:*

$$g(t,z) = \exp\left(\int_{s=0}^{t} \nu_s ds\right) z + \int_{s=0}^{t} \mu_s \exp\left(\int_{v=s}^{t} \nu_v dv\right) ds \tag{25}$$

$$h(t) = \int_{s=0}^{t} \exp\left(-2\int_{v=0}^{s} \nu_v dv\right) \sigma_s^2 ds$$

where $B_0 = W_0$, h is strictly increasing and $h(0) = 0$. As of time $t' \leq t$, W_t is normal with mean $\exp\left(\int_{s=t'}^{t} \nu_s ds\right) W_{t'} + \int_{s=t'}^{t} \mu_s \exp\left(\int_{v=s}^{t} \nu_v dv\right) ds$ and variance $\int_{s=t'}^{t} \exp\left(2\int_{v=s}^{t} \nu_v dv\right) \sigma_s^2 ds$. If, in addition, that there are constants $0 < N_1 < N_2 < \infty$ such that, for all t, $|\nu_t|, |\mu_t| < N_2$, $\int_{s=0}^{\infty} |\nu_s| ds < N_2$, $\sigma_t \in [N_1, N_2]$, and $\dot{\sigma}_t \leq N_2$, then:

1. *There are positive constants $\overline{\gamma}$ and $\underline{\gamma}$ such that for all t, t', and $z > z'$, $g(t,z) - g(t,z') \in \left[\underline{\gamma}(z-z'), \overline{\gamma}(z-z')\right]$ and for sufficiently small $|t-t'|$, $|g(t,z) - g(t',z)| \leq \overline{\gamma}(g(t,z)+1)|t-t'|$.*

2. *There are constants $\overline{\rho} \geq \underline{\rho} > 0$ such that for all $t > t'$, $h(t) - h(t') \in \left[\underline{\rho}(t-t'), \overline{\rho}(t-t')\right]$ and $|h'(t) - h'(t')| \leq \overline{\rho}|t-t'|$. ($h'$ is the derivative of h.)*

Proof of LEMMA 2. We first verify that $g(t, B_{h(t)})$ has the same infinitesimal drift and variance as W_t. Since both processes have continuous paths a.s., this will imply that they are identically distributed. By definition,

$$g(t, B_{h(t)}) = \exp\left(\int_{s=0}^{t} \nu_s ds\right) B_{h(t)} + \int_{s=0}^{t} \mu_s \exp\left(\int_{v=s}^{t} \nu_v dv\right) ds$$

$$\Longrightarrow d\left[g(t, B_{h(t)})\right] = \exp\left(\int_{s=0}^{t} \nu_s ds\right) dB_{h(t)} + \left[\nu_t g(t, B_{h(t)}) + \mu_t\right] dt$$

so that $Ed\left[g(t, B_{h(t)})\right] = \left(\nu_t g(t, B_{h(t)}) + \mu_t\right) dt$ and

$$\begin{aligned} E\left[dg(t, B_{h(t)})^2\right] &= \exp\left(2\int_{s=0}^{t} \nu_s ds\right) E\left[\left(dB_{h(t)}\right)^2\right] \\ &= \exp\left(2\int_{s=0}^{t} \nu_s ds\right) \left[h(t+dt) - h(t)\right] \\ &= \exp\left(2\int_{s=0}^{t} \nu_s ds\right) \exp\left(-2\int_{v=0}^{t} \nu_v dv\right) \sigma_t^2 dt = \sigma_t^2 dt \end{aligned}$$

proving that the two processes have the same distributions.

Since W is a Markov process,

$$W_t \underset{L}{=} \exp\left(\int_{s=t'}^{t} \nu_s ds\right) \widehat{B}_{\widehat{h}(t)} + \int_{s=t'}^{t} \mu_s \exp\left(\int_{v=s}^{t} \nu_v dv\right) ds$$

where $\widehat{h}(t) = \int_{s=t'}^{t} \exp\left(-2\int_{v=t'}^{s} \nu_v dv\right) \sigma_s^2 ds$, "$\underset{L}{=}$" denotes equality in law (distribution), and $\widehat{B}$ is another Brownian motion with zero drift and unit variance, satisfying $\widehat{B}_0 = W_{t'}$. The only stochastic term is $\widehat{B}_{\widehat{h}(t)}$, which is normal since $\widehat{B}$ is a Brownian motion. Hence, W_t is normal with mean $E_{t'}W_t = \exp\left(\int_{s=t'}^{t} \nu_s ds\right) W_{t'} + \int_{s=t'}^{t} \mu_s \exp\left(\int_{v=s}^{t} \nu_v dv\right) ds$
and variance

$$\begin{aligned} Var_{t'}W_t &= \exp\left(2\int_{s=t'}^{t} \nu_s ds\right) Var\left(\widehat{B}_{\widehat{h}(t)}\right) \\ &= \exp\left(2\int_{s=t'}^{t} \nu_s ds\right) \widehat{h}(t) = \int_{s=t'}^{t} \exp\left(2\int_{v=s}^{t} \nu_v dv\right) \sigma_s^2 ds \end{aligned}$$

For property 1, note that

$$|g(t,z) - g(t,z')| = \exp\left(\int_{s=0}^{t} \nu_s ds\right) |z - z'| \in \left[e^{-N_2}|z - z'|, e^{N_2}|z - z'|\right]$$

Moreover,[30]

$$\begin{aligned} |g(t',z) - g(t,z)| &= \left| g(t,z)\left(\exp\left(\int_{s=t}^{t'} \nu_s ds\right) - 1\right) + \int_{s=t}^{t'} \mu_s \exp\left(\int_{v=s}^{t'} \nu_v dv\right) ds \right| \\ &\le \left| g(t,z)\left(e^{N_2(t'-t)} - 1\right) + N_2 e^{N_2(t'-t)}(t'-t) \right| \\ &\le |g(t,z)|\, 2N_2(t'-t) + 2N_2^2(t'-t) \le 2N_2^2(|g(t,z)|+1)(t'-t) \end{aligned}$$

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[30]This is because

$$|g(t',z) - g(t,z)| = \left| \begin{array}{c} \exp\left(\int_{s=0}^{t} \nu_s ds\right)\left(\exp\left(\int_{s=t}^{t'} \nu_s ds\right) - 1\right) z \\ + \int_{s=0}^{t} \mu_s \exp\left(\int_{v=s}^{t} \nu_v dv\right)\left(\exp\left(\int_{v=t}^{t'} \nu_v dv\right) - 1\right) ds \\ + \int_{s=t}^{t'} \mu_s \exp\left(\int_{v=s}^{t'} \nu_v dv\right) ds \end{array} \right|$$

for small enough $|t - t'|$, as claimed. For property 2,

$$h(t) - h(t') = \int_{s=t'}^{t} \exp\left(-2\int_{v=0}^{s} \nu_v dv\right) \sigma_s^2 ds$$
$$\in \left[N_1^2 e^{-2N_2}(t - t'), N_2^2 e^{2N_2}(t - t')\right]$$

and

$$|h'(t) - h'(t')| = \left|\exp\left(-2\int_{v=0}^{t} \nu_v dv\right) \sigma_t^2 - \exp\left(-2\int_{v=0}^{t'} \nu_v dv\right) \sigma_{t'}^2\right|$$
$$\leq e^{2N_2}\left|\exp\left(-2\int_{v=t'}^{t} \nu_v dv\right) - 1\right| \sigma_t^2$$
$$+ \left|\exp\left(-2\int_{v=0}^{t} \nu_v dv\right)\left(\sigma_t^2 - \sigma_{t'}^2\right)\right|$$
$$\leq e^{2N_2} N_2^2 (e^{2N_2(t-t')} - 1) + e^{2N_2} N_2 (t - t')$$
$$\leq e^{2N_2} N_2^2 (3N_2(t - t')) + e^{2N_2} N_2 (t - t')$$

for sufficiently small $t - t'$. By the triangle inequality, this generalizes to any $t - t'$, so $|h'(t) - h'(t')| \leq \overline{\rho}\,|t - t'|$ for any $\overline{\rho} \geq e^{2N_2} N_2(3N_2^2 + 1)$. This proves property 2. Q.E.D.$_{\text{Lemma 2}}$

Proof of THEOREMS 1-5. Part 1 of the following Lemma proves that the relative value of being in mode 1 equals the expression given in Theorem 5. Part 2 proves some useful bounds on this difference. Part 3 establishes Theorem 5.

LEMMA 3 *Let $V_t^m = V^m(W, X, t)$ be the continuation payoff of a player who is in mode $m \in \{1, 2\}$ at state (W, X) at time t. Let k_v^m be the player's optimal switching rate conditional on being in m at time $v \geq t$. Then*

1.

$$V_t^1 - V_t^2 = E\int_{v=t}^{\infty} \exp\left(-\int_{s=t}^{v} (r + k_s^1 + k_s^2) ds\right) D(W_v, X_v, k_v^1, k_v^2) dv \tag{26}$$

2. For all states (W_t, X_t) and for any beliefs over the path $(X_v)_{v\geq t}$ that will result from any path $(W_v)_{v\geq t}$,

$$V_v^m - V_v^{m'} \leq E\int_{s=v}^{\infty} e^{-r(s-v)} \left(|u(m, W_s, X_s) - u(m', W_s, X_s)| + C\right) ds$$

for any $m, m' \in \{1, 2\}$. Moreover, there are positive constants c_0 and c_1 such that $|V_t^1 - V_t^2| \leq c_0 |W_t| + c_1$.

3. *For* $m \in \{1,2\}$, $k_v^m \in \operatorname{argmax}_{k\geq 0}\left(k(V_v^{m'}-V_v^m)-c^m(k,X_v)\right)$, *where* $m' = 1$ *if* $m=2$ *and vice-versa.*

Proof of LEMMA 3. For $m,m' \in \{1,2\}$, $m \neq m'$, the Bellman equation for V_v^m is

$$V_v^m \approx \left[\begin{array}{c} [u(m,W_v,X_v)-c^m(k_v^m,X_v)]\,dv \\ +k_v^m\cdot dv\cdot EV_{v+dv}^{m'} \\ +[1-k_v^m dv - r dv]\,EV_{v+dv}^m \end{array}\right] \tag{27}$$

This becomes exact as $dv \to 0$, proving part 3. Rearranging (27), we obtain

$$EdV_v^m = \left[-u(m,W_v,X_v)+c^m(k_v^m,X_v)-k_v^m V_v^{m'}+(k_v^m+r)V_v^m\right]dv$$

where $dV_v^m = V_{v+dv}^m - V_v^m$. Therefore,

$$E\left(dV_v^1-dV_v^2\right)=\left[\begin{array}{c} -D(W_v,X_v,k_v^1,k_v^2) \\ +(k_v^1+k_v^2+r)\left(V_v^1-V_v^2\right)\end{array}\right]dv \tag{28}$$

This expectation is as of time v. Now multiply both sides by $\exp\left[-\int_{s=t}^{v}(r+k_s^1+k_s^2)ds\right]$, integrate, and take the expectation as of time t, yielding (by iterated expectations)

$$\begin{aligned} &E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^1+k_s^2)ds\right)\left[dV_v^1-dV_v^2\right] \\ &=E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^1+k_s^2)ds\right)\left[\begin{array}{c} -D(W_v,X_v,k_v^1,k_v^2) \\ +(k_v^1+k_v^2+r)\left(V_v^1-V_v^2\right)\end{array}\right]dv \end{aligned}$$

Integrating by parts,

$$\begin{aligned} &E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^1+k_s^2)ds\right)\left[dV_v^1-dV_v^2\right] \\ &=E\left(\exp\left(-\int_{s=t}^{v}(r+k_s^1+k_s^2)ds\right)\left(V_v^1-V_v^2\right)\right)_{v=t}^{\infty} \\ &+E\int_{v=t}^{\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^1+k_s^2)ds\right)(k_v^1+k_v^2+r)\left(V_v^1-V_v^2\right)dv \end{aligned}$$

But

$$\left|E\left(\lim_{v\to\infty}\exp\left(-\int_{s=t}^{v}(r+k_s^1+k_s^2)ds\right)\left(V_v^1-V_v^2\right)\right)\right|\leq \lim_{v\to\infty}e^{-r(v-t)}E\left|V_v^1-V_v^2\right|.$$

We will now show that

$$\lim_{v\to\infty}e^{-r(v-t)}E\left|V_v^1-V_v^2\right|=0 \tag{29}$$

This will establish part 1.

For $m = 1, 2$, V_v^m is no greater than the continuation payoff from always being in the "right" action and paying the lowest possible switching cost of zero:

$$V_v^m \leq E \int_{s=v}^{\infty} e^{-r(s-v)} \max \{u(1, W_s, X_s), u(2, W_s, X_s)\} \, ds$$

If one chooses the lowest switching cost, the worst that can happen is that one is always in the wrong action; hence,

$$V_v^m \geq E \int_{s=v}^{\infty} e^{-r(s-v)} \min \{u(1, W_s, X_s), u(2, W_s, X_s)\} \, ds.$$

Thus,

$$\begin{aligned} \left|V_v^1 - V_v^2\right| &\leq E \int_{s=v}^{\infty} e^{-r(s-v)} \left|\Delta u(W_s, X_s)\right| ds \\ &\leq \frac{\max_{x \in [0,1]} \left|\Delta u(0, x)\right|}{r} \\ &\quad + E \int_{s=v}^{\infty} e^{-r(s-v)} \left|\Delta u(W_s, X_s) - \Delta u(0, X_s)\right| ds \end{aligned}$$

where $\Delta u(W_s, X_s) = u(1, W_s, X_s) - u(2, W_s, X_s)$. (This proves the first formula in part 2.) Since $c^m(\underline{K}^m, x) = 0$ for $m = 1, 2$ and for all x, $\Delta u(w, x) = D(w, x, \underline{K}^1, \underline{K}^2)$ is Lipschitz in w with constant $\overline{\alpha}$, so $\left|\Delta u(W_s, X_s) - \Delta u(0, X_s)\right| \leq \overline{\alpha} \left|W_s\right|$. But

$$\begin{aligned} E\left|W_s\right| = E\left[\sqrt{W_s^2}\right] &\leq \sqrt{E\left[W_s^2\right]} = \sqrt{\left[EW_s\right]^2 + \text{Var}\left(W_s\right)} \\ &\leq \sqrt{\left[EW_s\right]^2} + \sqrt{\text{Var}\left(W_s\right)} = \left|EW_s\right| + \sqrt{\text{Var}\left(W_s\right)} \end{aligned}$$

where all expectations are conditioned on W_v. Thus, by Lemma 2,

$$\begin{aligned} E\left|W_s\right| &\leq \exp\left(\int_{s'=v}^{s} \nu_{s'} ds'\right) \left|W_v\right| + \left|\int_{s'=v}^{s} \mu_{s'} \exp\left(\int_{v'=s'}^{s} \nu_{v'} dv'\right) ds'\right| \\ &\quad + \sqrt{\int_{s'=v}^{s} \exp\left(2 \int_{v'=s'}^{s} \nu_{v'} dv'\right) \sigma_{s'}^2 ds'} \\ &\leq e^{N_2} \left|W_v\right| + (s-v) N_2 e^{N_2} + e^{N_2} N_2 \sqrt{s-v} \end{aligned} \tag{30}$$

Hence, there are positive constants c_0, c_1, and c_2 such that

$$\begin{aligned} \left|V_v^1 - V_v^2\right| &\leq c_2 + \int_{s=v}^{\infty} e^{-r(s-v)} \overline{\alpha} \left(e^{N_2} \left|W_v\right| + (s-v) N_2 e^{N_2} + e^{N_2} N_2 \sqrt{s-v}\right) ds \\ &= c_2 + \frac{\overline{\alpha} e^{N_2} \left|W_v\right|}{r} + \frac{\overline{\alpha} N_2 e^{N_2}}{r^2} + \frac{\overline{\alpha} N_2 e^{N_2} \sqrt{\pi}}{2 r^{3/2}} = c_0 \left|W_v\right| + c_1 \end{aligned} \tag{31}$$

(establishing the second bound in part 2) so

$$\lim_{v\to\infty} e^{-r(v-t)} E\left|V_v^1 - V_v^2\right| \le c_4 \lim_{v\to\infty} e^{-r(v-t)} |W_v|$$

but by Chebyschev's inequality, for any $c_5 > 0$,

$$\Pr(e^{-r(v-t)} |W_v| > c_5) \le \frac{E|W_v|}{c_5 e^{r(v-t)}} \le \frac{1}{c_5} e^{N_2 - r(v-t)} \left(|W_t| + (v-t)N_2 + \sqrt{v-t}N_2\right)$$

which goes to 0 as $v \to \infty$, establishing (29). Q.E.D.$_{\text{Lemma 3}}$

By part 3 of Lemma 3, $BR^1(y,x) = \operatorname{argmax}_{k\in\left[\underline{K}^1,\overline{K}^1\right]}[-ky - c^1(k,x)]$ (resp., $BR^2(y,x) = \operatorname{argmax}_{k\in\left[\underline{K}^2,\overline{K}^2\right]}[ky - c^2(k,x)]$) is the set of optimal switching rates for a mode-1 (resp., mode-2) player when the relative value of being in mode 1 is y. Lemma 4 shows that these best response correspondences have a closed graph (part 1), satisfy the single crossing property (part 2), and have Lipschitz isoquants (part 3):

LEMMA 4 1. *(Closed Graph.)* $BR^2(y,x)$ *and* $BR^1(y,x)$ *are upper hemicontinuous in* y.

2. *(Single Crossing.) Suppose* $y < y'$. *If* $k \in BR^2(y,x)$, *and* $k' \in BR^2(y',x)$, *then* $k \le k'$. *If* $k \in BR^1(y,x)$, *and* $k' \in BR^1(y',x)$, *then* $k \ge k'$.

3. *(Lipschitz Isoquants.) Suppose* $y - y' > \eta|x - x'|$. *(η is defined in assumption A6.) Then* $\min BR^2(y,x) \ge \max BR^2(y',x')$ *and* $\max BR^1(y,x) \le \min BR^1(y',x')$.

Proof of LEMMA 4.

1. Fix x. Let $c(k)$ be shorthand for $c^1(k,x)$ or $c^2(k,x)$. We will show that if the function c is left-continuous, then $\zeta(y) = \operatorname{argmax}_{k\ge 0}(ky - c(k))$ is upper hemicontinuous. A similar argument holds for the function $\operatorname{argmax}_{k\ge 0}(-ky - c(k))$. Suppose there is a sequence $(y^n, k^n)_{n=1}^{\infty}$ such that $k^\infty = \lim_{n\to\infty} k^n$ and $y^\infty = \lim_{n\to\infty} y^n$ both exist and $k^n \in \zeta(y^n)$ for all n. Upper hemicontinuity means that $k^\infty \in \zeta(y^\infty)$ for all such sequences. We first show that $\lim_{n\to\infty} c(k^n) = c(k^\infty)$. This is trivial if c is continuous at k^∞. If not, we claim that there is an $I < \infty$ such that if $n > I$, then $k^n \le k^\infty$. By assumption, c is left continuous, so it must not be right continuous at k^∞. So let $\lim_{k\downarrow k^\infty} c(k) = c(k^\infty) + \varepsilon$ where $\varepsilon > 0$. For any $k^n > k^\infty$, since c is weakly increasing, $k^n y^n - c(k^n) \le k^n y^n - c(k^\infty) - \varepsilon$. Let I be large enough that $n > I$ implies $|k^n y^n - k^\infty y^n| < \varepsilon/2$. Then $k^n y^n - c(k^n) \le k^\infty y^n - c(k^\infty) - \varepsilon/2$, so $k^n \notin \zeta(y^n)$ after all. Therefore,

if $n > I$, then $k^n \le k^\infty$. Since c is left continuous, $c(k^\infty) = \lim_{n\to\infty} c(k^n)$, so $\lim_{n\to\infty} [k^n y^n - c(k^n)] = k^\infty y^\infty - c(k^\infty)$.

