Portfolio Liquidation and Security Design with Private Information†

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ABSTRACT. An issuer seeks to liquidate all or part of a portfolio of heterogeneous assets about which she has private information. In contrast with the single-asset case, greater optimism may lead her to sell *more* of a given asset. But if assets can be ranked by their informational sensitivity then the issuer first liquidates her least sensitive assets, which, under weak assumptions, are her more senior assets (following Myers's pecking order hypothesis). When the issuer can design new securities after learning her information then she has two equivalent, optimal strategies. She may design and sell, *ex post*, a single standard debt security whose face value is decreasing in her information. Or she may pool and tranche her assets *ex ante* into many prioritized debt securities and sell, *ex post*, those tranches whose seniority exceeds a threshold that is increasing in her information. In each case, the issuer retains more of her cash flow when information asymmetries are greater, as has been found in the case of no-documentation loans.

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1. Introduction

This paper studies how an issuer should sell assets about which she has private information. Examples include a firm that raises capital by selling debt, equity, and/or hybrid securities; an investor seeking to liquidate a portfolio; and a bank that sells claims to the repayments of loans that it has issued.

One can imagine various motivations for such a sale. A firm, for instance, may need cash to invest in a worthwhile project. A bank may seek funds in order to issue more loans or to comply with regulations. We capture these varied reasons by assuming, simply, that the issuer is less patient than investors. As this creates gains from trade, the issuer would sell her entire portfolio if information were symmetric. In the more realistic asymmetric information case, such an offer might be interpreted by investors as a negative signal. The issuer may thus choose to sell less, in order to signal optimism.

In most prior work on this problem, the issuer sells a single asset to investors. Examples include Leland and Pyle (1979), Myers and Majluf (1984), DeMarzo and Duffie (1999). In the equilibria of these models, the issuer signals optimism by retaining a larger portion of her single asset. DeMarzo (2005) studies a multiple-asset model, in which the quantity sold of each asset can be used to signal an independent dimension of the issuer's information. He shows that in that setting, the portfolio liquidation decision can be solved asset by asset as in the single-asset case.[1]

We depart from the above papers in assuming that the issuer holds many assets, but learns information about a one-dimensional common factor such that good news for one asset is good news for all. This condition holds, for example, if each asset's payout is a nondecreasing function of a common underlying cash flow, and "better" information raises the cash flow distribution in a first-order stochastic dominance sense. Other examples include a holder of a bond (or CDS) portfolio with information regarding volatility, or an issuer of mortgage-backed securities with information about default or pre-payment risk. While the model is stylized, it yields strong predictions about optimal asset sales, liquidity,

[1] He (2009) considers a two-asset version of Leland and Pyle (1979). The issuer has distinct information about each asset, but because the issuer is risk-averse the cost of retaining one asset depends on retention of the other. The CARA-Normal framework precludes analysis of security design, however, which is a primary focus of our paper.

and security design. These predictions are consistent with observed practice and are confirmed in the data.

In our base model, we characterize equilibria when the issuer has a fixed set of assets to sell.[2] We show by example that a more optimistic issuer may sell *more* of an asset at a higher price.[3] This contrasts with the single-asset case. Intuitively, the issuer most efficiently signals a given increase in her information by retaining those assets whose expected payouts (or returns) rise proportionally more as a result of her rosier expectations. But the identity of these "more informationally sensitive" assets may alternate as the issuer becomes progressively more optimistic. Hence she may retain some shares of one asset to signal a given level of optimism, only to sell this asset in its entirety and retain shares of a different asset in order to signal yet greater optimism.

In the remainder of the paper, we focus on settings in which the issuer's assets can be ordered ex ante in terms of their sensitivity to the issuer's information.[4] In this setting, as the issuer's optimism grows, she first retains her most informationally sensitive asset, then her second most, and so on. Thus, an issuer will not sell any portion of a given asset unless she also sells her entire holdings of her less informationally-sensitive assets.