Now suppose $k^\infty \notin \zeta(y^\infty)$. Then there is a k' and an $\varepsilon' > 0$ such that $k'y^\infty - c(k') > k^\infty y^\infty - c(k^\infty) + \varepsilon'$. We claim this implies $k^n \notin \zeta(y^n)$ for large enough n. Since $\lim_{n\to\infty} [k^n y^n - c(k^n)] = k^\infty y^\infty - c(k^\infty)$, for any $\varepsilon'' > 0$ there is an I' such that if $n > I'$, $|k^n y^n - c(k^n) - [k^\infty y^\infty - c(k^\infty)]| < \varepsilon''$. So $k'y^\infty - c(k') > k^n y^n - c(k^n) + \varepsilon' - \varepsilon''$ for all $n > I'$. But there is also an I'' such that if $n > I''$, $|k'y^\infty - k'y^n| < \varepsilon''$ (as k' is bounded by K). So $k'y^n - c(k') > k^n y^n - c(k^n) + \varepsilon' - 2\varepsilon''$. So setting $\varepsilon'' = \varepsilon'/3$, we know that if $n > \max\{I', I''\}$, then $k'y^n - c(k') > k^n y^n - c(k^n)$, so $k^n \notin \zeta(y^n)$ - a contradiction.

2. Suppose $y < y'$, $k \in BR^2(y, x)$, and $k' \in BR^2(y', x)$. Then $k'y' - c^2(k', x) \ge ky' - c^2(k, x)$ while $ky - c^2(k, x) \ge k'y - c^2(k', x)$. Subtracting, we obtain $(k' - k)(y' - y) \ge 0$, so $k' \ge k$.[31] The proof for BR^1 is analogous.

3. We will show this for BR^2; the proof for BR^1 is analogous. Let $k' = \max BR^2(y', x') > \underline{K}^2$. We will show that k' is strictly better than any lower switching rate at (y, x) by showing that $k'y - c^2(k', x) > ky - c^2(k, x)$ for all $k \in [\underline{K}^2, k')$; equivalently, we will show that $\varepsilon y > c^2(k', x) - c^2(k' - \varepsilon, x)$ for all $\varepsilon \in (0, k' - \underline{K}^2]$. By definition of k', $k'y' - c^2(k', x') \ge ky' - c^2(k, x')$ for all $k \in [\underline{K}^2, k')$. Letting $\varepsilon = k' - k$, $\varepsilon y' \ge c^2(k', x') - c^2(k' - \varepsilon, x')$ for all $\varepsilon \in (0, k' - \underline{K}^2]$ and so

$$\begin{aligned}
\varepsilon y &= \varepsilon y' + \varepsilon(y - y') \\
&\ge c^2(k', x') - c^2(k' - \varepsilon, x') + \varepsilon(y - y') > c^2(k', x') - c^2(k' - \varepsilon, x') + \varepsilon\eta\,|x - x'| \\
&= \left[c^2(k', x) - c^2(k' - \varepsilon, x)\right] + c^2(k', x') - c^2(k', x) \\
&- \left[c^2(k' - \varepsilon, x') - c^2(k' - \varepsilon, x)\right] + \varepsilon\eta\,|x - x'| \\
&\ge c^2(k', x) - c^2(k' - \varepsilon, x)
\end{aligned}$$

for all $\varepsilon \in (0, k' - \underline{K}^2]$ by assumption **A6**. Q.E.D.$_{\text{Lemma 4}}$

Let $Z_t = B_{h(t)}$. We now redefine the state space to be the set of triplets (t, Z_t, X_t) rather than $(W_t, X_t) = (g(t, Z_t), X_t)$. Since players know t, by (25) they can invert $g(t, Z_t)$ to discover Z_t. Let $D_v(Z_v, X_v, k_v^1, k_v^2)$ represent the relative payoff flow in mode 1 at time v:

$$D_v(Z_v, X_v, k_v^1, k_v^2) \triangleq D(g(v, Z_v), X_v, k_v^1, k_v^2) = D(W_v, X_v, k_v^1, k_v^2)$$

The iterative procedure begins by computing, at each state (t, Z_t, X_t), an upper bound $\Phi^0 = \Phi^0(t, Z_t, X_t)$ on $V_t^1 - V_t^2$, the relative value of being in mode 1.

[31] This relies on the fact that if $z' = z$, then $k' = k$.

We compute this bound using a belief that maximizes the relative value of being in mode 1: that all players will immediately switch to mode 1 and remain there forever.[32] By Lemma 3,

$$\Phi^0(t, Z_t, X_t) = E\int_{v=t}^{\infty} \exp\left(-\int_{s=t}^{v}(r + k_s^1 + k_s^2)ds\right) D_v(Z_v, 1, k_v^1, k_v^2)dv$$

where k_v^2, k_v^1, k_s^2, and k_s^1 are optimal given these beliefs.

Since there are $1 - X_t$ mode 2 players, who switch to mode 1 at a rate no greater than the maximum of $BR^2(\Phi^0(t, Z_t, X_t))$, and X_t mode 1 players, who switch *out of* mode 1 at a rate no less than the minimum of $BR^1(\Phi^0(t, Z_t, X_t))$,[33]

$$\begin{aligned}\dot{X}_t &\leq \max BR^2(\Phi^0(t, Z_t, X_t), X_t) \cdot (1 - X_t) - \min BR^1(\Phi^0(t, Z_t, X_t), X_t) \cdot X_t \\ &\triangleq \pi(\Phi^0(t, Z_t, X_t), X_t) \qquad (32)\end{aligned}$$

where we define $\pi(y, x)$ to be $\max BR^2(y, x) \cdot (1 - x) - \min BR^1(y, x) \cdot x$, the highest rate of change of X that is consistent with rational behavior when $X_t = x$ and the relative value of being in mode 1 is y.

Equation (32) implies, for any state (t, Z_t, X_t), a new upper bound $\Phi^1(t, Z_t, X_t)$ on the relative value of being in mode 1. Φ^1 is computed using the belief that most favors mode 1: that for all $v \geq t$, $\dot{X}_v$ will *equal* its the old upper bound, $\pi(\Phi^0(v, Z_v, X_v), X_v)$. For all $n \geq 1$, let $\Phi^n(t, Z_t, X_t)$ be the relative value of being in mode 1 on the belief that, at all times $v \geq t$,

$$\dot{X}_v = \pi(\Phi^{n-1}(v, Z_v, X_v), X_v) \qquad (33)$$

Let $\Phi^\infty(t, z, x) = \lim_{n\to\infty} \Phi^n(t, z, x)$.

A central fact used in our proof is that the dynamical system (33) has a unique solution for any n, including $n = \infty$. We prove this in a sequence of lemmas. For any t, t', and $v \geq t$, define $\phi(v, t, t')$ implicitly by

$$h(t' + \phi(v, t, t')) - h(t') = h(t + v) - h(t) \qquad (34)$$

where h is defined in Lemma 2. Let

$$\tau(t, t') = \max_{v\geq 0} |t' + \phi(v, t, t') - t - v| \qquad (35)$$

Lemma 5 proves four important properties of these functions.

[32] The model restricts players to arrival rates below K. The belief that players will all immediately jump to mode 1 thus gives an (unattainable) upper bound on the relative value of being in mode 1.

[33] The $\min$ and $\max$ of the respective sets exist since the cost functions are left continuous.

LEMMA 5 *For any t and t', let $dt = t' - t$. For all v:*

1. $\tau(t,t') \in \left[|dt|, \frac{\overline{\rho}}{\underline{\rho}}|dt|\right]$.
2. $|\phi(v,t,t') - v| \leq 2\frac{\overline{\rho}}{\underline{\rho}}|dt|$ *and* $|dt + \phi(v,t,t') - v| \leq \frac{\overline{\rho}}{\underline{\rho}}|dt|$.
3. $|\phi_1(v,t,t') - 1| \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 |dt|$.
4. $\tau(t+v, t'+\phi(v,t,t')) \leq \tau(t,t')$.

Proof of LEMMA 5. By (34), $\phi(0,t,t') = 0$, so $\tau(t,t') \geq |dt|$. By equation (34),

$$|h(t+v+dt+[\phi(v,t,t')-v]) - h(t+v)| = |h(t+dt) - h(t)| \tag{36}$$

The left hand side of (36) is at least $\underline{\rho}\,|dt + \phi(v,t,t') - v|$ while the right hand side is no greater than $\overline{\rho}\,|dt|$ by assumption A3. So $|dt + \phi(v,t,t') - v| \leq \frac{\overline{\rho}}{\underline{\rho}}|dt|$, which shows parts 1 and 2. Differentiating (34) with respect to v,

$$\begin{aligned}|\phi_1(v,t,t') - 1| &= \left|\frac{h'(t'+\phi(v,t,t')) - h'(t+v)}{h'(t'+\phi(v,t,t'))}\right| \\ &\leq \frac{\overline{\rho}}{\underline{\rho}}|dt + \phi(v,t,t') - v| \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 |dt|\end{aligned} \tag{37}$$

by part 2 of Lemma 2 and the prior computation. This shows part 3.

For part 4, let $t'' = t + v$ and $t''' = t' + \phi(v,t,t')$. Suppose that

$$s_0 = \text{argmax}_{s\geq 0}\,|t''' + \phi(s,t'',t''') - t'' - s|$$

We will show that

$$t''' + \phi(s_0,t'',t''') - t'' - s_0 = t' + \phi(s_0+v,t,t') - t - (s_0+v) \tag{38}$$

implying

$$\begin{aligned}\tau(t'',t''') &= |t''' + \phi(s_0,t'',t''') - t'' - s_0| \\ &= |t' + \phi(s_0+v,t,t') - t - (s_0+v)| \leq \tau(t,t')\end{aligned}$$

Substituting,

$$\begin{aligned}&t''' + \phi(s_0,t'',t''') - t'' - s_0 \\ &= t' + \phi(v,t,t') + \phi(s_0, t+v, t'+\phi(v,t,t')) - t - v - s_0\end{aligned}$$

This equals $t' + \phi(s_0 + v, t, t') - t - (s_0 + v)$ if

$$\phi(v, t, t') + \phi(s_0, t + v, t' + \phi(v, t, t')) = \phi(s_0 + v, t, t') \tag{39}$$

By repeatedly applying (34), we obtain

$$\begin{aligned} & h(t''' + \phi(s_0, t'', t''')) - h(t'' + s_0) = h(t''') - h(t'') \\ & = h(t' + \phi(v, t, t')) - h(t + v) = h(t') - h(t) \\ & = h(t' + \phi(s_0 + v, t, t')) - h(t + s_0 + v) \end{aligned}$$

But $t'' = t + v$. Thus, equating the first and last expressions,

$$h(t' + \phi(s_0 + v, t, t')) = h(t''' + \phi(s_0, t'', t'''))$$

Since h is strictly increasing by part 2 of Lemma 2,

$$\begin{aligned} t' + \phi(s_0 + v, t, t') &= t''' + \phi(s_0, t'', t''') \\ &= t' + \phi(v, t, t') + \phi(s_0, t + v, t' + \phi(v, t, t')) \end{aligned}$$

establishing (39). Q.E.D.$_{\text{Lemma 5}}$

For any y, let $f^2(y, x) = \max BR^2(y, x)$: the highest switching rate mode 2 players may choose if the relative value of being in mode 1 is y and $X_t = x$. Let $f^1(y, x) = -\min BR^1(y, x)$: the negative of the lowest switching rate mode-1 players may choose in the same situation. Equation (33) implies that

$$\dot{X}_v = f^2(\Phi^{n-1}(v, Z_v, X_v), X_v)(1 - X_v) + f^1(\Phi^{n-1}(v, Z_v, X_v), X_v)X_v$$

The following lemma will be used to show that this system has a unique solution. In reading it, one should interpret the function $F^m(v, z, x)$ for $m = 1, 2$ as $f^m(\Phi^{n-1}(v, z, x), x)$. Later we will show that this function indeed has the properties assumed in Lemma 6. Equation (40), which appears in the lemma, is just the integral version of (33) for these functions F^1 and F^2.

LEMMA 6 *Assume that $F^1(t, z, x)$ and $F^2(t, z, x)$ have the following properties:*

1. *They are weakly increasing in z.*
2. *There is a constant K such that $|F^m(t, z, x)| \leq K$ for $m = 1, 2$ and for all t, z, and x.*

3. *For $m = 1, 2$, there is a constant c_2 such that if*

$$z' - z > c_2(|x' - x| + \tau(t, t'))$$

and $|x' - x| \geq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t, t')$ *then* $F^m(t', z', x') \geq F^m(t, z, x)$.

Then for any $x_0 \in [0, 1]$, $\Upsilon > 0$, and almost every path $(Z_t)_{t\in[0,\Upsilon]}$ there exists a unique Lipschitz path $(X_t)_{t\in[0,\Upsilon]}$ such that

$$X_t = x_0 + \int_{s=0}^{t} \left(F^2(s, Z_s, X_s)(1 - X_s) + F^1(s, Z_s, X_s)X_s\right) ds \tag{40}$$

Proof of LEMMA 6. For any $N > 0$ let Υ_N be the first time t at which $|Z_t| > N$. We will prove that almost surely, for $t \leq 1/2$ and for any N, there exists a unique solution to the version of (40) that is killed when $|Z|$ reaches N:

$$X_t = x_0 + \int_{s=0}^{t\wedge\Upsilon_N} \Gamma(s, Z_s, X_s|F)ds \tag{41}$$

where $t \wedge \Upsilon_N = \min\{t, \Upsilon_N\}$, $F = (F^1, F^2)$, and

$$\Gamma(s, z, x|F) = F^2(s, z, x)(1 - x) + F^1(s, z, x)x \tag{42}$$

Since the same argument can be repeated for $t \in [1/2, 1]$ etc. and taking $N \to \infty$, this will prove the existence of a unique solution for all t. For brevity, we will write t in place of $t \wedge \Upsilon_N$.

We first prove existence. For any $\delta > 0$, define $X^\delta_t = x_0 + \int_{s=0}^{t} \Gamma^\delta_s ds$, where $\Gamma^\delta_s = \frac{1}{\delta}\int_{v=s-\delta}^{s} \Gamma(v, Z_v, X^\delta_v|F)dv$. (For $v \in [-\delta, 0)$, let $Z_v = Z_0$ and $X^\delta_v = x_0$.) Note that $\dot{X}^\delta_t = \frac{1}{\delta}\int_{v=t-\delta}^{t} \Gamma(v, Z_v, X^\delta_v|F)dv$; the right hand side is absolutely bounded by K, so this equation has a unique solution that is Lipschitz with constant K. Let $X_t = \limsup_{n\to\infty} Y^n_t$ where $Y^n_t = \sup_{m>n} X^{1/m}_t$. The supremum of an arbitrary family of Lipschitz functions with constant K is a Lipschitz function with the same constant, and the same is true for the limit of a sequence of such functions. Hence, for every n, the function $Y^n_t = \sup_{m>n} X^{1/m}_t$ is Lipschitz with constant K, and so is X_t. Moreover, for fixed t, there exists a subsequence $(m_j)_{j=1}^\infty$ such that $\lim_{j\to\infty} X^{1/m_j}_t = X_t$. By extracting further subsequences and then using the diagonal method we can obtain a subsequence $(m'_j)_{j=1}^\infty$ of the original sequence $(m_j)_{j=1}^\infty$ such that $\lim_{j\to\infty} X^{1/m'_j}_t = X_t$ for every rational $t \geq 0$ (and hence for every $t \geq 0$). The convergence is uniform on compact intervals because all functions X^{1/m'_j}_t are Lipschitz with constant K.

To finish the proof of existence, it remains only to show that (41) holds for $X_t = \lim_{j\to\infty} X_t^{1/m'_j}$. For any j,

$$\left| X_t - \left(x_0 + \int_{s=0}^{t} \Gamma(s, Z_s, X_s|F) ds \right) \right| \leq A_1^j + A_2^j + A_3^j \tag{43}$$

where

$$A_1^j = \left| X_t - \left(x_0 + \int_{s=0}^{t} \Gamma_s^{1/m'_j} ds \right) \right|$$
$$A_2^j = \left| \int_{s=0}^{t} \Gamma_s^{1/m'_j} ds - \int_{s=0}^{t} \Gamma(s, Z_s, X_s^{1/m'_j}|F) ds \right|$$
$$A_3^j = \int_{s=0}^{t} \left| \Gamma(s, Z_s, X_s^{1/m'_j}|F) - \Gamma(s, Z_s, X_s|F) \right| ds$$

Since $A_1^j = \left| X_t - X_t^{1/m'_j} \right|$, $\lim_{j\to\infty} A_1^j = 0$. Moreover,

$$\int_{s=0}^{t} \Gamma_s^{1/m'_j} ds = m'_j \int_{s=0}^{t} \int_{v=s-1/m'_j}^{s} \Gamma(v, Z_v, X_v^{1/m'_j}|F) dv ds$$
$$= \int_{v=0}^{t} \Gamma(v, Z_v, X_v^{1/m'_j}|F) dv + o(1/m'_j)$$

(reversing the order of integration), so that $\lim_{j\to\infty} A_2^j = 0$.

We now prove that $\lim_{j\to\infty} A_3^j = 0$. For $m = 1, 2$, all $t \in \Re$, $y \in [0, K]$ and $x \in [0, 1]$, let $H^m(t, y, x) = \inf\{z \in [-N, N] : F^m(t, z, x) > y\}$; if this set is empty, define $H^m(t, y, x) = N$. Let $c_4 = 3c_2 \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2$ and define $F^{12}(s, z, x) = F^1(s, z, x) + F^2(s, z, x)$.