We next apply this result to financial assets such as the debt, equity, and mezzanine tranches used in securitization deals. We assume, first, that the payout of each of the issuer's assets is a nondecreasing function of a common cash flow.[5] For instance, the issuer may be a firm that sells debt and equity that are secured by future operating profits, or a bank that wishes to sell tranches of a common loan pool. We assume, moreover, that an increase in the issuer's information lowers the hazard rate function of the underlying cash

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[2] In order to obtain a unique equilibrium, we assume the Intuitive Criterion of Cho and Kreps (1987): beliefs following an unexpected action must be concentrated only on types that could possibly hope to gain from the deviation. By allowing the issuer to underprice assets and ration their allocation, we eliminate implausible equilibria, even though no underpricing occurs in equilibrium.

[3] While such behavior seems rare, its intuition plays a key role in our subsequent, more realistic results.

[4] More precisely, we assume that if one asset's expected value rises proportionally more than another's for a given increase in the issuer's information, then it does so for *any* such increase.

[5] A common underlying cash flow is not needed for our preceding results, which assume only that the expected payout of each asset is nondecreasing in the issuer's information. For instance, each asset may be secured by a distinct cash flow, whose distribution is increasing in the issuer's information.

flow distribution. This *Hazard Rate Ordering* (HRO) property is stronger than first-order stochastic dominance, but weaker than the usual monotone likelihood ratio property.[6]

In this setting, we show that the issuer's senior assets are less informationally sensitive, and consequently more liquid, than her junior ones. Hence as her optimism grows, she will retain her junior assets first. For instance, the issuer will sell senior debt before selling junior debt, and junior debt before equity, as predicted by Myers's (1984) Pecking Order Hypothesis.[7] The result uses a novel and weak notion of seniority: one asset is senior to another if its payout grows at a slower rate than the other's in the underlying cash flow. While this ranks the various classes of debt by their priority in bankruptcy, it also extends to other common claims. For instance, equity is senior to a call option that is written on it.

The preceding results assume that the issuer is endowed with a fixed set of assets that she wishes to sell. We next permit security design: we let the issuer construct new securities that are secured by her initial assets. In prior work on security design, DeMarzo and Duffie (1999) studied the *ex-ante*, single-security case: the issuer designs a single monotone security, learns her information, and chooses a proportion of her security to sell. We study the general case. We assume the issuer has a given set of initial assets that are secured by a common underlying cash flow. She first designs any number of interim securities that are secured by her initial assets. She then learns her information and designs any number of *ex-post* securities that are secured by her interim securities. This general setting includes DeMarzo and Duffie (1999) as a special case.

In this setting, the issuer has two equivalent, optimal strategies.[8]

1. She can pool her initial assets, see her information, and issue a single standard debt security whose face value is decreasing in her information.

[6] See Shaked and Shanthikumar (2007), p. 43, Thm. 1.C.1.
[7] While Myers's hypothesis has mixed empirical support in the corporate finance literature (e.g. Fama and French (2002), Flannery and Rangan (2006), Opler *et al* (1999), and Shyam-Sunder and Myers (1999)), our context is more suited to the sale of asset-backed securities, where retention rates decline and liquidity improves with seniority (see e.g. Begley and Purnanandam (2017) and Franke and Krahnen (2007)).
[8] They are equivalent in the sense that they yield the same type-contingent revenue for the issuer and promise the same aggregate final payout to investors for any cash flow realization.

2. She can pool her initial assets, tranche the result into a maximal set of prioritized debt securities, see her information, and sell those tranches whose seniority exceeds a threshold that is increasing in her information.[9] This optimal strategy mirrors the usual structure of loan-pool securitization deals.[10]

In the first strategy, the issuer writes a single standard debt security *after* seeing her information. DeMarzo and Duffie (1999) assume the issuer must design her security *before* seeing her information. While they also find that debt is optimal, there is a key difference. With ex-ante debt, the issuer signals optimism by selling fewer shares, thus lowering the payout to investors by a constant proportion in *all* states. With ex-post debt, the issuer instead reduces the face value, which lowers investors' payout in nondefault states while leaving it unchanged in default states. Thus ex-post debt (or, equivalently, ex-ante tranching) provides a more efficient way to signal optimism, as it allows the issuer to retain more of the cash flow in those states whose probabilities have risen the most as the result of her greater optimism.[11]

The above security design results assume that the issuer's type and cash flow are discretely distributed. For wider applicability, we also study the limit as the issuer's information and her realized cash flows become continuous.[12] In this limit, the face value of the optimal *ex-post* debt security is given by a simple differential equation that has a unique solution. Moreover, this equation describes an equilibrium of the continuous model, and the issuer's expected profits in the discrete model converge uniformly to her expected profits in this equilibrium.[13] Hence, the continuous model is a good approximation to the discrete model.