CLAIM 2 *For any two states* (t, z, x) *and* (t', z', x')*, let* $|dt| = |t - t'|$, $|dz| = |z - z'|$*, and* $|dx| = |x - x'|$ *and* $\alpha = |dz| + c_4 (|dt| + |dx|)$*. For* $m = 1, 2$*:*

1. $F^m(t', z + c_4 [|dt| + |dx|], x') \geq F^m(t, z, x)$.
2. $H^m(t, y, x)$ *is Lipschitz in* t *and* x *with constant* c_4.
3.

$$|\Gamma(t, z, x|F) - \Gamma(t', z', x'|F)|$$
$$\leq \left| F^2(t, z, x) - F^2(t', z', x') \right| + \left| F^1(t, z, x) - F^1(t', z', x') \right| + 2K |dx|$$

and

$$|F^m(t, z, x) - F^m(t', z', x')| \leq F^m(t, z + \alpha, x) - F^m(t, z - \alpha, x)$$

4. *For any processes $Y \geq 0$ and $X^1 \in [0,1]$,*

$$\int_{s=0}^{t} \left[F^m\left(s, Z_s + Y_s, X_s^1\right) - F^m\left(s, Z_s, X_s^1\right)\right] ds$$
$$\leq \int_{y=-K}^{K} \int_{s=0}^{t} \mathbf{1}(H^m(s,y,X_s^1) - Z_s \in [0, Y_s]) ds dy$$

5. *Suppose that $\widehat{F}^m(t,z,x)$ satisfies the assumptions of Lemma 6 and for any (t,x), $\widehat{F}^m(t,z,x) = F^m(t,z,x)$ at all but a measure zero set of z's. Let $\widehat{H}^m(t,y,x) = \inf\{z \in [-N,N] : \widehat{F}^m(t,z,x) > y\}$. Then H^m and $\widehat{H}^m$ coincide everywhere.*

Proof of Claim. Part 1: Let $x'' = x + \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$, $z'' = z + c_2\left[\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 + 1\right]\tau(t,t')$, and $z' = z'' + c_2\left[\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t') + |dx|\right] \geq z'' + c_2\left[|x'' - x'|\right]$. (The inequality follows since $\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t') + |dx| = |x'' - x| + |x' - x| \geq |x'' - x'|$.) By assumption 3 of Lemma 6 and part 1 of Lemma 5,

$$F^m(t,z,x) \leq F^m(t',z'',x'') \leq F^m(t',z',x')$$

But $z' = z + c_2\left[2\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 + 1\right]\tau(t,t') + c_2|dx| \leq z + c_4[\tau(t,t') + |dx|]$ proving part 1.

Part 2: $|H^m(t',y,x') - H^m(t,y,x)| = \left| \begin{array}{c} \inf\{z \in [-N,N] : F^m(t',z,x') > y\} \\ -\inf\{z \in [-N,N] : F^m(t,z,x) > y\} \end{array} \right|$.
By part 1,

$$\begin{aligned} &\inf\{z \in [-N,N] : F^m(t,z,x) > y\} \\ &\quad \geq \inf\{z \in [-N,N] : F^m(t', z + c_4(|dx| + |dt|), x') > y\} \\ &\quad \geq \inf\{z \in [-N,N] : F^m(t',z,x') > y\} - c_4(|dx| + |dt|) \end{aligned}$$

Hence, $H^m(t',y,x') - H^m(t,y,x) \leq c_4(|dx| + |dt|)$. A symmetric argument shows that $H^m(t,y,x) - H^m(t',y,x') \leq c_4(|dx| + |dt|)$

Part 3: For $m = 1, 2$, let $F^m = F^m(t,z,x)$ and $F^{m\prime} = F^m(t',z',x')$. We have

$$\Gamma(t,z,x|F) - \Gamma(t',z',x'|F) \tag{44}$$
$$= F^2 \cdot (1-x) + F^1 \cdot x - F^{2\prime} \cdot (1-x') - F^{1\prime} \cdot x' \tag{45}$$
$$= (F^2 - F^{2\prime})(1-x) + F^{2\prime} \cdot (x' - x) + (F^1 - F^{1\prime})x + F^{1\prime} \cdot (x - x')$$

implying $|\Gamma(t,z,x|F) - \Gamma(t,z,x'|F)| \le |F^2 - F^{2\prime}| + |F^1 - F^{1\prime}| + 2K\,|dx|$ as claimed. By part 1, $F^{m\prime} \in [F^m(t,z-\alpha,x), F^m(t,z+\alpha,x)]$; clearly, F^m is in the same interval, so $|F^m - F^{m\prime}| \le F^m(t,z+\alpha,x) - F^m(t,z-\alpha,x)$, proving part 3.

Part 4: Since $F^m(s,z,x) = K - \int_{y=-K}^{K} \mathbf{1}(F^m(s,z,x) \le y)dy$,

$$\begin{aligned}
&\int_{s=0}^{t} \left[F^m\left(s, Z_s + Y_s, X_s^1\right) - F^m\left(s, Z_s, X_s^1\right)\right] ds \\
&= \int_{y=-K}^{K}\int_{s=0}^{t} \left[\mathbf{1}(F^m(s,Z_s,X_s^1) \le y) - \mathbf{1}(F^m(s,Z_s+Y_s,X_s^1) \le y))\right] dsdy \\
&\le \int_{y=-K}^{K}\int_{s=0}^{t} \mathbf{1}(Z_s \le H(s,y,X_s^1) \le Z_s + Y_s)dsdy \\
&= \int_{y=-K}^{K}\int_{s=0}^{t} \mathbf{1}(H(s,y,X_s^1) - Z_s \in [0,Y_s])dsdy
\end{aligned}$$

Part 5: Since both F^m and $\widehat{F}^m$ are weakly increasing in z, the sets $\{z \in [-N,N] : F^m(t,z,x) > y\}$ and $\{z \in [-N,N] : \widehat{F}^m(t,z,x) > y\}$ are each intervals of the form $(\zeta, N]$ or $[\zeta, N]$. Since F^m and $\widehat{F}^m$ agree almost everywhere, these intervals must also agree almost everywhere; hence, their infima must coincide. Q.E.D.$_{\text{Claim 2}}$

Let $K' = K + c_4$. By part 3 of Claim 2, the absolute value of the integrand in A_3^j is a Lipschitz function of $\left|X_s^{1/m_j'} - X_s\right|$, which goes uniformly to zero as $j \to \infty$, plus

$$\sum_{m\in\{1,2\}} \left[F^m\left(s, Z_s + c_4\left|X_s^{1/m_j'} - X_s\right|, X_s\right) - F^m\left(s, Z_s - c_4\left|X_s^{1/m_j'} - X_s\right|, X_s\right)\right]$$

Thus, by part 4 of Claim 2,

$$\lim_{j\to\infty} A_3^j \le \lim_{j\to\infty} \sum_{m\in\{1,2\}} \int_{y=-K}^{K}\int_{s=0}^{t} 1\left(\begin{array}{c} H^m(s,y,X_s) - Z_s \\ \in \left[0, 2c_4\left|X_s^{1/m_j'} - X_s\right|\right) \end{array}\right) dsdy$$

Since Brownian motion has a jointly continuous local time ([30, p. 310]),

$$\lim_{j\to\infty} \int_{s=0}^{t} 1\left(-Z_s \in \left[0, 2c_4\left|X_s^{1/m_j'} - X_s\right|\right)\right) ds = 0$$

almost surely. But by part 2 of Claim 2, for $m = 1, 2$, $H^m(s,y,X_s)$ is Lipschitz in s with constant $c_4' = c_4(1+K)$. Thus, by the Girsanov Theorem [28], the law of $H^m(s, y, X_s) - Z_s$ is mutually absolutely continuous with the law of $-Z_s$. Consequently, $\lim_{j\to\infty} A_3^j = 0$ almost surely. This proves existence.

We now prove uniqueness. Let X_t^+ and X_t^- be the maximal and minimal solutions to (41). Define $Y_t = X_t^+ - X_t^-$. By part 3 of Claim 2, for some constant c,

$$Y_t \leq c\int_{s=0}^{t} Y_s ds + \sum_{m\in\{1,2\}} \int_{s=0}^{t} \left[F^m(s, Z_s + c_4 Y_s, X_s^-) - F^m(s, Z_s - c_4 Y_s, X_s^-)\right] ds \tag{46}$$

so that by part 4,

$$Y_t \leq c\int_{s=0}^{t} Y_s ds + \sum_{m\in\{1,2\}} \int_{y=-K}^{K} \int_{s=0}^{t} \mathbf{1}(H^m(s, y, X_s^-) - Z_s \in [0, 2c_4 Y_s])dy$$

Since Z has zero drift, the probability distribution over $(H^m(s, y, X_s^-) - Z_s)_{s\geq 0}$ is the same as the probability distribution over $(Z_s + H^m(s, y, X_s^-))_{s\geq 0}$. Hence, if there is a positive probability that $Y_t > 0$, then this also occurs with positive probability if Y_t instead satisfies

$$Y_t \leq c\int_{s=0}^{t} Y_s ds + \sum_{m\in\{1,2\}} \int_{y=-K}^{K} \int_{s=0}^{t} \mathbf{1}(Z_s + H^m(s, y, X_s^-) \in [0, c_4 Y_s])dsdy \tag{47}$$

(redefining c_4 to be twice the old c_4.) We will show that if (47) holds, Y_t is identically zero for all $t \in [0, 1/2 \wedge \Upsilon_N]$ almost surely.

Let $(\Omega, \mathcal{F}, P^z)$ be the probability space associated with Z when $Z_0 = z$, and let $(\mathcal{F}_u)_{u\geq 0}$ be the filtration generated by Z.[34] Since each $H^m(s, y, X_s^-)$ has paths that are Lipschitz-continuous in s with constant c_4', by the Girsanov theorem (Øksendal [28]), for any $u \in [0, t]$, and for any positive A and α,

$$E_{\mathcal{F}_u}\left[\int_{s=u}^{t} \mathbf{1}(Z_s \in [0, As^\alpha])ds\right] = E_{\mathcal{F}_u}\left[\int_{s=u}^{t} \mathbf{1}\left(\begin{array}{c} Z_s + H^m(s, y, X_s^-) \\ \in [0, As^\alpha] \end{array}\right) \cdot M_s^y\, ds\right] \tag{48}$$

where $E_{\mathcal{F}_u}$ denotes the expectation conditional on $\mathcal{F}_u$ and

$$M_s^y = \exp\left(\begin{array}{c} -\int_{v=u}^{s}\left[\frac{dH^m(v,y,X_v^-)}{dv}\right] dZ_v \\ -\frac{1}{2}\int_{v=u}^{s}\left[\frac{dH^m(v,y,X_v^-)}{dv}\right]^2 dv \end{array}\right) \geq c_5 \exp\left(-\int_{v=u}^{s}\left[\frac{dH^m(v, y, X_v^-)}{dv}\right] dZ_v\right)$$

[34] Ω is the set of possible sample paths $(Z_t)_{t\geq 0}$; $\mathcal{F}$ is the σ-algebra of measurable subsets of Ω; for any $S \in \mathcal{F}$ and constant z, $P^z(S)$ is the probability, conditional on $Z_0 = z$, that the sample path will be in S. $\mathcal{F}_u$ is the σ-algebra that contains information about Z_v for $v \leq u$ but no information about Z_v for $v > u$.

where $c_5 = \exp\left(-\frac{1}{4}\left[c_4'\right]^2\right)$ (as $s \le \frac{1}{2}$ and $\left|\frac{dH^m(v,y,X_v^-)}{dv}\right| \le c_4' = c_4(1+K)$). But

$$\int_{v=u}^{s}\left[\frac{dH^m(v,y,X_v^-)}{dv}\right]dZ_v = \int_{v=u}^{s}\left[\frac{dH^m(v,y,X_v^-)}{dv}\right]dB_{h(v)}$$
$$= \int_{v=h(u)}^{h(s)}\left[\frac{dH^m\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]dB_v$$

Hence, for any $\lambda > 0$,

$$\Pr_{\mathcal{F}_u}\left(\min_{s\in[u,t]}\left\{-\int_{v=u}^{s}\left[\frac{dH^m(v,y,X_v^-)}{dv}\right]dZ_v\right\} < -\lambda\right)$$
$$= \Pr_{\mathcal{F}_u}\left(\min_{s\in[u,t]}\left\{-\int_{v=h(u)}^{h(s)}\left[\frac{dH^m\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]dB_v\right\} < -\lambda\right)$$
$$\le \Pr_{\mathcal{F}_u}\left(\max_{s\in[u,t]}\left|\int_{v=h(u)}^{h(s)}\left[\frac{dH^m\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]dB_v\right| > \lambda\right)$$

where $\Pr_{\mathcal{F}_u}$ is the probability conditional on $\mathcal{F}_u$. The integral in the last line is a martingale and B_v is a Brownian motion with zero drift and unit variance. Hence, by Doob's martingale inequality (Øksendal [28, p. 33]), the last line is no greater than

$$\frac{E_{\mathcal{F}_u}\left[\int_{v=h(u)}^{h(s)}\left[\frac{dH^m\left(h^{-1}(v),y,X_{h^{-1}(v)}^-\right)}{dh^{-1}(v)}\right]^2 dv\right]}{\lambda^2} \le \frac{\left(c_4'\right)^2 h(s)}{\lambda^2} \le \frac{c_6}{\lambda^2}$$

where $c_6 = c_3^2 h(1/2)$ (as $s \le \frac{1}{2}$ and $h' > 0$). Thus, for any sufficiently small $m \in (0, c_5)$,

$$\Pr_{\mathcal{F}_u}(\min_{s\in[u,t]} M_s^y < m) \le \Pr_{\mathcal{F}_u}\left(\min_{s\in[u,t]}\left\{\exp\left(-\int_{v=u}^{s}\left[\frac{dH^m(v,y,X_v^-)}{dv}\right]dZ_v\right) < \frac{m}{c_5}\right\}\right)$$
$$\le \Pr_{\mathcal{F}_u}\left(\min_{s\in[u,t]}\left\{\int_{v=u}^{s}\left[-\frac{dH^m(v,y,X_v^-)}{dv}\right]dZ_v\right\} < -\ln\left(\frac{c_5}{m}\right)\right)$$
$$\le \frac{c_6}{\ln\left(\frac{c_5}{m}\right)^2} \triangleq n(m)$$

where $n(m)$ is independent of y and u and $\lim_{m\to 0} n(m) = 0$. Thus, for any

$m > 0$,

$$\begin{aligned}
&E_{F_u}\left[\int_{y=-K}^{K}\int_{s=u}^{t}\mathbf{1}(Z_s + H^m(s,y,X_s^-) \in [0, As^\alpha])\, M_s^y\, ds\, dy\right] \\
&\geq m(1-n(m))E_{F_u}\left[\int_{y=-K}^{K}\int_{s=u}^{t}\mathbf{1}(Z_s + H^m(s,y,X_s^-) \in [0, As^\alpha])\, ds\, dy\right] \\
&= m(1-n(m))E_{F_u}\left[C_t - C_u\right]
\end{aligned}$$

where $C_t = \int_{y=-K}^{K}\int_{s=0}^{t}\mathbf{1}(Z_s + H^m(s,y,X_s^-) \in [0, As^\alpha])\, ds\, dy$. Hence, by (48), there is a positive constant c_7, independent of A, α, and u, such that

$$E_{F_u}\left[C_t - C_u\right] \leq c_7 \cdot E_{F_u}\left[\int_{s=u}^{t}\mathbf{1}(Z_s \in [0, As^\alpha])ds\right]$$

Let $\left(\widehat{F}_s\right)_{s\geq 0}$ be the filtration generated by B where $Z_t = B_{h(t)}$. Since B is a Brownian motion, there is a constant c_8 such that

$$\Pr_{F_s}(Z_t \in dy \mid\ ) = \Pr_{\widehat{F}_s}(B_{h(t)} \in dy) \leq \frac{c_8}{(h(t)-h(s))^{1/2}}dy \leq \frac{c_8/\underline{\rho}^{1/2}}{(t-s)^{1/2}}dy \tag{49}$$

Using this fact, the argument of Lemma 2.14 in Bass and Burdzy [1] implies that there exist constants c_9 and c_{10}, independent of A and α, such that

$$\Pr(C_t > \lambda) \leq c_9 \exp(-c_{10}\lambda\alpha^{1/8}/(At^{\alpha+1/4}))$$

Using this fact, the argument of Lemma 2.15 of Bass and Burdzy implies that given $\zeta > 0$ there exist constants c_{11} and c_{12} such that if $\alpha \geq 1$, $A, A_0 > 0$, $A_0/A > \zeta$, and $A_1 = \alpha + 1/8$, then $\Pr(C_t \geq A_0 t^{A_1}$ for some $t \leq 1/2) \leq c_{11}\exp(-c_{12}A_0\alpha^{1/8}/A)$. Armed with this result, it is straightforward to adapt the argument in Lemma 2.17 of Bass and Burdzy to show that $\Pr(Y_t \neq 0$ for some $t \in [0, 1/2 \wedge \Upsilon_N]) = 0$. By induction on t and letting $N \to \infty$, we then have $Y_t = 0$ for all t almost surely. This proves uniqueness. Q.E.D.$_{\text{Lemma 6}}$

The following two lemmas prove important comparative statics properties of the solution to (40).

LEMMA 7 *1. Suppose that $(X_t^1)_{t\in[0,\Upsilon]}$ and $(X_t^2)_{t\in[0,\Upsilon]}$ are Lipschitz solutions to equation (40) corresponding to pairs of functions (F_1^1, F_1^2) and (F_2^1, F_2^2) that satisfy the properties of (F_1^1, F_1^2) in Lemma 6 and such that $F_1^m(t,z,x) \geq F_2^m(t,z,x)$ for $m = 1, 2$ and for all (t,z,x). Suppose the solutions $(X_t^1)_{t\in[0,\Upsilon]}$ and $(X_t^2)_{t\in[0,\Upsilon]}$ are defined relative to the same Brownian motion sample path, $(Z_t)_{t\in[0,\Upsilon]}$. Assume also that $X_0^1 \geq X_0^2$. Then $X_t^1 \geq X_t^2$ for all $t \in [0,\Upsilon]$ almost surely.*

2. *Suppose, in addition, that for any* (t, x) *and* $m = 1, 2$, $F_1^m(t, z, x) = F_2^m(t, z, x)$ *at all but a measure zero set of* z*'s. If* $X_0^1 = X_0^2$, *then* $X_t^1 = X_t^2$ *for all* $t \in [0, \Upsilon]$, *almost surely.*

Proof of LEMMA 7. Part 1: Let $Y_t = \max\{0, X_t^2 - X_t^1\}$ and $F_1^{12}(s, z, x) = F_1^1(s, z, x) + F_1^2(s, z, x)$. Then there is a $c > 0$ such that

$$\begin{aligned}
\dot{Y}_t &= \left[\Gamma(t, Z_t, X_t^2|F_2) - \Gamma(t, Z_t, X_t^1|F_1)\right] \mathbf{1}(X_t^2 \geq X_t^1) \\
&\leq \left[\Gamma(t, Z_t, X_t^2|F_1) - \Gamma(t, Z_t, X_t^1|F_1)\right] \mathbf{1}(X_t^2 \geq X_t^1) \\
&\leq cY_t + \sum_{m\in\{1,2\}} \left[\begin{array}{c} F_1^m\left(t, Z_t + c_4(X_t^2 - X_t^1), X_t^1\right) \\ -F_1^m\left(t, Z_t - c_4(X_t^2 - X_t^1), X_t^1\right) \end{array} \right] \mathbf{1}(X_t^2 \geq X_t^1) \\
&= cY_t + \sum_{m\in\{1,2\}} \left[F_1^m\left(t, Z_t + c_4 Y_t, X_t^1\right) - F_1^m\left(t, Z_t - c_4 Y_t, X_t^1\right)\right]
\end{aligned}$$

(The second inequality follows from part 3 of Claim 2.) This implies that equation (46) holds for this Y_t, with X_s^1 substituted for X_s^- and F_1^m substituted for F^m. The argument following equation (46) now applies verbatim to show that Y_t is identically zero.