[9] In both strategies, the issuer pools her initial assets. Intuitively, any function of these assets' payouts can be rewritten as a function solely of the underlying cash flow, so pooling them does not limit the issuer's subsequent actions in any way.

[10] For empirical examples, see Begley and Purnanandam (2017) and Franke and Krahnen (2007). We are aware of one alternative explanation for this structure: Dang, Gorton, and Holmström (2015) show that an uninformed issuer may retain an equity tranche to discourage information gathering by investors.

[11] Nachman and Noe (1994) also show that debt is the optimal ex post security design when an issuer must raise a *fixed* amount of cash. The all-or-nothing nature of the financing problem leads to a pooling equilibrium in which each type of issuer sells standard debt with the same face value. A pooling equilibrium with debt also emerges when an informed issuer sells to a single large investor (Biais and Mariotti 2005); here, the issuer pools in order to shield her informational rents from the investor.

[12] Applications of these continuum results include section 4.2 of DeMarzo (2005) and section 7.2 of Frankel and Jin (2015). DeMarzo (2005) cites an earlier version of this paper, DeMarzo (2003), for these results.

[13] Manelli (1996, 1997) studies a general sequence of finite signaling games (which have finite type and message spaces) that converges to a continuous signaling game that, like ours, has compact type and message

The continuous solution can be described heuristically as follows. We normalize the investors' discount factor to one and let $\delta \in (0,1)$ denote the issuer's discount factor. The issuer's optimism is indexed by her type $t \geq 0$. On seeing *t*, the issuer sells a single *ex post* debt security, secured by her cash flow, whose face value D_t is determined as follows. The most pessimistic issuer (whose type *t* is zero) sells everything: the face value D_0 of her debt security equals the highest possible value of her underlying cash flow. As her optimism grows, she lowers the face value at the rate

$$\frac{dD_t}{dt} = -\frac{v_2(D_t, t)}{(1-\delta)\left[1-\pi(D_t, t)\right]} < 0, \tag{1}$$

where $v(D,t)$ is the conditional expected payout of a standard debt security with face value D, $\pi(D,t)$ is the conditional default probability of this security, and subscripts denote partial derivatives.[14] As the equilibrium is separating, competition among investors drives the security's price to its conditional expected payout $v(D_t, t)$.

In equation (1), the rate at which a more optimistic issuer lowers her face value is equal to the benefit-cost ratio: the informational benefit she receives from the decrease, divided by the social cost of retaining more of the cash flow.[15] This equation serves as a building block in the applied models of DeMarzo (2005) and Frankel and Jin (2015). Its optimality was first derived heuristically in an earlier version of this paper, DeMarzo (2003), in a setting in which the issuer designs a single security after seeing her information. In the

spaces. Manelli (1996) shows that any sequence of equilibria of the finite games has a convergent subsequence that converges to some equilibrium of the continuous game. Similarly, Manelli (1997) shows that if a sequence of equilibria of the finite games, each of which satisfies the Never a Weak Best Response criterion of Kohlberg and Mertens (1986), converges to some equilibrium of the continuous game, then this limiting equilibrium satisfies the same criterion. However, because Manelli's finite games have finite message space while ours has an infinite message space - the set of monotone securities - his results cannot be directly applied to our setting.

[14] More precisely, $v(D,t) = E\left[\min\{D,Y\} \mid t\right]$ and $\pi(D,t) = \Pr(Y < D \mid t)$ where *Y* is the issuer's underlying cash flow, and $v_2(D,t) = \partial v(D,t)/\partial t$.