Part 2: we will prove that

$$\left|\int_{s=0}^{t} \Gamma(s, Z_s, X_s^1|F_1)ds - \int_{s=0}^{t} \Gamma(s, Z_s, X_s^1|F_2)ds\right| = 0 \tag{50}$$

This implies that X_t^1 is a solution to (40) defined relative to F_2; by uniqueness, $X_t^1 = X_t^2$. To see (50), consider any (s, z, x); for $m = 1, 2$, and $n = 1, 2$, let $F_n^m(x)$ represent $F_n^m(s, z, x)$. We have

$$\begin{aligned}
0 &\leq \Gamma(s, z, x|F_1) - \Gamma(s, z, x|F_2) \\
&= F_1^2(x)(1 - x) + F_1^1(x)x - F_2^2(x)(1 - x) - F_2^1(x)x \\
&\leq F_1^2(x) - F_2^2(x) + F_1^1(x) - F_2^1(x)
\end{aligned}$$

Thus,

$$0 \leq \sum_{m\in\{1,2\}} \int_{s=0}^{t} \left[F_1^m(s, Z_s, X_s^1) - F_2^m(s, Z_s, X_s^1)\right] ds$$

As in the proof of part 4 of Claim 2,

$$\begin{aligned}
&\int_{s=0}^{t} \left[F_1^m\left(s, Z_s, X_s^1\right) - F_2^m\left(s, Z_s, X_s^1\right)\right] ds \\
&= \int_{s=0}^{t} \int_{y=-K}^{K} \left[\mathbf{1}(F_2^m(s, Z_s, X_s^1) \leq y) - \mathbf{1}(F_1^m(s, Z_s, X_s^1) \leq y)\right] dyds
\end{aligned}$$

Let $H_j^m(t,y,x) = \inf\{z \in [-N,N] : F_j^m(t,z,x) > y\}$ for $j = 1,2$. By part 5 of Claim 2, H_1^m and H_2^m coincide everywhere. Moreover, $F_j^m(s,Z_s,X_s^1) \leq y$ implies $H_j^m(s,y,X_s^1) \geq Z_s$ and $F_j^m(s,Z_s,X_s^1) > y$ implies $H_j^m(s,y,X_s^1) \leq Z_s$. So $\mathbf{1}(F_2^m(s,Z_s,X_s^1) \leq y) - \mathbf{1}(F_1^m(s,Z_s,X_s^1) \leq y) \neq 0$ only if $H_2^m(s,y,X_s^1) \geq Z_s \geq H_1^m(s,y,X_s^1)$. Since $H_1^m = H_2^m$, this implies $Z_s = H_1^m(s,y,X_s^1)$. As $H_1^m(s,y,X_s^1)$ has paths that are Lipschitz-continuous in s,

$$\int_{y=-K}^{K}\int_{s=0}^{t}\left[\mathbf{1}(F_2^m(s,Z_s,X_s^1) \leq y) - \mathbf{1}(F_1^m(s,Z_s,X_s^1) \leq y)\right] dsdy$$
$$\leq \int_{y=-K}^{K}\int_{s=0}^{t}\mathbf{1}(Z_s = H_1^m(s,y,X_s^1))dsdy = \int_{y=-K}^{K} 0dy = 0 \text{ a.s.}$$

by the Girsanov theorem (Øksendal [28]). Q.E.D$_{\text{LEMMA 7}}$

LEMMA 8 *Suppose that $(X_t)_{t\in[0,\Upsilon]}$ is the unique Lipschitz solution of (40), where F^1, F^2 satisfy the assumptions of Lemma 6. Let $\widetilde{X}_t^{z,x}$ be the solution to (40) starting from $\widetilde{X}_0^{z,x} = x_0 + x$ and corresponding to $\widetilde{Z}_t = Z_t + z$. ($F^1, F^2$ remain the same in parts 1 and 2 of this lemma).*

1. *If $z, x > 0$ then $\widetilde{X}_t^{z,0} \geq X_t$ and $\widetilde{X}_t^{0,x} \geq X_t$ for all $t \in [0,\Upsilon]$ almost surely.*
2. *As z and x go to 0, the processes $\widetilde{X}_t^{z,x}$ converge almost surely to X_t, uniformly on $[0,\Upsilon]$.*
3. *Suppose that for $n = 1,2,\ldots$, (F_n^1, F_n^2) have the properties of (F^1, F^2) in Lemma 6, for the same constant c_2. Fix some x_0 and z_0. For each n, let $\widehat{X}_t^n$ be the solution to (40) on $t \in [0,\Upsilon]$ with (F_n^1, F_n^2) appearing in place of (F^1, F^2). If $\lim_{n\to\infty} F_n^m = F^m$ for $m = 1,2$, then the solutions $\widehat{X}_t^n$ converge to X_t, the solution of (40) corresponding to (F^1, F^2).*

Proof of LEMMA 8. We will deduce part 1 from Lemma 7. For $m = 1,2$, define $\widetilde{F}^m$ by $\widetilde{F}^m(t,Z_t,X_t) = F^m(t,Z_t + z,X_t) = F^m(t,\widetilde{Z}_t,X_t)$. Since $\widetilde{F}^m \geq F^m$, Lemma 7 implies that $\widetilde{X}_t^{z,0} \geq X_t$. The assertion $\widetilde{X}_t^{0,x} \geq X_t$ follows directly from Lemma 7.

For part 2, take any sequence $\{(z_n,x_n)\}$ such that $z_n \to 0$ and $x_n \to 0$ as n goes to infinity. For a fixed t, there exists a subsequence $\{(z_{n_j},x_{n_j})\}$ such that $\widetilde{X}_t^{z_{n_j},x_{n_j}}$ converges. By extracting further subsequences and then using the diagonal method we can obtain a subsequence $\{(z'_n,x'_n)\}$ of the original sequence $\{(z_n,x_n)\}$ such that $\widetilde{X}_s^{z'_n,x'_n}$ converges to a limit X_s^* for every rational $s > 0$. The convergence is uniform on compact sets because all functions $\widetilde{X}_s^{z'_n,x'_n}$ are Lipschitz with constant K. We see that X_s^* must be a solution to (40) by the following argument. Let $\widetilde{F}_n^m(t,z+z_n,x) = F^m(t,z,x)$ and let $X_s^n = \widetilde{X}_s^{z'_n,x'_n}$. For any n,

$$\left|X_t^* - \left(x_0 + \int_{s=0}^t \Gamma(s,Z_s,X_s^*|F)ds\right)\right| \le A_1^n + A_2^n + A_3^n$$

where

$$A_1^n = \left|X_t^* - \left(x_0 + \int_{s=0}^t \Gamma(s,Z_s,X_s^n|F_n)ds\right)\right|$$
$$A_2^n = \left|\int_{s=0}^t \Gamma(s,Z_s,X_s^n|F_n)ds - \int_{s=0}^t \Gamma(s,Z_s,X_s^*|F)ds\right|$$
$$A_3^n = \left|\int_{s=0}^t \Gamma(s,Z_s,X_s^n|F)ds - \int_{s=0}^t \Gamma(s,Z_s,X_s^*|F)ds\right|$$

Since $A_1^n = |X_t^* - X_t^n|$, $\lim_{n\to\infty} A_1^n = 0$. Since $F_n \to F$, $\lim_{n\to\infty} A_2^n = 0$. One can prove that $\lim_{n\to\infty} A_3^n = 0$ by the same argument used to prove that $A_3^j \to 0$ in Lemma 6. By uniqueness, $X_s^* = X_s$ for all s. Since the same is true for any initial sequence $\{(z_n,x_n)\}$, we conclude that $\widetilde{X}_t^{z,x}$ converges to X_t almost surely, uniformly on compact time intervals.

The proof of part 3 is completely analogous to that for part 2. One can show that for every subsequence of $\widehat{X}_t^n$, there is a further subsequence which converges and, moreover, it converges to a solution of (40). The argument is finished by invoking the uniqueness of the solution. Q.E.D$_{\text{LEMMA 8}}$

The following two lemmas imply that if $F^m(v,z,x) = f^m(\Phi^{n-1}(v,z,x),x)$ for $m = 1,2$, then (F^1,F^2) satisfies the assumptions of Lemma 6, so there is a unique solution to (33).

LEMMA 9 *1. The functions $f^1(y,x)$, $f^2(y,x)$, and $\pi(y,x)$ are weakly increasing in y and right-continuous in y.*

2. For $m = 1,2$, if $y' - y > \eta\,|x' - x|$, then $f^m(y',x') \ge f^m(y,x)$.

Proof of LEMMA 9. This is an easy consequence of Lemma 4.

LEMMA 10 *For each $n \ge 0$, including $n = \infty$, and for all (t,z,x), and (t',z',x'),*

(i) $\Phi^n(t,z,x)$ is strictly increasing in z;

(ii) there is a constant c_2*, independent of* n*, such that if* $g(t,z)$ *and* $g(t',z')$ *are both in* $[\underline{w},\overline{w}]$*,* $z'-z > c_2(|x'-x| + \tau(t,t'))$*, and* $|x'-x| \geq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$*, then* $\Phi^n(t',z',x') > \Phi^n(t,z,x) + \eta\,|x'-x|$*.*

(iii) $\Phi^n(t,z,x)$ *is weakly decreasing in* n*;*

(iv) for all $\varepsilon > 0$ *there is a* $\delta > 0$*, independent of* (t,z,x)*, such that if*

$$\max\{|dx|\,,|dt|\,,|dz|\} < \delta$$

then $\Phi^n(t',z',x') - \Phi^n(t,z,x) < \varepsilon$*.*

Proof of LEMMA 10. We prove (i-iii) by induction. (iii) holds for $n=0$ if we define Φ^{-1} to be ∞. By Lemma 3, for any n,

$$\Phi^n(t,z,x) = E\int_{v=0}^{\infty} \exp\left(-\int_{s=0}^{v}(r + k^1_{t+s} + k^2_{t+s})ds\right)\left[D_{t+v}(Z_{t+v}, X_{t+v}, k^1_{t+v}, k^2_{t+v})\right]dv \tag{51}$$

(The expectation is conditioned on (Z_t, X_t) equalling (z,x).)

Let $(b_v)_{v\geq 0}$ be a fixed Brownian sample path with $b_0 = 0$. We compare $\Phi^n(t',z',x')$ to $\Phi^n(t,z,x)$ path by path, so that the continuation path of Z from time t (t') on begins at z (z') and its changes are given by $(b_v)_{v\geq 0}$ with time suitably transformed. For the path starting at (t,z,x), let $Z_{t+v} = z + b_{h(t+v)-h(t)}$; for the path starting at (t',z',x'), let $Z'_{t'+v} = z' + b_{h(t'+v)-h(t')}$. Let $dx = x'-x$, $dz = z'-z$, and $dt = t'-t$. Let

$$D'_v = D_{t'+\phi(v,t,t')}\left(Z'_{t'+\phi(v,t,t')}, X'_{t'+\phi(v,t,t')}, k^{1\prime}_{t'+\phi(v,t,t')}, k^{2\prime}_{t'+\phi(v,t,t')}\right)$$

Using the change of variables $v = \phi(\widehat{v},t,t')$, and then replacing $\widehat{v}$ by v (noting $\phi(0,t,t')=0$), we obtain

$$\begin{aligned}
&\Phi^n(t',z',x') \\
&= E\int_{v=0}^{\infty} \exp\left(-\int_{s=0}^{v}(r + k^{1\prime}_{t'+s} + k^{2\prime}_{t'+s})ds\right)\left[D_{t'+v}(Z'_{t'+v}, X'_{t'+v}, k^{1\prime}_{t'+v}, k^{2\prime}_{t'+v})\right]dv \\
&= E\int_{\phi(\widehat{v},t,t')=0}^{\infty} \exp\left(-\int_{s=0}^{\phi(\widehat{v},t,t')}(r + k^{1\prime}_{t'+s} + k^{2\prime}_{t'+s})ds\right) D'_{\widehat{v}}\cdot\phi_1(\widehat{v},t,t')d\widehat{v} \\
&= E\int_{v=0}^{\infty} \exp\left(-\int_{s=0}^{\phi(v,t,t')}(r + k^{1\prime}_{t'+s} + k^{2\prime}_{t'+s})ds\right) D'_{v}\cdot\phi_1(v,t,t')dv
\end{aligned}$$

For small (dt,dz,dx), the choices $\left(k^{1\prime}_{t'+s}, k^{2\prime}_{t'+s}\right)_{s\geq 0}$ and $(k^{1\prime}_{t'+\phi(s,t,t')}, k^{2\prime}_{t'+\phi(s,t,t')})_{s\geq 0}$ must give approximately the same expected payoffs to being in mode 1 and mode 2 as $\left(k^1_{t+s}, k^2_{t+s}\right)_{s\geq 0}$ by the envelope theorem. Thus, letting $k_{t+v} = \left(k^1_{t+v}, k^2_{t+v}\right)$ and $\varsigma_v = \exp\left(-\int_{s=0}^{v}(r + k^1_{t+s} + k^2_{t+s})ds\right)$,

$$\begin{aligned}
&\Phi^n(t', z', x') - \Phi^n(t, z, x) \\
&= E\int_{v=0}^{\infty} \zeta_{\phi(v,t,t')} \cdot D_{t'+\phi(v,t,t')}\begin{pmatrix} Z'_{t'+\phi(v,t,t')}, \\ X'_{t'+\phi(v,t,t')}, k_{t+v}\end{pmatrix}\phi_1(v,t,t')dv \\
&- E\int_{v=0}^{\infty} \zeta_v \cdot D_{t+v}\left(Z_{t+v}, X_{t+v}, k_{t+v}\right)dv
\end{aligned}$$

to first order, by (51). By definition,

$$Z'_{t'+\phi(v,t,t')} = z' + b_{h(t'+\phi(v,t,t'))-h(t')} = z' + b_{h(t+v)-h(t)} = Z_{t+v} + dz$$

Hence, $\Phi^n(t', z', x') - \Phi^n(t, z, x) = A_1 + A_2 + A_3$ where

$$\begin{aligned}
A_1 &= E\int_{v=0}^{\infty} \zeta_{\phi(v,t,t')} \cdot D_{t'+\phi(v,t,t')}\begin{pmatrix} Z'_{t'+\phi(v,t,t')}, \\ X'_{t'+\phi(v,t,t')}, k_{t+v}\end{pmatrix}\phi_1(v,t,t')dv \\
&\quad - E\int_{v=0}^{\infty} \zeta_v \cdot D_{t+v}\left(Z'_{t'+\phi(v,t,t')}, X'_{t'+\phi(v,t,t')}, k_{t+v}\right)dv \\
A_2 &= E\int_{v=0}^{\infty} \zeta_v \cdot \begin{pmatrix} D_{t+v}\left(Z_{t+v} + dz, X'_{t'+\phi(v,t,t')}, k_{t+v}\right) \\ -D_{t+v}\left(Z_{t+v} + dz, X_{t+v}, k_{t+v}\right)\end{pmatrix}dv \\
A_3 &= E\int_{v=0}^{\infty} \zeta_v \cdot \left(D_{t+v}\left(Z_{t+v} + dz, X_{t+v}, k_{t+v}\right) - D_{t+v}\left(Z_{t+v}, X_{t+v}, k_{t+v}\right)\right)dv
\end{aligned}$$

Let $\overline{|w|} = \max\{|\underline{w}|, |\overline{w}|\}$. By (35), Lemmas 2 and 5, part 2 of Lemma 3, and equation (30), there is a $c > 0$, independent of (t, t', x, x', z, z'), such that for small enough $|dt|$, $|A_1| \leq c\,|dt|$ if $g(t, z)$ and $g(t', z')$ are both in $[\underline{w} - \lambda, \overline{w} + \lambda]$. Let Υ be a (strictly positive) lower bound on $E\int_{v=0}^{\infty} e^{-(r+2K)v}1\left(g(t+v, Z_{t+v}) \in [\underline{w}, \overline{w}]\right)dv$ over all starting points $g(t, Z_t) \in [\underline{w}, \overline{w}]$. By axioms **A3** and **A4** and equation (24),

$$\begin{aligned}
A_2 &\geq \frac{\beta}{r}\min_{(b_v)_{v\geq 0}}\left\{0, \min_{v\in[0,T-t]}\left(X'_{t'+\phi(v,t,t')} - X_{t+v}\right)\right\} \\
A_3 &\geq \alpha\underline{\gamma}\Upsilon dz
\end{aligned}$$

where $\min_{(b_v)_{v\geq 0}}$ denotes the minimum over all possible paths $(b_v)_{v\geq 0}$. Thus,

$$A_2 + A_3 \geq \alpha\underline{\gamma}\Upsilon dz + \frac{\beta}{r}\min_{(b_v)_{v\geq 0}}\left\{0, \min_{v\in[0,T-t]}\left(X'_{t'+\phi(v,t,t')} - X_{t+v}\right)\right\}$$

By the above inequalities,

$$\Phi^n(t',z',x') - \Phi^n(t,z,x)$$
$$\geq \underline{\alpha}\gamma\Upsilon dz + \frac{\beta}{r} \min_{(b_v)_{v\geq 0}} \left\{0, \min_{v\in[0,T-t]} \left(X'_{t'+\phi(v,t,t')} - X_{t+v}\right)\right\} - c|dt|$$

For $n = 0$, X and X' are identically zero for any dx, so $\Phi^0(t,z,x)$ is strictly increasing in z and independent of x (part i). Moreover, since $|dt| \leq \tau(t,t')$,

$$\frac{1}{\underline{\alpha}\gamma\Upsilon}\left[\Phi^n(t',z',x') - \Phi^n(t,z,x)\right]$$
$$\geq dz - \frac{c}{\underline{\alpha}\gamma\Upsilon}\tau(t,t') + \frac{\beta}{r\underline{\alpha}\gamma\Upsilon} \min_{(b_v)_{v\geq 0}} \left\{0, \min_{v\geq 0} \left(X'_{t'+\phi(v,t,t')} - X_{t+v}\right)\right\} \tag{52}$$

Let $c_2 = \frac{c}{\underline{\alpha}\gamma\Upsilon} + \frac{1}{\underline{\alpha}\gamma\Upsilon}\left(\eta + \frac{\beta}{r}\right)$. To prove (ii), it remains to show that if $dz \geq c_2\left(\tau(t,t') + |dx|\right)$ and $|dx| \geq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$ then $\min\left\{0, \min_{v\geq 0}\left(X'_{t'+\phi(v,t,t')} - X_{t+v}\right)\right\}$ is not less than $-|dx|$ since then

$$\frac{1}{\underline{\alpha}\gamma\Upsilon}\left[\Phi^n(t',z',x') - \Phi^n(t,z,x) - \eta|dx|\right]$$
$$\geq dz - \frac{c}{\underline{\alpha}\gamma\Upsilon}\tau(t,t') - \frac{1}{\underline{\alpha}\gamma\Upsilon}\left(\eta + \frac{\beta}{r}\right)|dx|$$
$$> dz - c_2\left[\tau(t,t') + |dx|\right] > 0$$

This is trivial for the case $n = 0$ since X and X' are identically zero for any dx. Let $dX_{t+v} = X'_{t'+\phi(v,t,t')} - X_{t+v}$. For small enough $\varepsilon > 0$, we will show that $d\left(dX_{t+v}\right)/dv \geq 0$ whenever $dX_{t+v} \in [-|dx| - \varepsilon, -|dx|]$. Since X has continuous paths, this will imply $dX_{t+v} \geq -|dx|$, so $dX_{t+v} \wedge 0 \geq -|dx|$, proving (ii).

To see why $d\left(dX_{t+v}\right)/dv \geq 0$ in this range, recall that by Lemma 5, $\tau(t + v, t' + \phi(v,t,t')) \leq \tau(t,t')$, so $dz > c_2(|dx| + \tau(t+v, t'+\phi(v,t,t')))$. Thus, if $dX_{t+v} \in [|dx| - \varepsilon, |dx|]$ for small enough ε, $dz > c_2(dX_{t+v} + \tau(t+v, t'+\phi(v,t,t')))$ and $\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t+v, t'+\phi(v,t,t')) \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t') \leq |dX_{t+v}|$. By the induction hypothesis,

$$\eta\left|dX_{t+v}\right| < m_v$$
$$\triangleq \Phi^{n-1}\left(t' + \phi(v,t,t'), Z_{t+v} + dz, X'_{t'+\phi(v,t,t')}\right)$$
$$- \Phi^{n-1}\left(t+v, Z_{t+v}, X_{t+v}\right)$$

Define $\widehat{x}' = X_{t+v}$, $\widehat{x} = \widehat{x}' + dX_{t+v}$, $y' = \Phi^{n-1}(t+v, Z_{t+v}, X_{t+v})$, and $y = y' + m_v$. Recall that $f^2(y,\widehat{x}) = \max BR^2(y,\widehat{x})$ and $f^1(y,\widehat{x}) = -\min BR^1(y,\widehat{x})$,

$$\begin{aligned}
& d\left(dX_{t+v}\right)/dv = \pi(y,\widehat{x})\phi_1(v,t,t') - \pi(y',\widehat{x}') \\
& = \begin{pmatrix} f^2(y,\widehat{x})\cdot(1-\widehat{x}) \\ +f^1(y,\widehat{x})\cdot\widehat{x} \end{pmatrix} (\phi_1(v,t,t')-1) \\
& + f^2(y,\widehat{x})\cdot(1-\widehat{x}) - f^2(y',\widehat{x}')\cdot(1-\widehat{x}') \\
& + f^1(y,\widehat{x})\cdot\widehat{x} - f^1(y',\widehat{x}')\cdot\widehat{x}' \\
& \geq \begin{pmatrix} f^2(y,\widehat{x})\cdot(1-\widehat{x}) \\ +f^1(y,\widehat{x})\cdot\widehat{x} \end{pmatrix} (\phi_1(v,t,t')-1) \\
& - \left(f^2(y,\widehat{x}) - f^1(y,\widehat{x})\right) dX_{t+v}
\end{aligned}$$

where the inequality follows from part 2 of Lemma 9. This is nonnegative if $|\phi_1(v,t,t')-1| \leq |dx|$ since by hypothesis $dX_{t+v} \leq -|dx|$. By Lemma 5, $|\phi_1(v,t,t')-1| \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 |dt| \leq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t') \leq |dx|$. This proves that $d\left(dX_{t+v}\right)/dv \geq 0$ and hence (ii) holds for finite $n > 0$.

Now consider (i) for finite $n > 0$. The relative payoff of being in mode 1, $\Phi^n(t,z,x)$, is computed assuming players believe that, in the future, $\dot{X}_v$ will equal $f^2(\Phi^{n-1}(v,Z_v,X_v),X_v)(1-X_v) + f^1(\Phi^{n-1}(v,Z_v,X_v),X_v)X_v$ (equation (33)). By induction and Lemma 9, for $m = 1,2$, $f^m(\Phi^{n-1}(t,z,x),x)$ has the properties of $F^m(t,z,x)$ assumed in Lemma 8. Hence, if $dt = 0$ and both dx and dz are nonnegative, then $X'_{t'+\phi(v,t,t')} = X'_{t+v} \geq X_{t+v}$. By (52) and the envelope theorem, $\Phi^n(t,z,x)$ is strictly increasing in z and weakly increasing in x, proving (i) for finite $n > 0$.

For (iii) with finite $n > 0$, we know by induction that $\Phi^{n-1}(t,z,x) \leq \Phi^{n-2}(t,z,x)$ and that for $m = 1,2$, both $f^m(\Phi^{n-1}(t,z,x),x)$ and $f^m(\Phi^{n-2}(t,z,x),x)$ satisfy the assumptions of $F^m(t,z,x)$ in Lemma 6. Hence, $\Phi^n(t,z,x) \leq \Phi^{n-1}(t,z,x)$ by Lemma 7.[35]

For the case $n = \infty$, $f^m(\Phi_N^\infty(t,z,x),x) = f^m(\lim_{n\to\infty}\Phi^n(t,z,x),x)$ satisfies the properties of F^m in Lemma 6 as each $f^m(\Phi^n(t,z,x),x)$ does for $n < \infty$, and these properties clearly hold in the limit. In particular, c_2 is independent of n, so if

$$z' - z > c_2(|x'-x| + \tau(t,t'))$$

and $|x'-x| \geq \left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$ then $\Phi_N^\infty(t',z',x') \geq \Phi_N^\infty(t,z,x) + \eta\,|dx|$, whence $f^m(\Phi_N^\infty(t',z',x'),x') \geq f^m(\Phi_N^\infty(t,z,x),x')$ by Lemma 9. Thus, by Lemma 8, if

[35] Evaluate them path-by-path in Z and use the envelope theorem to show that a lower X must lower the relative payoff to playing R; then apply Lemma 7.

$dt = 0$ and both dx and dz are nonnegative, then $dX_{t+v} \geq 0$ for all $v \geq 0$. This shows (i). For (ii), if $z' - z > c_2(|dx| + \tau(t,t'))$ and $|dx| \geq \left(\frac{\bar{\rho}}{\underline{\rho}}\right)^2 \tau(t,t')$ then there is an $\varepsilon > 0$ such that $(z' - \varepsilon) - z > c_2(|dx| + \tau(t,t'))$, whence $\Phi^n(t', z' - \varepsilon, x') > \Phi^n(t, z, x) + \eta\,|dx|$ for all n, so $\Phi_N^\infty(t', z' - \varepsilon, x') \geq \Phi_N^\infty(t, z, x) + \eta\,|dx|$; by part (i), $\Phi_N^\infty(t', z', x') > \Phi_N^\infty(t, z, x) + \eta\,|dx|$.

We now show (iv) for $n = 0, 1, \ldots$. Consider the decomposition used above:

$$\begin{aligned} \Phi^n(t', z', x') - \Phi^n(t, z, x) &= A_1 + A_2 + A_3 \\ &\leq c|dt| + A_2 + \overline{\alpha\gamma}\,|dz|\,/r \end{aligned}$$

Moreover,

$$A_2 \leq \beta E \int_{v=0}^{T} e^{-rv}\,|dX_{t+v}|\,dv$$

Define $F^m(t,z,x) = f^m(\Phi^{n-1}(t,z,x), x)$ and $\Gamma(t,z,x) = F^2(t,z,x)(1-x) + F^1(t,z,x)x$. Note that

$$\begin{aligned} & d\left(dX_{t+v}\right)/dv \\ & = \Gamma\left(t' + \phi(v,t,t'), Z_{t+v} + dz, X'_{t'+\phi(v,t,t')}\right)\phi_1(v,t,t') \\ & - \Gamma\left(t+v, Z_{t+v}, X_{t+v}\right) \\ & = B_1 + B_2 + B_3 \end{aligned}$$

where

$$\begin{aligned} B_1 &= \Gamma\left(t' + \phi(v,t,t'), Z_{t+v} + dz, X'_{t'+\phi(v,t,t')}\right)\left[\phi_1(v,t,t') - 1\right] \\ B_2 &= \Gamma\left(t' + \phi(v,t,t'), Z_{t+v} + dz, X'_{t'+\phi(v,t,t')}\right) - \Gamma\left(t+v, Z_{t+v}, X'_{t'+\phi(v,t,t')}\right) \\ B_3 &= \Gamma\left(t+v, Z_{t+v}, X'_{t'+\phi(v,t,t')}\right) - \Gamma\left(t+v, Z_{t+v}, X_{t+v}\right) \end{aligned}$$

Assume WLOG that $dX_{t+v} \geq 0$. (A symmetric argument holds for $dX_{t+v} \leq 0$.) By Lemma 5, $|B_1| \leq K\left(\frac{\bar{\rho}}{\underline{\rho}}\right)^2 |dt|$. Recall that $F^1 \leq 0$ and $F^2 \geq 0$. Thus, by (45) and part 3 of Claim 2,

$$B_2 + B_3 \leq \sum_{m\in\{1,2\}} \left[F^m(t+v, Z_{t+v} + \alpha_v, X_{t+v}) - F^m(t+v, Z_{t+v} - \alpha_v, X_{t+v})\right]$$

where $\alpha_v = |dz| + c_4\left(\frac{\bar{\rho}}{\underline{\rho}}\,|dt| + |dX_{t+v}|\right)$. By part 4 of Claim 2,

$$\frac{d}{dv}\left(dX_{t+v}\right) \leq K\left(\frac{\bar{\rho}}{\underline{\rho}}\right)^2 |dt| + \sum_{m\in\{1,2\}} \int_{y=-K}^{K} \mathbf{1}\left(H^m(t+v, y, X_{t+v}) - Z_{t+v} \in [-\alpha_v, \alpha_v]\right) dy \tag{53}$$

and thus

$$dX_{t+v} \leq dx + K\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 |dt|\, v + \int_{s=0}^{v} \sum_{m\in\{1,2\}} \int_{y=-K}^{K} \mathbf{1}\left(\begin{array}{c} H^m(t+s, y, X_{t+s}) - Z_{t+s} \\ \in [-\alpha_s, \alpha_s] \end{array}\right) dyds \tag{54}$$

We now use the following mathematical result.

PROPOSITION 1 *Let B_t be a Brownian motion and let $Z_t = B_{h(t)}$ where $h : \Re^+ \to \Re^+$ and there are constants $\overline{\rho} \geq \underline{\rho} > 0$ such that for all $t > t'$, $h(t) - h(t') \in \left[\underline{\rho}(t-t'), \overline{\rho}(t-t')\right]$ and $|h'(t) - h'(t')| \leq \overline{\rho}\,|t-t'|$. Let X_t be a process adapted to Z_t with Lipschitz paths. Let a_0, a_1, a_2 be nonnegative and let K, n, q, r be strictly positive constants. Consider another process $Y \geq 0$ where $Y_0 = a_0$ and*

$$Y_t - Y_0 \leq a_1 t + \int_{s=0}^{t} \int_{w=-K}^{K} \mathbf{1}\left(\begin{array}{c} H(s, w, X_s) - Z_s \in \\ [-a_2 - q(a_1 + Y_s), a_2 + q(a_1 + Y_s)] \end{array}\right) dwds \tag{55}$$

where $H : \Re^+ \times [-K, K] \times [0,1] \to [-n, n]$ is Lipschitz in s and X_s and weakly increasing in w. Then there is a function f such that for any Y satisfying (55), $E\int_{t=0}^{\infty} e^{-rt} Y_t dt \leq f(a_0, a_1, a_2)$, and such that for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $\max\{a_0, a_1, a_2\} < \delta$ then $f(a_0, a_1, a_2) < \varepsilon$.

Proof of Proposition 1. We make use of the following Lemma:

LEMMA 11 *For every $\lambda < \infty$ and $K_1 > 0$, there exists a stopping time $T > 0$, such that for every Lipschitz function with constant λ and every $\varepsilon > 0$,*

$$\int_0^T \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(B_s) ds \leq K_1 \varepsilon.$$

Proof of Lemma 11. For a Lipschitz function $f(t)$, we define the local time L_t^f of B_t on f by the formula

$$L_t^f = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(B_s) ds.$$

We will write L_t^ε if $f(t) \equiv \varepsilon$. It is well known that there exists a version of L_t^ε which is jointly continuous in ε and t, and, moreover, for every $\varepsilon > 0$ and $t \geq 0$,

$$\int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(B_s) ds = \int_{-\varepsilon}^{\varepsilon} L_t^x dx \tag{56}$$

Let $\widetilde{B}_t = B_t - f(t)$ and

$$\widetilde{L}_t^f = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(\widetilde{B}_s) ds.$$

For a fixed Lipschitz function f, the process $\widetilde{B}_t$ has a distribution mutually absolutely continuous with the distribution of the Brownian motion, so (56) applies to $\widetilde{B}_t$, i.e.,

$$\int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(\widetilde{B}_s) ds = \int_{-\varepsilon}^{\varepsilon} \widetilde{L}_t^x dx.$$

This is equivalent to

$$\int_0^t \mathbf{1}_{(f(s)-\varepsilon, f(s)+\varepsilon)}(B_s) ds = \int_{-\varepsilon}^{\varepsilon} L_t^{f+x} dx.$$

The last formula holds for a fixed f, a.s. for all $\varepsilon > 0$ and $t \geq 0$. Let $\mathcal{F}$ be a countable family of Lipschitz functions with constant λ which is dense in the supremum norm in the space of all Lipschitz functions with constant λ. Then the last formula holds a.s. for all $f \in \mathcal{F}$, $\varepsilon > 0$ and $t \geq 0$.

By Theorem 3.6 and Remark 3.7 of Bass and Burdzy [2], a.s., the function $(f, t) \to L_t^f$ is jointly continuous if f ranges over Lipschitz functions with constant λ and it is bounded for every fixed $t < \infty$. Fix some t and then fix an ω for which both statements are true. Take any Lipschitz function $g(t)$ with constant λ, fix some $\varepsilon > 0$, and consider an arbitrarily small $\delta > 0$. Let $\{f_n\}$ be a sequence of functions in $\mathcal{F}$ which converges uniformly to g. Then for sufficiently large n,

$$\int_0^t \mathbf{1}_{(g(s)-\varepsilon, g(s)+\varepsilon)}(B_s) ds \leq \int_0^t \mathbf{1}_{(f_n(s)-\varepsilon-\delta, f_n(s)+\varepsilon+\delta)}(B_s) ds = \int_{-\varepsilon-\delta}^{\varepsilon+\delta} L_t^{f_n+x} dx.$$

By the continuity of $f \to L_t^f$ and dominated convergence, the last integral converges to $\int_{-\varepsilon-\delta}^{\varepsilon+\delta} L_t^{g+x} dx$, so we see that

$$\int_0^t \mathbf{1}_{(g(s)-\varepsilon, g(s)+\varepsilon)}(B_s) ds \leq \int_{-\varepsilon-\delta}^{\varepsilon+\delta} L_t^{g+x} dx.$$

Since δ is arbitrarily small,

$$\int_0^t \mathbf{1}_{(g(s)-\varepsilon, g(s)+\varepsilon)}(B_s) ds \leq \int_{-\varepsilon}^{\varepsilon} L_t^{g+x} dx.$$

The lower bound can be proved analogously and so we obtain a.s. simultaneously for all Lipschitz functions g with constant λ, all $\varepsilon > 0$ and $t \geq 0$,

$$\int_0^t \mathbf{1}_{(g(s)-\varepsilon, g(s)+\varepsilon)}(B_s) ds = \int_{-\varepsilon}^{\varepsilon} L_t^{g+x} dx \tag{57}$$

Recall that for a fixed t, $M_t = \sup_f L_t^f < \infty$ where f ranges over Lipschitz functions with constant λ. This and (57) imply that

$$\int_0^t \mathbf{1}_{(g(s)-\varepsilon, g(s)+\varepsilon)}(B_s) ds \leq 2M_t \varepsilon \tag{58}$$

It follows from the joint continuity of $(f,t) \to L_t^f$ that $M_t \to 0$ as $t \to 0$. Hence, $T = \inf\{t \geq 0 : M_t \geq K_1/2\} > 0$. This and (58) prove the lemma. Q.E.D.$_{\text{Lemma 11}}$

We now complete the proof of Proposition 1.

Step 1. Let $\alpha(t)$ be the inverse of $h(t)$, i.e., $\alpha(t) = \inf\{s \geq 0 : h(s) \geq t\}$. Then (55) is equivalent to

$$\begin{aligned} &Y_t - Y_0 \\ &\leq a_1 t \\ &+ \int_{w=-K}^{K} \int_{s=0}^{h(t)} \mathbf{1} \left(\begin{array}{c} H(\alpha(s), w, X_{\alpha(s)}) - Z_{\alpha(s)} \in \\ [-a_2 - q(a_1 + Y_{\alpha(s)}), a_2 + q(a_1 + Y_{\alpha(s)})] \end{array} \right) \alpha'(s) ds dw \end{aligned}$$

Consider a processes $\widehat{Y}_t$ with $\widehat{Y}_0 = Y_0 = a_0$, defined by

$$\begin{aligned} &\widehat{Y}_t - \widehat{Y}_0 \\ &= a_1 t \\ &+ \int_{w=-K}^{K} \int_{s=0}^{h(t)} \mathbf{1} \left(\begin{array}{c} H(\alpha(s), w, X_{\alpha(s)}) - B_s \in \\ [-a_2 - q(a_1 + Y_{\alpha(s)}), a_2 + q(a_1 + Y_{\alpha(s)})] \end{array} \right) \alpha'(s) ds dw \end{aligned}$$

Step 2. Fix an arbitrarily large $t_0 < \infty$. We will show that Y_t converges to 0 on $[0, t_0]$ a.s., as a_0, a_1 and a_2 go to 0. Since Y_t is monotone, the convergence is necessarily uniform.

It follows from the assumptions on h that $\alpha'(s) \leq 1/\underline{\rho}$.

For every fixed w, the process $s \to H(s, w, X_{\alpha(s)})$ has Lipschitz trajectories with some constant λ. We will apply Lemma 1 with this constant λ and $K_1 = \underline{\rho}/(16Kq)$. Recall the random variable T from the proof of Lemma 1 and let $T_1 = T$, and for $k \geq 2$,

$$T_k = \inf\{t \geq T_{k-1} : \sup_f L_t^f - L_{T_{k-1}}^f \geq \underline{\rho}/(32Kq)\},$$

where f ranges over Lipschitz functions with constant λ. By the strong Markov property, the random variables $T_k - T_{k-1}$ are i.i.d. so for some random but finite k_0, we have $T_{k_0} > t_0$.