[15] In particular, the numerator captures the issuer's benefit of convincing investors that she is more optimistic, while the denominator equals lost social surplus from a unit decrease in the face value: the expected decline $1-\pi$ in the payout to investors, multiplied by the issuer's holding cost per unit reduction in the payout, $1-\delta$. When the informational benefit of lowering the face value is large relative to the cost, imitation by lower types is more profitable and thus, to be deterred, requires a larger decline in the face value.

current version, it is shown to be optimal even if the issuer can design any number of securities, both before and after she becomes informed.[16]

In equilibrium, a higher face value signals that the issuer is pessimistic and thus leads investors to lower their valuation of her cash flow. For this reason, if an investor of a *given* type chooses a higher face value, her securitization revenue rises by less than the true increase in value of her security. This illiquidity induces high-value issuers to retain more equity. Begley and Purnanandam (2017) confirm this result empirically by showing that when the equity (retained) tranche of a pool of residential mortgage-backed securities (RMBS's) makes up a larger proportion of the pool's face value, the loans in the pool have lower subsequent delinquency rates conditional on observables and the securities that are sold fetch higher prices conditional on their credit ratings. This is implied by (1): the issuer's face value D_t is lower, and so her equity tranche is larger, when her type t is higher which, in the case of a loan pool, is naturally interpreted as a lower expected default rate.

Begley and Purnanandam (2017) also find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation ("no-doc") loans, controlling for other loan observables. Equation (1) implies this as well if we assume that no-doc loans are more fragile: that both the default risk $\pi\left(D_t,t\right)$ and the informational sensitivity $v_2\left(D_t,t\right)$ of the issuer's security are greater for no-doc loans. On this assumption, equation (1) implies a more rapid decline in the face value D_t in the case of no-doc loans and thus a larger equity tranche for any $t>0$.[17] Intuitively, as the value of her security is more sensitive to her information, a no-doc issuer gains more from convincing investors of her greater optimism. In order to deter imitation by more pessimistic types, she must therefore send a costlier signal as her optimism grows. She does so by increasing the size of her equity tranche more rapidly.

[16] That is, an issuer in this more general setting still prefers to issue a single *ex-post* debt security with face value given by (1).

[17] This relies on the fact that the lowest-type issuer ($t=0$) always chooses the minimum equity tranche, regardless of the proportion of no-doc loans in the pool. Since her equity tranche rises faster in t in the case of no-doc loans, it is larger for any $t>0$.

The rest of the paper is organized as follows. The base model with a fixed set of assets is studied in section 2. The general security design game is analyzed in section 3. We relax a monotonicity assumption in section 4 and conclude in section 5.

2. The Asset Sale Game

Section 2.1 sets out a base model (the "Asset Sale game") in which a privately informed issuer is endowed with a fixed portfolio of assets. Selling all her assets is efficient as the issuer is relatively impatient; for instance, she may have attractive alternative investments or face liquidity or capital requirements. However, because there is asymmetric information, she may retain some of her assets in order to signal that she is optimistic and thus to obtain higher prices.

Signaling games often have multiple equilibria even with one-dimensional signals. The problem is potentially worse in our setting as the signal is multidimensional: the quantity of each asset to offer for sale. Nevertheless, we show in section 2.2 that the game has a unique equilibrium that satisfies the Intuitive Criterion.[18]

In the Asset Sale game, the most pessimistic issuer sells her entire portfolio; as her information rises, she retains those assets that she prizes the most relative to lower types, as this is the most efficient way to distinguish herself from these types. Since these most-prized assets may alternate, an issuer of intermediate optimism may retain an asset that a more optimistic issuer sells in its entirety. This nonmonotonic behavior, which cannot occur in the single-asset case, appears in two computed examples in section 2.3.

In section 2.4, we add a mild assumption that rules out such nonmonotonicity: that the issuer's assets can be ranked globally in terms of their informational sensitivity. That is, if one asset's expected value rises proportionally more than another's for a given increase in the issuer's type, then it does so for *any* such increase. We show that as the issuer's information rises, she first retains her most informationally sensitive asset, then her second most, and so on. Thus, an issuer will not sell any portion of an asset unless she also sells all of her less informationally sensitive assets in their entirety.

[18] The uniqueness argument will rely on an assumption that the issuer can set a price cap for each asset.

Section 2.5 then relates this prediction to commonly observed financial assets such as debt, equity, and the prioritized tranches of securitized loan pools. We show that under a mild distributional assumption – the hazard rate ordering property – senior assets are less informationally sensitive than junior ones. Thus, by the preceding result, she will sell her senior assets first. Importantly, this result relies on a novel and weak notion of seniority: one asset is senior to another if its payout rises more slowly in the underlying cash flow – i.e., if its relative claim to the cash flow is stronger when the cash flow is low. This weak notion of seniority lets us extend Myers's (1984) pecking-order hypothesis to a much larger class of assets. For instance, we find that a stock is senior to a call option written on the stock and thus will be sold before the option in order to raise cash.