Consider arbitrarily small $\varepsilon > 0$. We will assume without loss of generality that K and q are larger than 1. Suppose that

$$\begin{aligned} a_0 &\leq \varepsilon/(8Kq), \\ a_1 &\leq \min(\varepsilon/(32\alpha(t_0)Kq), \varepsilon/(8Kq)), \\ \text{and } a_2 &\leq \varepsilon/(8K). \end{aligned}$$

Let $S = \inf\{s \geq 0 : \widehat{Y}_{\alpha(s)} \geq \varepsilon/(4Kq)\}$, $U = \min(S, T_1, t_0)$ and consider a process $\widetilde{Y}_t$ with $\widetilde{Y}_0 = Y_0 = a_0$, defined by

$$\begin{aligned} &\widetilde{Y}_t - \widetilde{Y}_0 \\ &= a_1 t \\ &+ \int_{w=-K}^{K} \int_{s=0}^{\min\{U,h(t)\}} \mathbf{1}\begin{pmatrix} H(\alpha(s), w, X_{\alpha(s)}) - B_s \in \\ [-a_2 - q(a_1 + Y_{\alpha(s)}), a_2 + q(a_1 + Y_{\alpha(s)})] \end{pmatrix} \alpha'(s) ds dw \end{aligned}$$

Our assumptions on a_j's imply that for $s \leq U$,

$$\begin{aligned} a_2 + q(a_1 + Y_{\alpha(s)}) &\leq a_2 + q(a_1 + \widehat{Y}_{\alpha(s)}) \\ &\leq \varepsilon/(8K) + q(\min(\varepsilon/(32\alpha(t_0)Kq), \varepsilon/(8Kq)) + \varepsilon/(4Kq)) \\ &\leq \varepsilon/(2K) \end{aligned}$$

Hence, using Lemma 1,

$$\begin{aligned} &\int_{s=0}^{\min\{U,h(t)\}} \mathbf{1}\begin{pmatrix} H(\alpha(s), w, X_{\alpha(s)}) - B_s \in \\ [-a_2 - q(a_1 + Y_{\alpha(s)}), a_2 + q(a_1 + Y_{\alpha(s)})] \end{pmatrix} \alpha'(s) ds \\ &\leq \int_{s=0}^{\min\{U,h(t)\}} \mathbf{1}\begin{pmatrix} H(\alpha(s), w, X_{\alpha(s)}) - B_s \in \\ [-\varepsilon/(2K), \varepsilon/(2K)] \end{pmatrix} \alpha'(s) ds \leq \varepsilon/(32K^2 q) \end{aligned}$$

It follows that for t such that $h(t) \leq U$,

$$\begin{aligned} \widetilde{Y}_t &\leq \widetilde{Y}_0 + a_1 t + \int_{w=-K}^{K} \varepsilon/(32K^2 q) dw \\ &\leq \varepsilon/(8Kq) + \alpha(t_0)\varepsilon/(32\alpha(t_0)Kq) + \varepsilon/(16Kq) \leq (7/32)\varepsilon/(Kq) \end{aligned}$$

Note that $\widehat{Y}_t = \widetilde{Y}_t$ if $h(t) \leq U$, i.e., $t \leq \alpha(U)$. We will show that $S \geq U$. Suppose otherwise, i.e., $S < U$. Then $\widehat{Y}_{\alpha(S)} = \varepsilon/(4Kq)$. This contradicts the fact that $\widehat{Y}_{\alpha(S)} = \widetilde{Y}_{\alpha(S)} \leq \widetilde{Y}_{\alpha(U)} < (7/32)\varepsilon/(Kq)$. Hence, $S \geq U$, and so for $t < \min(T_1, t_0)$ we have $Y_t \leq \widehat{Y}_t = \widetilde{Y}_t \leq (7/32)\varepsilon/(Kq) < \varepsilon$. We conclude that by choosing sufficiently small a_j's, we can make Y_t smaller than an arbitrary $\varepsilon > 0$ for all $t \leq \min(T_1, t_0)$.

Note that

$$\widehat{Y}_{T_1+t} - \widehat{Y}_{T_1}$$
$$= a_1 t$$
$$+ \int_{s=h(T_1)}^{h(T_1+t)} \int_{w=-K}^{K} \mathbf{1} \left(\begin{array}{c} H(\alpha(s), w, X_{\alpha(s)}) - B_s \in \\ [-a_2 - q(a_1 + Y_{\alpha(s)}), a_2 + q(a_1 + Y_{\alpha(s)})] \end{array} \right) \alpha'(s) dw ds$$

The strong Markov property allows us to apply the above argument to the function $\widehat{Y}_t$ on the interval $[T_1, T_2]$. In other words, if a_j's are sufficiently small then $\widehat{Y}_t$ (and consequently Y_t) is smaller than an arbitrary $\varepsilon > 0$ for all $t \leq \min(T_2, t_0)$. An inductive procedure allows us to extend the claim to all $t \leq t_0$.

Step 3. Fix an arbitrarily small $\delta > 0$. Assume without loss of generality that $a_1 < 1$. Note that for all $t \geq 0$,

$$Y_t - Y_0 \leq a_1 t + 2Kt \leq (2K+1)t, \tag{59}$$

and so

$$E \int_0^\infty e^{-rt} Y_t dt \leq \int_0^\infty e^{-rt}(2K+1)t dt < \infty.$$

Suppose that t_0 is so large that

$$\int_{t_0}^\infty e^{-rt}(2K+1)t dt < \delta/2.$$

We have shown in Step 1 that for any sequence of parameters (a_0^n, a_1^n, a_2^n) converging to $(0,0,0)$, the corresponding processes Y_t^n converge uniformly to zero on $[0, t_0]$, a.s. In view of (59), we can apply the bounded convergence theorem to see that

$$\lim_{n\to\infty} E \int_0^{t_0} e^{-rt} Y_t^n dt = 0.$$

For sufficiently large n, $E \int_0^{t_0} e^{-rt} Y_t^n dt < \delta/2$ and so $E \int_0^\infty e^{-rt} Y_t^n dt < \delta$. Q.E.D.$_{\text{Proposition 1}}$.

We apply Proposition 1 by letting $Y_v = dX_{t+v}$, $a_0 = dx$, $a_1 = \max\left\{ c_4 \frac{\overline{\rho}}{\underline{\rho}}, K\left(\frac{\overline{\rho}}{\underline{\rho}}\right)^2 \right\} |dt|$, $a_2 = |dz|$, and $q = c_4$. Thus equation (55) corresponds to (54), implying that for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $\max\{|dx|, |dt|, |dz|\} < \delta$ then $A_2 < \varepsilon$. Thus, for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $\max\{|dx|, |dt|, |dz|\} < \delta$ then $\Phi^n(t', z', x') - \Phi^n(t, z, x) < \varepsilon$. Q.E.D.$_{\text{Lemma 10}}$

By Lemma 9 and part (iii) of Lemma 10,

$$\dot{X}_v \leq \lim_{n\to\infty} \pi(\Phi^{n-1}(v, Z_v, X_v), X_v) = \pi(\Phi^\infty(v, Z_v, X_v), X_v) \tag{60}$$

Moreover, $\dot{X}_v = \pi(\Phi^\infty(v, Z_v, X_v), X_v)$ is an equilibrium: if $\dot{X}_v$ is expected to equal $\pi(\Phi^\infty(v, Z_v, X_v), X_v)$ for all $v \geq t$, players' best responses lead $\dot{X}_t$ to equal $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$. The reasoning is as follows. Let

$$\begin{aligned}\dot{X}_v^n &= \pi(\Phi^{n-1}(v, Z_v, X_v), X_v)\\ &= f^1(\Phi^{n-1}(v, Z_v, X_v))(1 - X_v) + f^2(\Phi^{n-1}(v, Z_v, X_v))X_v\end{aligned}$$

By Lemmas 9 and 10, for all n and for $a = 1, 2$, $f^a(\Phi^{n-1}(v, Z_v, X_v))$ has the properties of F^a assumed in Lemmas 6-8, so for any path $(Z_v)_{v\geq t}$ there is a unique Lipschitz solution $(X_v^n)_{v\geq t}$ to this dynamical system.[36] By Lemmas 9 and part (iii) of 10, $\lim_{n\to\infty} f^a(\Phi^n(v, Z_v, X_v)) = f^a(\Phi^\infty(v, Z_v, X_v))$ for $a = 1, 2$. Let $X_v^\infty = \lim_{n\to\infty} X_v^n$. By part 3 of Lemma 8, $(X_v^\infty)_{v\geq t}$ is the unique solution to $\dot{X}_v = \pi(\Phi^\infty(v, Z_v, X_v), X_v)$. This implies that $(X_v^\infty)_{v\geq t}$ is a best response when the relative value of being in mode 1 for any (v, Z_v, X_v) is $\Phi^\infty(v, Z_v, X_v)$. It remains to show that $\Phi^\infty(t, Z_t, X_t)$ is the relative value of being in mode 1 if for any $(Z_v)_{v\geq t}$ players expect $(X_v^\infty)_{v\geq t}$. By the envelope theorem and (26), the value of being in mode 1 is a continuous function of the path of X. But $X_v^\infty = \lim_{n\to\infty} X_v^n$, so $\Phi^\infty(t, Z_t, X_t) = \lim_{n\to\infty} \Phi^n(t, Z_t, X_t)$ must be the relative value of being in mode 1 when X follows $(X_v^\infty)_{v\geq t}$.

This proves that $\dot{X}_t = \pi(\Phi^\infty(t, Z_t, X_t), X_t)$ is both an upper bound on $\dot{X}_t$ and the equilibrium with the highest path of X for any path of Z. We now iterate from below: we construct a growing sequence of *lower* bounds on $\dot{X}_t$. Each lower bound in the sequence is now some *translation* of $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$, the upper bound on $\dot{X}_t$. We will show that the limit of this sequence of lower bounds coincides with the upper bound. This will imply that the equilibrium $\dot{X}_t = \pi(\Phi^\infty(t, Z_t, X_t), X_t)$ is in fact the unique equilibrium of the model.

Since $\pi(y, x)$ is right continuous in y (Lemma 9) and $\Phi^\infty(t, z, x)$ is nondecreasing and continuous in z (Lemma 10), the upper bound on $\dot{X}_t$, $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$, is right continuous in Z_t. Let $\widetilde{\pi}(y, x) = \lim_{\varepsilon\downarrow 0} \pi(y - \varepsilon, x)$ be the the left continuous (in y) version of π. By part (iv) of Lemma 10, $\widetilde{\pi}(\Phi^\infty(t, Z_t, X_t), X_t)$ is left continuous in Z_t; it is the left continuous version of the upper bound on $\dot{X}_t$.

We iterate with translations of this $\widetilde{\pi}(\Phi^\infty(t, Z_t, X_t), X_t)$. Let $\lambda_0 > 0$ be large enough that regardless of their expectations for $(X_v)_{v\geq t}$, players at state (t, Z_t, X_t) must choose switching rates that yield a rate of change of X_t that is at least $\widetilde{\pi}(\Phi^\infty(t, Z_t - \lambda_0, X_t), X_t)$. There must be such a λ_0 by the existence of dominance regions and the assumption that the integral of the absolute drift terms is finite ($\int_{s=0}^\infty |\nu_s|\, ds < N_2$). To see this, consider the following three cases:

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[36]Property 3 of lemma 6 holds by part (ii) of Lemma 10 since if $g(t, z) \notin [\underline{w}, \overline{w}]$, then $f^a(\Phi^{n-1}(t, z, x))$ is locally constant.

1. $W_t = g(t, Z_t) > \overline{w}$: then players must choose switching rates that yield the highest feasible $\dot{X}_t$ (which is $\overline{K}^2(1 - X_t)$), so the result is trivial;

2. $W_t = g(t, Z_t) < \underline{w}$: then players must choose switching rates that yield the lowest feasible $\dot{X}_t$ (which is $-\overline{K}^1 X_t$); but $g(t, Z_t - \lambda_0) < \overline{w}$, so at $(t, Z_t - \lambda_0, X_t)$ they must also do so as well;

3. $W_t = g(t, Z_t) \in [\underline{w}, \overline{w}]$: then by equation (25) and since $\int_{s=0}^{\infty} |\nu_s|\, ds < N_2$, if $\lambda_0 > (\overline{w} - \underline{w})\, e^{N_2}$ then $g(t, Z_t - \lambda_0) < \underline{w}$, so players at $(t, Z_t - \lambda_0, X_t)$ must choose switching rates that yield the lowest feasible $\dot{X}_t$ $(= -\overline{K}^1 X_t)$; thus, the property holds here as well.

Let λ_n be the infimum of constants λ such that if players believe that $\dot{X}_v$ will be at least $\widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda_{n-1}, X_v), X_v)$ for all $v \geq t$, they must choose switching rates that yield an $\dot{X}_t$ that is at least $\widetilde{\pi}(\Phi^\infty(t, Z_t - \lambda, X_t), X_t)$.

More precisely, let $\Phi^\infty_\lambda(t, Z_t, X_t)$ be the relative value of being in mode 1 on the belief that, for all $v \geq t$, $\dot{X}_v$ will equal the translation of the left continuous (LC) version of the upper bound on $\dot{X}_v$ by λ, $\widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda, X_v), X_v)$. (Note that $\Phi^\infty_0(t, Z_t, X_t) = \Phi^\infty(t, Z_t, X_t)$.) Let $\underline{\pi}(y, x) = \min BR^2(y, x)(1 - x) - \max BR^1(y, x)x$: the lowest possible $\dot{X}_t$ when $X_t = x$ and the relative value of being in mode 1 is y. When the relative value of being in mode 1 is $\Phi^\infty_\lambda(t, Z_t, X_t)$, the rate of change $\dot{X}_t$ must be at least $\underline{\pi}(\Phi^\infty_\lambda(t, Z_t, X_t), X_t)$. For $n \geq 1$, let λ_n be the infimum of numbers λ such that, for all states (t, z, x), $\underline{\pi}(\Phi^\infty_{\lambda_{n-1}}(t, z, x), x)$ (the lowest possible rate of change when others are expected to play according to the translation of the LC version of the upper bound downward by λ_{n-1}) is at least $\widetilde{\pi}(\Phi^\infty(t, z - \lambda, x), x)$, the translation of the LC version of the upper bound downward by λ.

By construction, $\lambda_0 \geq \lambda_1$. By Lemma 7, for any path $(Z_v)_{v \geq t}$, the solution $(X_v)_{v \geq t}$ to the equation $\dot{X}_v = \widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda, X_v), X_v)$ is weakly decreasing in λ; thus, by Lemma 3, $\lambda_1 \geq \lambda_2$. Continuing by induction, $\lambda_{n-1} \geq \lambda_n$ for all n. Let $\lambda_\infty = \lim_{n \to \infty} \lambda_n$. We know that $\dot{X}_t$ cannot lie above $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$ nor below $\widetilde{\pi}(\Phi^\infty(t, Z_t - \lambda_\infty, X_t), X_t)$.

We now show that $\lambda_\infty = 0$. For any (t, z, x) and any λ, let $S^\lambda(t, z, x)$ stand for the situation in which players choose switching rates at state $(t, Z_t, X_t) = (t, z + \lambda, x)$ and believe that $\dot{X}_v$ will equal $\widetilde{\pi}(\Phi^\infty(v, Z_v - \lambda, X_v), X_v)$ for all $v \geq t$. The initial rate of change of X_t in situation $S^\lambda(t, z, x)$ is $\widetilde{\pi}(\Phi^\infty(t, z, x), x)$, independent of λ. The relative value of being in mode 1 in situation $S^\lambda(t, z, x)$ is just $\Phi^\infty_\lambda(t, z + \lambda, x)$.

For any $\lambda, \lambda' \in [0, \lambda_\infty]$, the distribution of continuation paths $(Z_v - Z_t)_{v \geq t}$ in situations $S^\lambda(t, z, x)$ and $S^{\lambda'}(t, z, x)$ is the same since $Z_v = B_{h(v)}$ where h is a fixed function and B is a Brownian motion. And given a continuation path of Z, the continuation path of X is determined by the same dynamical system: $X_t = x$ and $\dot{X}_v$ equals $\widetilde{\pi}(\Phi^\infty(v, Z_v - Z_t + z, X_v), X_v)$, independent of λ. By Lemmas 9 and parts (i) and (ii) of 10, for $a = 1, 2$, $F^a(v, Z_v, X_v) = f^a(\Phi^\infty(v, Z_v - Z_t + z, X_v))$ has the properties assumed in Lemma 6, so this dynamical system has a unique solution for each λ. So for any $\lambda, \lambda' \in [0, \lambda_\infty]$, players in situations $S^\lambda(t, z, x)$ and $S^{\lambda'}(t, z, x)$ expect the same distribution of continuation paths of the state, $(Z_v - Z_t, X_v - X_t)_{v\geq t}$. Fix any sample path $(z_v, x_v)_{v\geq t}$; since X_t is independent of λ, this sample path in situation λ has the same probability as the sample path $(z_v + \lambda' - \lambda, x_v)_{v\geq t}$ in situation $S^{\lambda'}(t, z, x)$. Hence, by Lemma 3 and the envelope theorem,

$$\frac{d\left[\Phi^\infty_\lambda(t, z + \lambda, x)\right]}{d\lambda} = E\int_{v=t}^{\infty} \exp\left(-\int_{s=t}^{v}(r + k_s^{1\lambda} + k_s^{2\lambda})ds\right)\frac{\partial D_v(Z_v, X_v, k_v^1, k_v^2)}{\partial Z_v}dv \tag{61}$$

where $k_s^{1\lambda}$ and $k_s^{2\lambda}$ are the optimal switching rates in situation $S^\lambda(t, z, x)$.[37] By Lemma 2 and equation (24), $\partial D_v(Z_v, X_v, k_v^1, k_v^2)/\partial Z_v > \underline{\alpha}e^{-N_2}$ whenever $g(v, Z_v) \in [\underline{w}, \overline{w}]$. Since $k_s^{1\lambda} + k_s^{2\lambda} \leq 2K$,

$$\frac{d\left[\Phi^\infty_\lambda(t, z + \lambda, x)\right]}{d\lambda} \geq \underline{\alpha}e^{-N_2}E\int_{v=t}^{\infty} e^{-(r+2K)(v-t)}1\left(g(v, Z_v) \in [\underline{w}, \overline{w}]\right)dv \geq \underline{\alpha}e^{-N_2}\Upsilon(c) \tag{62}$$

where $c > 0$ is is any constant such that $g(t, z + \lambda) \in [\underline{w} - c, \overline{w} + c]$ and $\Upsilon(c) > 0$ is the minimum expected discounted (at rate $r + 2K$) amount of time $v > t$ that $g(v, Z_v)$ is expected to spend in the non-dominance region $[\underline{w}, \overline{w}]$, given that $g(t, Z_t)$ is within c of this region (i.e., that $g(t, Z_t) \in [\underline{w} - c, \overline{w} + c]$). $\Upsilon(c)$ is positive because the variance and drift of W are bounded in absolute value. Importantly, $\underline{\alpha}e^{-N_2}\Upsilon(c)$ is independent of (t, λ, z, x), as long as $g(t, z + \lambda) \in [\underline{w} - c, \overline{w} + c]$.