2.1. The Base Model

The participants are a single issuer and a continuum of investors. All participants are risk-neutral and fully rational. The issuer is endowed with a portfolio of *n* assets represented by the vector $a \in \Re_+^n$, where $a_i > 0$ is the number of shares she owns of asset *i*. A holder of one share of asset $i \in \{1,\ldots,n\}$ is entitled to the random future payout F_i. The issuer has private information about these payouts, which is summarized by her type $t \in \{0,\ldots,T\}$. Conditional on the issuer's type, asset *i* has an expected payout per share of $f_i(t) \stackrel{d}{=} E[F_i \mid t]$.[19] Let $F = (F_1,\ldots,F_n) \in \Re_+^n$ denote the vector of random asset payouts and let $f(t) = E[F \mid t] \in \Re_+^n$ denote the vector of expected payouts conditional on *t*. We refer to (a, f) as the issuer's *endowment*.

We assume throughout that (a) the issuer's information can be ordered so that higher types *t* are more optimistic about the expected payout of each asset; (b) an asset's expected conditional payout is never negative (e,g., because of limited liability); and (c) even the most pessimistic issuer thinks that her portfolio has a positive expected value:[20]

[19] More precisely, the payout of asset *i* is a function $F_i(\omega)$ of an exogenous, unknown random state ω. Then $f_i(t)$ is simply the expectation of $F_i(\omega)$ with respect to the conditional distribution of ω given *t*.

[20] In the case of 2 types and 2 assets, property (a) is not needed; see section 4.

ASSUMPTION A (MONOTONE EXPECTED PAYOUTS). For $t > s$, $f(t) \geq f(s)$. In addition, $f(0) \geq 0$ and $af(0) > 0$.

This information structure is natural when the issuer is informed about a common factor that affects all of her assets. For instance, the issuer might own a portfolio of securities backed by a common pool of assets, such as the debt and equity of a single firm or the prioritized debt tranches of a loan pool, and have private information about the future value of the underlying pool.[21]

On seeing her information t, the issuer chooses a quantity $q_i \in [0, a_i]$ of each asset i to offer for sale. Let $q = (q_1, \ldots, q_n) \in \Re_+^n$ denote the row vector of chosen quantities, where $0 \leq q \leq a$.[22] The issuer may also set a price cap $\overline{p}_i \in [0, \infty]$ for each asset i.[23] If the market clearing price for asset i exceeds $\overline{p}_i$, she charges $\overline{p}_i$ and rations the asset.[24] Let $\overline{p} = (\overline{p}_1, \ldots, \overline{p}_n) \in \Re_+^n$ denote the vector of price caps.

The issuer is less patient than the investors: she discounts future cash flows at some rate $\delta \in (0,1)$, while investors' discount factor is normalized to one.[25] For instance, the issuer

[21] The case of asset-backed securities is further developed in section 2.5. Other natural examples are a loan portfolio where the issuer has private information about market volatility or prepayment risk, and a portfolio of equities within an industry when the issuer is privately informed about future industry profits.

[22] For vectors x and y, the notation "$x \leq y$" means that for all i, $x_i \leq y_i$.

[23] There is a superficial similarity between the price caps in our model and the price-quantity menu chosen in Biais and Mariotti (2005). In the latter paper, a single investor chooses a price-quantity pair from a menu that the issuer specifies. Hence the issuer chooses one of the prices that the investor pays. In our model, in contrast, the price caps never bind. Rather, their role is merely to rule out implausible equilibria. Intuitively, a type t issuer who deviates could choose price caps low enough that any lower type must lose from the deviation. By the Intuitive Criterion, investors must then think her type is t or more so they cannot bid less than her valuation of the assets. This can make the deviation profitable, thus ruling out the equilibrium.

[24] We could also allow the issuer to set reserve or minimum prices for the securities. However, since investors would refuse to buy overpriced securities, extending the strategy space in this way would play no role in equilibrium. (In other auction environments, a reserve price is useful to extract additional surplus from buyers. Our model differs in that investors are homogeneous and uninformed. Hence they earn no surplus even absent a reserve price.)

[25] More precisely, let the unnormalized discount factors of the investors and the issuer be $\hat{\delta} \in (0,1]$ and $\tilde{\delta} \in (0, \hat{\delta})$, respectively, and let $\hat{f}$ be the undiscounted expected payoffs of the assets. Then our model corresponds to $f = \hat{\delta}\hat{f}$ and $\delta = \tilde{\delta}/\hat{\delta}$; that is, we interpret f as the conditional expected present value of the asset using the investors' discount rate, and δ as the issuer's relative discount factor.

may face capital requirements or need cash to invest in worthwhile projects. This difference in discount factors is the source of gains from trade in the model.

We assume the issuer has no other collateral which can be used to raise funds apart from her portfolio (a, f). We also assume, for now, that she is restricted to selling only her existing securities – in Section 3 we will relax this assumption and allow her to design new securities (such as secured debt) using her existing securities as collateral.

The investors have a common, positive prior over the different possible realizations of the issuer's type *t*. On seeing the issuer's sale decision $(q, \overline{p})$, they form posterior beliefs $\mu(t \mid q, \overline{p})$. Investors are risk-neutral, behave competitively, and have deep pockets, so their demand for asset *i* is perfectly elastic at the price $\sum_t f_i(t)\,\mu(t \mid q, \overline{p})$. Given the vector $\overline{p}$ of price caps, the realized prices of the assets are thus given by the vector

$$p(q, \overline{p}) = \overline{p} \wedge \sum_t f(t)\,\mu(t \mid q, \overline{p}), \tag{2}$$

where $x \wedge y$ denotes the componentwise minimum of vectors *x* and *y*.

Suppose an issuer of type *t* sells the quantities *q* of her assets at prices $p = (p_1, \ldots, p_n) \in \Re_+^n$. She receives revenue *qp* from the asset sale, plus the discounted expected payout $\delta(a-q) f(t)$ of her retained assets.[26] Her total payoff – the sum of these terms – can be written as $U(t, q, p) = \delta a f(t) + q(p - \delta f(t))$.

Our equilibrium concept is as follows.

> **ASSET SALE EQUILIBRIUM.** A *perfect Bayesian equilibrium* for the Asset Sale game is an issuance strategy $(q(t), \overline{p}(t))$ for the issuer, a price response function $p(q, \overline{p})$, and a posterior belief function $\mu(t \mid q, \overline{p})$ for the investors, such that the following conditions hold.[27]

[26] The concatenation of two vectors is always to be interpreted as a dot product; e.g., $pq = p \cdot q$.

[27] The arguments of these functions are included for clarity. For instance, $q(t)$ should be interpreted as a function $q:\{0, \ldots, T\} \to \Re_+^n$ that specifies the quantity vector chosen by each type of issuer.

1. Payoff Maximization: for any type t, the issuer's sale decision $(q(t),\overline{p}(t))$ solves $\max_{q',\overline{p}'} U(t,q',p(q',\overline{p}'))$ subject to $0 \le q' \le a$.

2. Competitive Pricing: for any sale decision $(q,\overline{p})$, the price vector $p(q,\overline{p})$ satisfies equation (2).

3. Rational Updating: investors' posterior beliefs $\mu(t\,|\,q,\overline{p})$ are given by Bayes' rule if $(q,\overline{p})$ is chosen by some type of issuer in equilibrium.

Substituting the equilibrium price function $p(t)\overset{d}{=}p\left(q(t),\overline{p}(t)\right)$ for the price vector p in $U(t,q,p)$ and omitting the fixed term $\delta af(t)$, we obtain the issuer's equilibrium payoff function $u(t)\equiv q(t)\left[p(t)-\delta f(t)\right]$, which we refer to as the *outcome* of the equilibrium.[28] It equals the additional surplus that a type-t issuer receives from selling assets to investors.