By definition, λ_∞ is the infimum of numbers λ such that for all states (t, z, x), $\underline{\pi}(\Phi^\infty_{\lambda_\infty}(t, z, x), x)$ (the lowest possible rate of change at (t, z, x) when others are

[37]By the envelope theorem, equation (61) holds path-by-path (i.e., if $(Z_v - Z_t)_{v\geq t}$ is held constant as λ is varied); but the distribution of these paths is the same in all situations S_λ, so the equality holds in expectation as well. The envelope theorem applies even though $k_s^{R\lambda}$ and $k_s^{L\lambda}$ need not be continuous in λ. By construction, $k_s^{R\lambda}$ and $k_s^{L\lambda}$ are left continuous and monotonically increasing in λ; hence, either λ is a point of continuity of $k_s^{R\lambda}$ and $k_s^{L\lambda}$, in which case $d\lambda$ can be chosen small enough that $k_s^{R,\lambda+\varepsilon}$ and $k_s^{L,\lambda+\varepsilon}$ are close to $k_s^{R\lambda}$ and $k_s^{L\lambda}$ for $\varepsilon \in [0, d\lambda]$, or else λ is a point of right-discontinuity of either $k_s^{R\lambda}$ or $k_s^{L\lambda}$, in which case $d\lambda$ can be chosen small enough that $k_s^{R,\lambda+\varepsilon}$ and $k_s^{L,\lambda+\varepsilon}$ are close to $\lim_{\varepsilon\downarrow 0} k_s^{R,\lambda+\varepsilon}$ and $\lim_{\varepsilon\downarrow 0} k_s^{L,\lambda+\varepsilon}$, respectively, for $\varepsilon \in [0, d\lambda]$. Since the sample path $(Z)_{v\geq t}$ changes continuously as λ is varied, $\lim_{\varepsilon\downarrow 0} k_s^{R,\lambda+\varepsilon}$ and $\lim_{\varepsilon\downarrow 0} k_s^{L,\lambda+\varepsilon}$ must give the same payoffs to R and L as $k_s^{R\lambda}$ and $k_s^{L\lambda}$ do at λ. Thus, (61) holds at points of discontinuity in λ if we reinterpret $k_s^{R\lambda}$ and $k_s^{L\lambda}$ as $\lim_{\varepsilon\downarrow 0} k_s^{R,\lambda+\varepsilon}$ and $\lim_{\varepsilon\downarrow 0} k_s^{L,\lambda+\varepsilon}$, respectively, which suffices for equation (62).

expected to play according to the translation of the left continuous version of the upper bound on $\dot{X}_t$ downward by λ_∞) is at least $\widetilde{\pi}(\Phi^\infty(t, z-\lambda, x), x)$ (the translation of the left continuous version of the upper bound on $\dot{X}_t$ downward by λ). Hence, for any $\varepsilon > 0$ there must be a state $(t^\varepsilon, z^\varepsilon, x^\varepsilon)$ such that

$$\underline{\pi}(\Phi^\infty_{\lambda_\infty}(t^\varepsilon, z^\varepsilon, x^\varepsilon), x^\varepsilon) < \widetilde{\pi}(\Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon), x^\varepsilon). \tag{63}$$

Otherwise, the infimum could be no greater than $\lambda_\infty - \varepsilon$, a contradiction. Since (63) implies that players at $(t^\varepsilon, z^\varepsilon, x^\varepsilon)$ may choose switching rates that differ from those chosen at $(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon)$, either z^ε or $z^\varepsilon - \lambda_\infty + \varepsilon$ must lie in the non-dominance region, so each can be no further than $\lambda_0 > \lambda_\infty - \varepsilon$ away from the non-dominance region.

We now show by contradiction (63) cannot hold for all $\varepsilon > 0$ unless $\lambda_\infty = 0$. By part (iv) of Lemma 10, for all $\varepsilon' > 0$ there is a $\delta > 0$, independent of $(t^\varepsilon, z^\varepsilon, x^\varepsilon)$, such that if $\varepsilon < \delta$ then $\Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon) < \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty, x^\varepsilon) + \varepsilon$. By (62),

$$\begin{aligned}\Phi^\infty_{\lambda_\infty}(t^\varepsilon, z^\varepsilon, x^\varepsilon) &= \Phi^\infty_{\lambda_\infty}(t^\varepsilon, (z^\varepsilon - \lambda_\infty) + \lambda_\infty, x^\varepsilon) \\ &\geq \Phi^\infty_0(t^\varepsilon, (z^\varepsilon - \lambda_\infty) + 0, x^\varepsilon) + \underline{\alpha} e^{-N_2}\Upsilon(\lambda_0)\lambda_\infty \\ &= \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty, x^\varepsilon) + \underline{\alpha} e^{-N_2}\Upsilon(\lambda_0)\lambda_\infty \\ &\geq \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon) - \varepsilon + \underline{\alpha} e^{-N_2}\Upsilon(\lambda_0)\lambda_\infty\end{aligned}$$

For $\varepsilon < \underline{\alpha} e^{-N_2}\Upsilon(\lambda_0)\lambda_\infty$, $\Phi^\infty_{\lambda_\infty}(t^\varepsilon, z^\varepsilon, x^\varepsilon) > \Phi^\infty(t^\varepsilon, z^\varepsilon - \lambda_\infty + \varepsilon, x^\varepsilon)$. By part 2 of Lemma 4, this contradicts (63). This shows that $\lambda_\infty = 0$.

Consequently, the equilibrium is unique wherever $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$ is continuous in Z_t. By Lemma 9, $\pi(\Phi^\infty(t, z, x), x)$ is weakly increasing in z and bounded, so for any (t, x), $\pi(\Phi^\infty(t, z, x), x)$ is almost everywhere continuous in z. Hence, by part 2 of Lemma 7, with probability one the path of X that results from any path of Z does not depend on whether players play according to the right or left continuous version of $\pi(\Phi^\infty(t, Z_t, X_t), X_t)$. (Intuitively, $\dot{X}_t$ is almost always the same and it is bounded, so $X_t = X_0 + \int_{v=0}^t \dot{X}_v dv$ is the same in the two cases.) Thus, for almost any path of Z, there is a unique equilibrium path of X.

Theorems 2-5 are immediate from the above arguments. The proof that the equilibrium is time-independent with Brownian shocks is available from the corresponding author on request. Q.E.D.$_{\text{Theorems 1-5}}$

Proof of THEOREM 6. Fix a time t. Let k and k' be switching rates chosen in mode 2 at states (w, x) and (w', x), where $w > w'$. Define $y' = V^1(w', x) - V^2(w', x)$ and $y = V^1(w, x) - V^2(w, x)$. By Relative Payoff Monotonicity, $y > y'$. By the Switching Rate Rule, $k'y' - c^2(k', x) \geq ky' - c^2(k, x)$ while $ky - c^2(k, x) \geq$

$k'y - c^2(k', x)$. Subtracting, we obtain $(k' - k)(y' - y) \geq 0$, so $k' \geq k$. The proof for mode 1 is analogous. Q.E.D.$_{\text{Theorem 6}}$

Proof of THEOREM 7. This follows directly from Theorem 6. Q.E.D.$_{\text{Theorem 7}}$

Proof of THEOREM 8. For

$$k^2(1 - X) - k^1 X = C \tag{64}$$

we must have $X = \frac{k^2 - C}{k^2 + k^1}$. Given the interval from which C is drawn, the expression $\frac{k^2 - C}{k^2 + k^1}$ must be strictly increasing in k^2 and strictly decreasing in k^1. This establishes that $\underline{X} \leq \overline{X}$, with equality only if $\underline{K}^1 = \overline{K}^1$ and $\underline{K}^2 = \overline{K}^2$. It also implies that X is maximized subject to $\dot{X} = C$ when $(k^1, k^2) = \left(\underline{K}^1, \overline{K}^2\right)$; substituting, its maximum value is $\overline{X}$. Analogously, its minimum value is $\underline{X}$. Consequently, $\dot{X}$ can feasibly equal C for all X in $[\underline{X}, \overline{X}] \cap [0, 1]$ as claimed. If $X > \overline{X}$, then

$$\dot{X} < k^2(1 - \overline{X}) - k^1 \overline{X} \leq \overline{K}^2(1 - \overline{X}) - \underline{K}^1 \overline{X} = C$$

as claimed. The proof that $\dot{X} > C$ for all $X < \underline{X}$ is analogous. Q.E.D.$_{\text{Theorem 8}}$

Proof of THEOREM 9. Let the set of optimal switching rates in mode m at state (w, x) and time t be $k^m(w, x, t)$. Now define $\max \dot{X}(w, x, t)$ to be the maximum rate of change of X at state (w, x) at time t in equilibrium: $\max \dot{X}(w, x, t) = \max k^2(w, x, t)(1 - x) - \min k^1(w, x, t)x$. Similarly define

$$\min \dot{X}(w, x, t) = \min k^2(w, x, t)(1 - x) - \max k^1(w, x, t)x,$$

the minimum rate of change of X in equilibrium at (w, x) at time t. (While there can be more than one rate of change at certain states, this does not give rise to multiple equilibrium outcomes since this set has measure zero.) Theorem 9 follows from the following two facts:

1. For all $x \in [\underline{X}, \overline{X}]$, let $\underline{\Omega}(x, t)$ be the minimum value of w such that

   $$\max \dot{X}(w, x, t) \geq C.$$

   The Lower Isorate Curve, $\underline{\Omega}_t$, is just the set of states (w, x) satisfying $w = \underline{\Omega}(x, t)$. By Relative Payoff Monotonicity and the Single Crossing Property (Lemma 4, part 2) $\min \dot{X}(w, x, t) \geq C$ at all states (w, x) to the right of this curve and $\max \dot{X}(w, x, t) < C$ at all states (w, x) to the left.

2. For all $x \in [\underline{X}, \overline{X}]$, let $\overline{\Omega}(x,t)$ be the maximum value of w such that

$$\min \dot{X}(w,x,t) \le C.$$

The Upper Isorate Curve, $\overline{\Omega}_t$, is just the set of states (w,x) satisfying $w = \overline{\Omega}(x,t)$. By Relative Payoff Monotonicity and the Single Crossing Property (Lemma 4, part 2) $\max \dot{X}(w,x,t) \le C$ at all states (w,x) to the left of this curve and $\min \dot{X}(w,x,t) > C$ at all states (w,x) to the right.

Q.E.D.$_{\text{Theorem 9}}$

Proof of THEOREM 10. We first prove a property we will call Switching Rate Continuity. Fix a switching rate k and a time t. The following maxima and minima exist by Payoff Continuity and the closed graph property (Lemma 4, part 1, p. 45).

1. the minimum W_t such that at state (W_t, X_t), mode-2 agents are willing to choose a switching rate of k or greater.

2. the maximum W_t such that at state (W_t, X_t), mode-2 agents are willing to choose a switching rate of k or less.

3. the maximum W_t such that at state (W_t, X_t), mode-1 agents are willing to choose a switching rate of k or greater.

4. the minimum W_t such that at state (W_t, X_t), mode-1 agents are willing to choose a switching rate of k or less.

We first prove that these maxima and minima are all continuous functions of X_t. This is the property of Switching Rate Continuity. Since the four proofs are essentially the same, we present the first only. Let $F(x)$ be the minimum w such that at state (w,x), mode-2 agents are willing to choose a switching rate of k or greater. Given any x, we must show that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x' - x| < \delta$, then $|F(x') - F(x)| < \varepsilon$. Fix $w = F(x) - \varepsilon$. By definition of F, since $w < F(x)$, there is a $k' < k$ such that

$$k' \left(V^1(w,x) - V^2(w,x)\right) - c^2(k',x) > k \left(V^1(w,x) - V^2(w,x)\right) - c^2(k,x)$$

If x' is close enough to x, then by Payoff Continuity and the continuity of c^2 in x, the same inequality holds with x' in place of x. Thus, $w = F(x) - \varepsilon < F(x')$ as well. Thus, for any x and $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x' - x| < \delta$, then $F(x') > F(x) - \varepsilon$. We conclude by showing that for any x and $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x' - x| < \delta$, then $F(x') < F(x) + \varepsilon$. Fix x and $\varepsilon > 0$.

By definition of F, for any $w > F(x)$, a mode-2 agent at state (w, x) is willing to choose an update rate of at least k: for any $k' < k$,

$$k\left(V^1(w,x) - V^2(w,x)\right) - c^2(k,x) \geq k'\left(V^1(w,x) - V^2(w,x)\right) - c^2(k',x)$$
$$\text{or } (k-k')\left(V^1(w,x) - V^2(w,x)\right) \geq c^2(k,x) - c^2(k',x)$$

Now consider $w' = w + \varepsilon/2$. By Payoff Continuity, there is a $\delta' > 0$ such that if $|x' - x| < \delta'$, then

$$\left|V^1(w,x') - V^2(w,x') - \left[V^1(w,x) - V^2(w,x)\right]\right| < c\varepsilon/4$$

where c is the "strictly positive constant" in the statement of Theorem 4.[38] Thus, by Relative Payoff Monotonicity, if $|x' - x| < \delta'$ then for any $k' < k$,

$$\begin{aligned}(k-k')\left(V^1(w',x') - V^2(w',x')\right) &\geq (k-k')\left(V^1(w,x') - V^2(w,x') + c\varepsilon/2\right)\\ &\geq (k-k')\left(V^1(w,x) - V^2(w,x) + c\varepsilon/4\right)\\ &\geq c^2(k,x) - c^2(k',x) + (k-k')c\varepsilon/4\\ &\geq c^2(k,x') - c^2(k',x') + (k-k')c\varepsilon/4 - \eta(k-k')\delta'\end{aligned}$$

where the first inequality follows from Relative Payoff Monotonicity and the last inequality follows from axiom **A6**. Thus, by selecting $\delta' = \min\{\delta', c\varepsilon/4\eta\}$, we have

$$k\left(V^1(w',x') - V^2(w',x')\right) - c^2(k,x') \geq k'\left(V^1(w',x') - V^2(w',x')\right) - c^2(k',x')$$

for all $k' < k$: k is at least as good as any $k' < k$ at the state $(w + \varepsilon/2, x')$ for all x' close enough to x. Thus, $w + \varepsilon/2 \geq F(x')$. Since this holds for any $w > F(x)$, $F(x') < F(x) + \varepsilon$ as claimed: F is continuous. This proves Switching Rate Continuity.

By Payoff Continuity, $\underline{\Omega}(x,t)$ and $\overline{\Omega}(x,t)$ are continuous functions of t. We now show that $\underline{\Omega}(x,t)$ is a continuous function of x as well; the proof for $\overline{\Omega}(x,t)$ is analogous. Let us suppose that $\underline{\Omega}(x,t)$ is not continuous at x. Then for each $\varepsilon > 0$, there are x' arbitrarily close to x such that $|\underline{\Omega}(x,t) - \underline{\Omega}(x',t)| > \varepsilon$. Define $w = \underline{\Omega}(x,t)$. For such x', either (i) $\max \dot{X}(w',x') \geq C$ for some $w' < w - \varepsilon$ or (ii) $\max \dot{X}(w',x') < C$ for some $w' > w + \varepsilon$. Assume case (i), which implies $\max \dot{X}(w - \varepsilon, x') \geq C$. Since $\max \dot{X}(w,x)$ is right-continuous in w by Lemma 4, there is some $\delta > 0$ such that $\max \dot{X}(w - \varepsilon/2, x) \leq C - \delta$. By Switching Rate Continuity, by taking x' close enough to x, we can find w'' arbitrarily close

[38] Since x is fixed, this follows from Payoff Continuity alone; uniform continuity in x is not required.

to $w - \varepsilon/2$ such that $\max k^2(w'', x') \leq \max k^2(w - \varepsilon/2, x)$ and $\min k^1(w'', x') \geq \min k^1(w - \varepsilon/2, x)$. But then

$$\begin{aligned}\max \dot{X}(w'', x') &= \max k^2(w'', x')(1 - x') - \min k^1(w'', x')x' \\ &\leq \max k^2(w - \varepsilon/2, x)(1 - x') - \min k^1(w - \varepsilon/2, x)x' \\ &\leq \max \dot{X}(w - \varepsilon/2, x) + 2K\,|x' - x| \leq C - \delta + 2K\,|x' - x|\end{aligned}$$

which is strictly less than C for $|x' - x|$ small enough. But since w'' is arbitrarily close to $w - \varepsilon/2$, it can be taken to be strictly greater than $w - \varepsilon$; this implies that $\max \dot{X}(w - \varepsilon, x') < C$, a contradiction. The proof of case (ii) is analogous. This shows that $\underline{\Omega}(x, t)$ is a continuous function of x. Finally, under **A2′**, payoffs are independent of time, which implies by the Switching Rate Rule that $\dot{X}$ (and thus the two Isorate Curves) depends only on the state (W_t, X_t). Q.E.D.$_{\text{Theorem 10}}$

Proof of THEOREM 11. Consider the Isorate Curves at some fixed time t for some given rate of increase C of X.

1. Suppose the first sufficient condition holds: at least one of $\underline{K}^1$, $\underline{K}^2$, and C is not zero. Suppose $\underline{\Omega}(x_0, t) < \overline{\Omega}(x_0, t)$ at some $x_0 \in (0, 1)$. Then there are $w' > w''$ such that $\min \dot{X}(w', x_0) \leq C \leq \max \dot{X}(w'', x_0)$. By Relative Payoff Monotonicity, the relative value of being in mode 1 is strictly increasing in w. By Lemma 4, this implies that

$$\begin{aligned}&\min k^2(w', x_0)(1 - x_0) - \max k^1(w', x_0)x_0 \\ &= C \\ &= \max k^2(w'', x_0)(1 - x_0) - \min k^1(w'', x_0)x_0\end{aligned}$$

   By Switching Rate Monotonicity, there must be constants λ_1 and λ_2 such that at the state (w, x_0) for all $w \in [w'', w']$, $\min k^1(w, x_0) = \max k^1(w, x_0) = \lambda_1$ and $\min k^2(w, x_0) = \max k^2(w, x_0) = \lambda_2$. By Switching Rate Continuity (see proof of Theorem 10), for $m = 1, 2$ and for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x - x_0| < \delta$, then $\min k^m(w, x) = \max k^m(w, x) = \lambda_m$ for all $w \in [w'' + \varepsilon, w' - \varepsilon]$. But since one of C, $\underline{K}^1$, or $\underline{K}^2$ is not zero, either $\lambda_1 > 0$ or $\lambda_2 > 0$. Hence, for any $w \in [w'' + \varepsilon, w' - \varepsilon]$, $\max \dot{X}(w, x) < C$ for all $x > x_0$ such that $|x - x_0| < \delta$ and $\min \dot{X}(w, x) > C$ for all $x < x_0$ such that $|x - x_0| < \delta$. But this contradicts Isorate Curve Continuity (Theorem 10).