2.2. Uniqueness and Computation

Like most signaling games, the above Asset Sale game has multiple equilibria. In order to obtain a unique prediction, we must use a signaling game refinement. The weakest such refinement is the Intuitive Criterion of Cho and Kreps (1987). In our context, this criterion states that if investors see an out-of-equilibrium sale decision $(q,\overline{p})$, their beliefs put weight only on those types who could possibly expect to gain from the deviation. More precisely, say a type-t issuer deviates to $(q,\overline{p})$. A sufficient condition for her to lose from this deviation is that her equilibrium payoff $u(t)$ exceeds her maximum payoff $q\left[\overline{p}-\delta f(t)\right]$ from the deviation – or, equivalently, that[29]

$$q\overline{p} < u(t)+\delta qf(t). \tag{3}$$

[28] The outcome captures all payoff-relevant features of the equilibrium since competition drives investors' payoffs to zero. However, two equilibria may involve different actions but the same outcome; e.g., if the issuer is endowed with two identical assets 1 and 2, then the outcome depends only on the sum q_1+q_2 of quantities sold for each type t and not on the individual quantities q_1 and q_2.

[29] If type t issues q^t for the prices p^t in equilibrium, we can rewrite (3) as $q\overline{p} < q^t p^t + \delta(q-q^t)f(t)$, where the right hand side is the opportunity cost of deviating to $(q,\overline{p})$: the sum of type t's original revenue and her

The Intuitive Criterion states that on seeing the deviation $(q, \overline{p})$, investors must put zero probability on type *t* if this type is not willing to choose $(q, \overline{p})$ but some other type might be: if condition (3) holds for *t* but fails for some other type *s*. That is:

> **THE INTUITIVE CRITERION.** A perfect Bayesian equilibrium of the asset sale game with posterior belief function $\mu(\cdot \mid \cdot, \cdot)$ and outcome $u(\cdot)$ is *intuitive* if, on seeing any quantity vector $0 \le q \le a$ and price cap vector $\overline{p}$, investors' posterior probability $\mu(t \mid q, \overline{p})$ is zero for any type *t* that satisfies (3) as long as there is some type *s* for which the inequality is reversed: for which $q\overline{p} \ge u(s) + \delta q f(s)$.

An equilibrium that satisfies the Intuitive Criterion will be called *intuitive*, as will investors' beliefs in that equilibrium. If an outcome $u(\cdot)$ is supported by an intuitive equilibrium, we will call the outcome intuitive as well.

Clearly, a simple way ensure intuitive beliefs is for investors to respond to a deviation $(q, \overline{p})$ by putting all of their weight on a type *t* for which the opportunity cost $u(t) + \delta q f(t)$ of the given deviation is at a minimum. Since no type has a lower opportunity cost of deviating to $(q, \overline{p})$ than this type *t*, there is no price vector $p \le \overline{p}$ that makes some type *s* willing to deviate to $(q, \overline{p})$ if it does not also make type *t* willing to choose this deviation: beliefs are intuitive. Since technically more than one type *t* may minimize the opportunity cost $u(t) + \delta q f(t)$ of deviating, we can break ties by putting all of investors' weight on the lowest such type. Summarizing, a sufficient condition for the belief function μ to be intuitive is that, following a deviation $(q, \overline{p})$, it satisfies

$$\mu\left(\tau\left(q\right) \mid q, \overline{p}\right) = 1 \tag{4}$$

where

$$\tau\left(q\right) = \min\left\{\arg\min_t \left[u\left(t\right) + \delta q f\left(t\right)\right]\right\} \tag{5}$$

is the lowest type *t* that minimizes $u\left(t\right) + \delta q f\left(t\right)$.

---

discounted cost of transferring q units of each asset to investors rather than q^t. Type *t* will not deviate to $(q, \overline{p})$ if this opportunity cost exceeds her maximum revenue $q\overline{p}$ from doing so, i.e. if (3) holds.

We will show that any intuitive equilibrium of the Asset Sale game satisfies the following property.

> **Fair Pricing.** An equilibrium is *fairly priced* if, for all i and t, $q_i(t) > 0$ implies $p_i(q(t), \overline{p}(t)) = f_i(t)$.

That is, the price assigned to any asset i that is sold in equilibrium equals the conditional expected payout of this asset. This property is related, but not identical, to the notion of a separating equilibrium. Unlike separation, fair pricing implies that the issuer's price caps never bind and that investors' payoffs are identically zero, so social welfare is given by the issuer's payoff function $u(t)$. And unlike fair pricing, separation implies that investors can infer the issuer's actual information t in equilibrium, while fair pricing implies only that they can infer the conditional expected payout of any asset that the issuer chooses to sell – which is all they need in order to compute the asset's fair-market value.