2. Suppose the first sufficient condition fails but the second one holds. By Relative Payoff Monotonicity, for any x there is a unique value w of the payoff parameter such that the relative value of being in mode 1, $V^1 - V^2$, is zero at (w, x). At any state (w', x) where $w' > w$ ($w' < w$), this relative

value is strictly positive (negative). Hence, by the second sufficient condition and the fact that $\underline{K}^1 = \underline{K}^2 = 0$, $\dot{X}$ must be strictly positive (strictly negative): the Isorate curves for $C = 0$ coincide. Q.E.D.$_{\text{Theorem 11}}$

Proof of THEOREM 12. By assumption, both switching cost functions are weakly increasing and left-continuous in the switching rate and Lipschitz in X_t. Substituting, $D(w, x, k^1, k^2) = D(w, x, \delta, \theta) = \omega f(w, x) + c^A(\theta, x)$. Strategic Complementarities holds by assumption that $f(w, x)$ is increasing and Lipschitz in both arguments, and c^A is weakly increasing and Lipschitz in x. Payoff Monotonicity holds since $f(w, x)$ is strictly increasing in w. There is a bounded effect of x on marginal cost by assumption. Consider now the upper Dominance Region. By taking W_t arbitrarily high, we can ensure that the surplus will remain positive for an arbitrarily long time with arbitrarily high probability, so the optimal layoff rate is zero and the optimal hiring rate is $\overline{\theta}$ for an arbitrarily long time. Thus, by equation (18), $V_t^F - V_t^U$ can be made arbitrarily close to

$$\lim_{w\to\infty} E\left[\int_{v=t}^{\infty} \exp\left(-\int_{s=t}^{v}\left[r+\delta+\overline{\theta}\right]ds\right)\left(\omega f(W_v, X_v) + c^A(\overline{\theta}, X_v)\right)dv \;\middle|\; W_t = w\right]$$

This is at least $\frac{\omega \lim_{w\uparrow+\infty} f(w,0)}{r+\delta+\overline{\theta}}$. By assumption, for any X, $c_\theta^A(\overline{\theta}, X) < \frac{\omega \lim_{w\uparrow+\infty} f(w,0)}{r+\delta+\overline{\theta}}$. The Switching Rate Rule now implies that $\theta_t = \overline{\theta}$ is strictly dominant for high enough W_t. The proof that there exists a lower Dominance Region is analogous. Q.E.D.$_{\text{Theorem 12.}}$

Proof of THEOREM 13. From standard results in probability theory,[39] player i's posterior over θ is $\theta_i^{\text{post}} \sim N\left(\frac{x_i}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2}\right)$ and so her posterior over $x_j = \theta_i^{\text{post}} + \varepsilon_j$, the sum of two independent random variables, is distributed $N\left(\frac{x_i}{1+\sigma^2}, \frac{\sigma^2}{1+\sigma^2} + \sigma^2\right)$. Let the c.d.f. of this distribution be $F(x_j|x_i)$.

We will look for cutoff equilibria in which a player plays R if her signal exceeds some threshold x^* and L otherwise. For this to be an equilibrium, a player with

[39] Suppose we have a variable $x_0 \sim N(\overline{x}, V_0)$ to estimate. We observe the variables $x_j = x_0 + \varepsilon_j$ for $j = 1, ..., J$, where each $\varepsilon_j \sim N(0, V_j)$ is independent of every $\varepsilon_{j'}$ and of x_0. Define the precision of variable j to be $w_j = 1/V_j$. Then the posterior distribution of x_0 is

$$x_0^{\text{posterior}} \sim N\left(\overline{x} + \frac{\sum_{j=1}^{J} w_j\left[x_j - \overline{x}\right]}{\sum_{j=0}^{J} w_j}, \frac{1}{\sum_{j=0}^{J} w_j}\right)$$

signal $x_i = x^*$ must be indifferent between R and L. A player i who thinks her opponent will play R with probability p gets the expected payoff $cx_i + p$ from playing R and $1 - p$ from playing L. The relative payoff from playing R is thus $cx_i + 2p - 1$. Since $p = 1 - F(x^*|x_i)$, in a threshold equilibrium with cutoff x^* a player's relative payoff from playing R if her signal is x_i is $f(x_i, x^*) = cx_i + 2\left[1 - F(x^*|x_i)\right] - 1$. x^* is a threshold equilibrium if $f(x^*, x^*) = 0$, $f(x, x^*) \geq 0$ for $x > x^*$, and $f(x, x^*) \leq 0$ for $x < x^*$. Since f is strictly increasing in its first argument, the last two conditions always hold.

By inspection, f is continuous, so $f(x, x) > 0$ for $x > 1/c$ and $f(x, x) < 0$ for $x < -1/c$. In addition, $F(0|0) = 1/2$, so $f(0, 0) = 0$. Thus, if for small enough c we can show that $f(x, x)$ is decreasing in x when $x = 0$, there must be points $1/c \geq x^1 > 0 > x^2 \geq -1/c$ such that $f(x^n, x^n) = 0$ for $n = 1, 2$. As long as $\sigma^2 > 0$, $\frac{dF(x|x)}{dx}\Big|_{x=0} < 0$, since $F(0|0) = 1/2$ and $F(x|x) < 1/2$ for $x > 0$. Thus, by choosing c close enough to zero, one obtains $\frac{df(x,x)}{dx}\Big|_{x=0} = c - 2\,\frac{dF(x|x)}{dx}\Big|_{x=0} < 0$, as claimed. Q.E.D.$_{\text{Theorem 13}}$

Proof of LEMMA 1. The argument of Lemma 3 implies that

$$\left|V_w^1 - V_w^2\right| \leq \frac{\max_{x\in[0,1]} |\Delta u(0, x)|}{r} + E\int_{s=w}^{\infty} e^{-r(s-w)}\overline{\alpha}\,|W_s|\,ds$$

and

$$E\,|W_s| \leq |EW_s| + \sqrt{\text{Var}\,(W_s)}$$

where all expectations are conditioned on W_w. If $\nu = 0$, then as of time $w < s$, $EW_s = W_w$ and $\text{Var}(W_s) = \sigma^2(s - w)$, respectively. Thus,

$$|EW_s| \leq |W_w| + \sigma\sqrt{s - w}$$

Hence,[40]

$$\begin{aligned}\left|V_w^1 - V_w^2\right| &\leq \frac{\max_{x\in[0,1]} |\Delta u(0, x)|}{r} + \int_{s=w}^{\infty} e^{-r(s-w)}\overline{\alpha}\left(|W_w| + \sigma\sqrt{s - w}\right) ds \\ &\leq \frac{\max_{x\in[0,1]} |\Delta u(0, x)|}{r} + \overline{\alpha}\left[\frac{|W_w|}{r} + \sigma + \frac{\sigma}{r^2}\right] \\ &= c_0\,|W_w| + c_1\end{aligned}$$

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[40]This uses the fact that for any $z \geq 0$, $\sqrt{z} \leq 1$ if $z \leq 1$ and $\sqrt{z} \leq z$ if $z \geq 1$, and $e^{-rz} \leq 1$, so

$$\int_{s=v}^{\infty} e^{-r(s-v)}\sqrt{s - v}ds \leq 1 + \frac{1}{r^2}$$

This establishes the first bound in part 1. If $\nu > 0$, then by Lemma 2, as of time $w \le s$, W_s is normal with mean $e^{-\nu(s-w)}W_w + \left(1 - e^{-\nu(s-w)}\right) W_{\text{mean}}$ and variance $\frac{\sigma^2}{2\nu}\left(1 - e^{-2\nu(s-w)}\right)$. Thus,

$$|EW_s| \le |W_w| + |W_{\text{mean}}| \left(1 - e^{-\nu(s-w)}\right) + \sigma\sqrt{\frac{1 - e^{-2\nu(s-w)}}{2\nu}}$$

Hence,[41]

$$\begin{aligned}
|V_w^1 - V_w^2| &\le \frac{\max_{x\in[0,1]} |\Delta u(0,x)|}{r} \\
&\quad + \int_{s=w}^{\infty} e^{-r(s-w)}\overline{\alpha}\left(|W_w| + |W_{\text{mean}}| \left(1 - e^{-\nu(s-w)}\right) + \sigma\sqrt{\frac{1 - e^{-2\nu(s-w)}}{2\nu}}\right) ds \\
&\le \frac{\max_{x\in[0,1]} |\Delta u(0,x)|}{r} + \overline{\alpha}\left[\frac{|W_w| + |W_{\text{mean}}|}{r} - \frac{|W_{\text{mean}}|}{r+\nu} + \frac{\sigma}{r\sqrt{2\nu}}\right] \\
&= c_0 |W_w| + c_1'
\end{aligned}$$

This establishes the second bound in part 1, for the case of mean reversion.

We now prove part 2. First suppose $\nu = 0$. Denote by $\Delta V(w)$ the relative value of being in mode 1 if an agent believes that all other agents will remain in mode 2 forever and she picks optimal switching rates given these beliefs. Since this is the scenario that makes mode 1 the least appealing, it yields a lower bound on $V_t^1 - V_t^2$. By the Upper Bounds, $\Delta V(0) \ge -c_1$. By the Envelope Theorem and (26), for any $\varepsilon > 0$, $\Delta V(w+\varepsilon) - \Delta V(w) \ge \frac{\alpha\varepsilon}{r+2K}$, since increasing the starting value W_t by some $\varepsilon > 0$ is equivalent to adding ε to W_w for all times $w \ge t$ but retaining the same probability distribution over sample paths $(W_w)_{w\ge t}$.[42] Thus, $\Delta V(w) \ge -c_1 + \frac{\alpha w}{r+2K}$ as claimed. The proof for $V_t^2 - V_t^1$ is analogous.

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[41] This uses the fact that

$$\begin{aligned}
\int_{s=w}^{\infty} e^{-r(s-w)}\overline{\alpha}\sigma\sqrt{\frac{1 - e^{-2\nu(s-w)}}{2\nu}} ds &= \frac{\overline{\alpha}\sigma}{\sqrt{2\nu}}\int_{s=w}^{\infty} e^{-r(s-w)}\sqrt{1 - e^{-2\nu(s-w)}} ds \\
&\le \frac{\overline{\alpha}\sigma}{\sqrt{2\nu}}\int_{s=w}^{\infty} e^{-r(s-w)} ds = \frac{\overline{\alpha}\sigma}{r\sqrt{2\nu}}
\end{aligned}$$

[42] This application of the Envelope Theorem is valid even though the switching rates need not be continuous in the payoff parameter. This is because at points where a small change in b causes a switching rate to jump from k to k' at some date $w \ge t$ for some sample path $(B_w)_{w\ge t}$, the agent is indifferent between the two rates k and k', so the jump does not affect the relative value of being in mode 1.

Now suppose $\nu > 0$. By the Upper Bounds, $\Delta V(0) \geq -c_1'$. By Lemma 2, increasing the starting value W_t by some $\varepsilon > 0$ is now equivalent to adding $\varepsilon e^{-\nu(w-t)}$ to W_w for all times $w \geq t$ and retaining the same probability distribution over paths of W. (I.e., the new payoff parameter follows the path $\varepsilon e^{-\nu(w-t)} + W_w$ where the probability distribution over the paths $(W_w)_{w \geq t}$ is unchanged.) Hence, by the Envelope Theorem and (26), $\Delta V(w+\varepsilon) - \Delta V(w) \geq \frac{\alpha\varepsilon}{r+2K+\nu}$, so $\Delta V(w) \geq -c_1' + \frac{\alpha w}{r+2K+\nu}$. The proof for $V_t^2 - V_t^1$ is analogous. Q.E.D.$_{\text{Lemma 1}}$

References

[1] Bass, Richard F., and Krzysztof Burdzy. 1999. "Stochastic Bifurcation Models." *Annals of Probability* 27:50-108.

[2] Bass, Richard F., and Krzysztof Burdzy. 2001. "The supremum of Brownian local times on Hölder curves." Forthcoming, *Ann. Inst. Henri Poincaré*.

[3] Bianchi, Marco, and Gylfi Zoega. 1998. "Unemployment Persistence: Does the Size of the Shock Matter?" *Journal of Applied Econometrics* 13:283-304.

[4] Blanchard, Olivier J., and Stanley Fisher. 1990. *Lectures on Macroeconomics*. Cambridge, Massachusetts: MIT.

[5] Blanchard, Olivier J., and Lawrence Summers. 1987. "Hysteresis in Unemployment." *European Economic Review* 31:288-295.

[6] Burdzy, K., D.M. Frankel, and A. Pauzner. 2001. "Fast Equilibrium Selection by Rational Players Living in a Changing World." *Econometrica* 69:163-189.

[7] Carlsson, H., and E. van Damme. 1993. "Global Games and Equilibrium Selection." *Econometrica* 61:989-1018.

[8] Cass, David, and Karl Shell. 1983. "Do Sunspots Matter?" *Journal of Political Economy* 91:193-227.

[9] Chamley, Christophe. 1999. "Coordinating Regime Switches." *Quarterly Journal of Economics* 114:869-905.

[10] Chauvet, Marcelle, Chinhui Juhn, and Simon Potter. 2002. "Markov Switching in Disaggregate Unemployment Rates." *Empirical Economics* 27:205-232.

[11] Clemente, Jesús, Luis Lanaspa, and Antonio Montañés. 2002. "The Unemployment Structure of the U.S. States." Mimeo.

[12] Frankel, David M., Stephen Morris, and Ady Pauzner. 2000. "Equilibrium Selection in Global Games with Strategic Complementarities." Forthcoming, *Journal of Economic Theory*.

[13] Frankel, David M., and Ady Pauzner. 2000. "Resolving Indeterminacy in Dynamic Settings: the Role of Shocks." *Quarterly Journal of Economics* 115:283-304.

[14] Froot, Kenneth A., David S. Scharfstein, and Jeremy C. Stein. 1992. “Herd on the Street: Informational Inefficiencies in a Market with Short-Term Speculation.” *Journal of Finance* XLVII:1461-1484.

[15] Fudenberg, D., and J. Tirole, 1991. *Game Theory*. Cambridge: MIT Press.

[16] Hamilton, James D., 1989. “A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle.” *Econometrica* 57:357-384.

[17] Howitt, Peter. 1988. “Business Cycles with Costly Search and Recruiting.” *Quarterly Journal of Economics* 103:147-166.

[18] Howitt, Peter, and R. Preston McAfee. 1992. “Animal Spirits.” *American Economic Review* 82:493-507.

[19] Kajii, Atsushi, and Stephen Morris. 1997. “The Robustness of Equilibria to Incomplete Information.” *Econometrica* 65:1283-1309.

[20] Lucas, Robert E., Jr. 1978. “Asset Prices in an Exchange Economy.” *Econometrica* 46:1429-1445.

[21] Matsuyama, Kiminori. 1991. “Increasing Returns, Industrialization, and Indeterminacy of Equilibrium.” *Quarterly Journal of Economics* 106:617-650.

[22] P. Milgrom and J. Roberts. 1990. “Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities.” *Econometrica* 58:1255-1277.

[23] Morris, S. (1999) “Potential Methods in Interaction Games.”

[24] Morris, S., R. Rob and H. S. Shin (1995). “p-Dominance and Belief Potential.” *Econometrica* 63:145-157.

[25] Morris, S. and H. S. Shin (1998). “Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks.” *American Economic Review* 88:587-597.

[26] Morris, S., and H.S. Shin (2000). "Rethinking Multiple Equilibria in Macroeconomic Modeling." NBER Macroeconomic Annual 2000, pp. 139-182.

[27] Morris, S., and H. S. Shin (2004). “Coordination Risk and the Price of Debt.” *European Economic Review* 48:133-153.

[28] Øksendal, Bernt. 1998. *Stochastic Differential Equations: An Introduction with Applications.* Berlin: Springer-Verlag.

[29] Pissarides, Christopher A. 1992. "Loss of Skill During Unemployment and the Persistence of Employment Shocks." *Quarterly Journal of Economics* 107:1371-1391.

[30] Rogers, L.C.G. 1989. “A Guided Tour Through Excursions.” *Bulletin of the London Mathematical Society* 21:305-341.

[31] Rogers, L.C.G., and D. Williams. 2000. *Diffusions, Markov Processes and Martingales. Volume 1: Foundations.* U.K.: Cambridge University Press.

[32] Shiller, Robert J. 1989. *Market Volatility.* Cambridge, Mass.: MIT Press.

[33] Shell, Karl. 1977. “Monaie et Allocation Intertemporelle.” CEPREMAP memorandum.

[34] Skalim, Joakim, and Timo Teräsvirta. 1999. “Modelling Asymmetries and Moving Equilibria in Unemployment Rates.” Stockholm School of Economics, Working Paper Series in Economics and Finance, No. 262.

[35] Zeeman, E.C. 1971. *The Geometry of Catastrophe.* Mathematics Institute, University of Warwick.

Symbol	Definition	First Used
t	time	p. 5
δ	attrition rate (HM)	p. 5
X	proportion of jobs that are filled (HM)	p. 5
X	proportion of players in mode 1	p. 6
$f(X)$	worker-firm surplus - no shocks (HM)	p. 5
$f(W,X)$	worker-firm surplus - shocks (HM)	p. 5
ω	firms' fixed fraction of surplus (HM)	p. 5
θ	firm hiring intensity (HM)	p. 5
$\overline{\theta}$	maximum hiring intensity (HM)	p. 5
$c^A(\theta,X)$	firm hiring cost function (HM)	p. 5
W	the (stochastic) payoff parameter	p. 5
E_t	expectation as of time t	p. 6
r	constant discount rate	p. 6
ϕ_s	indicator function for filled job (HM)	p. 6
k^m	switching rate of players in mode $m=1,2$	p. 6
$\left[\underline{K}^m, \overline{K}^m\right]$	interval from which players choose k^m	p. 6
K	upper bound on switching rate in any mode	p. 6
$c^m(k^m,X)$	switching cost function in mode $m=1,2$	p. 7
$u(m,W,X)$	direct utility flow in mode $m=1,2$	p. 7
$\mu(t,W)$	drift of general Ito process	p. 8
$\sigma^2(t,W)$	variance of general ito process	p. 8
μ_t	constant term in W's drift under assumption **A2**	p. 8
ν_t	linear coefficient in W's drift under assumption **A2**	p. 8
σ_t^2	variance term in W's drift under assumption **A2**	p. 8
B	Brownian motion with zero drift and unit variance	p. 8
N_1, N_2	Bounds on parameters of W (Assumption **A2**)	p. 8
μ, σ	Constant drift and variance of W under **A2**′	p. 9

Table 1: Notation used in body of paper, part 1.

Symbol	Definition	First Used
$D(W, X, k^1, k^2)$	Relative payoff flow in mode 1	p. 9
β	Upper bound of effect of X on relative payoff flow in mode 1	p. 9
$\overline{\alpha}$	Upper bound of effect of W on relative payoff flow in mode 1	p. 9
(w_1, w_2)	Interval over which D strictly increasing in W	p. 9
$\overline{w}$	Boundary of upper dominance region	p. 10
$\underline{w}$	Boundary of lower dominance region	p. 10
η	Upper bound on effect of X on marginal switching costs	p. 10
c_k^m	Marginal cost of raising switching rate when in mode $m = 1, 2$	p. 10
$V_t^m = V^m(W, X, t)$	Continuation payoff of player who is locked into mode $m = 1, 2$	p. 11
$V^1 - V^2$	Relative value of being in mode 1	p. 4.1
$\dot{X}$	Rate of increase of X	p. 12
C	Rate of increase of X used in definition of Isorate curves	p. 14
$[\underline{X}, \overline{X}]$	Interval in which X can feasibly rise at some given rate	p. 14
$V^F = V^F(W, X, t)$	Continuation payoff of firm with filled position in HM	p. 19
$V^U = V^U(W, X, t)$	Continuation payoff of firm with unfilled position in HM	p. 19

Table 2: Notation used in body of paper, part 2.