We now show that there exists a unique intuitive outcome of the Asset Sale game, and that this outcome is supported by a fairly priced equilibrium. Moreover, this equilibrium maximizes the payoff of an issuer of each type t within the set of fairly priced equilibria and hence is efficient within this set (since, in a fairly priced equilibrium, investors must break even on average).

To construct the equilibrium, we recursively calculate the maximum surplus $u^*(t)$ achievable by each type subject to fair pricing and to the incentive constraint that no lower type would choose to imitate the given type:

> **Recursive Linear Program (RLP).** For type $t = 0$, let $q^*(0) = a$ and let $u^*(0) = (1-\delta)af(0)$ denote the gains from selling all of the issuer's endowment. For any higher type $t > 0$, define[30]
>
> $$u^*(t) = \max_{0 \le q \le a} (1-\delta) q\, f(t) \quad \text{s.t.} \quad q\big(f(t) - \delta f(s)\big) \le u^*(s) \quad \text{for all } s < t,$$

[30] Any type s that chooses the same q as type t will earn revenue of $qf(t)$, and have cost $q\delta f(s)$, for total surplus $q(f(t)-\delta f(s))$, which must not exceed $u^*(s)$ in order to maintain incentive compatibility.

and let $q^*(t)$ denote any quantity vector that solves the given maximization problem and yields the payoff $u^*(t)$.

We will now show that the unique intuitive outcome is the solution $u^*(\cdot)$ to the RLP.[31] For each t, let type t's quantity vector $q^*(t)$ be any solution to RLP and let type t's price cap vector be any solution to $\overline{p}^*(t) > f(t)$ (which will imply that the price caps do not bind if the equilibrium is fairly priced). Let the posterior beliefs $\mu^*(\cdot \mid q, \overline{p})$ be given by Bayes's Rule if $(q, \overline{p})$ is chosen by some type in equilibrium and by (4) and (5) if not (with u replaced by u^* in (5)). Finally, let the market price function $p^*(q, \overline{p})$ be the result of substituting μ^* for μ in equation (2). Since beliefs μ^* are intuitive by construction, e^* satisfies the Intuitive Criterion if it is an equilibrium.

The main result of this section is that e^* is an equilibrium, is fairly priced, yields the outcome u^*, and is efficient (and optimal for the issuer) within the set of fairly priced equilibria. Moreover, any intuitive equilibrium of the Asset Sale game has the same outcome u^* as e^*. Finally, in e^*, a higher quantity sold of any asset cannot raise the price of any asset.

PROPOSITION 1. The RECURSIVE LINEAR PROGRAM has a unique solution u^*, which is strictly positive and nonincreasing in the issuer's type t, and is the unique intuitive outcome. The profile e^* defined above is an intuitive equilibrium with outcome u^*. Moreover, for each type t, $u^*(t)$ is both the highest attainable issuer's payoff and the maximum social welfare in any fairly priced equilibrium. Finally, for any quantity vectors $q' \geq q$ (whether or not chosen in equilibrium), no asset price is higher under q' than q: $p^*(q', \overline{p}) \leq p^*(q, \overline{p})$.

PROOF: Appendix. ◆

[31] Whether u^* corresponds to an equilibrium is not yet obvious; we have not, for example, checked whether the incentive constraints for high types not to mimic low types are satisfied. We verify this below.

2.3. Nonmonotonic Issuance

In the single-asset case, a more optimistic issuer retains more of her asset in order to distinguish herself from more pessimistic types (e.g., Leland and Pyle 1979, DeMarzo and Duffie 1999). With multiple assets, this monotonicity property can fail to hold. We show this via two computed examples. The first is a simple example with three types of issuer, whose solution can be easily calculated by hand.

> **EXAMPLE 1.** Assume the issuer owns one share each of two assets: $a_1 = a_2 = 1$. The issuer has three possible types $t = 0,1,2$ and her discount factor is $\delta = 1/2$. The conditional expected payouts of the assets are $f(0) = (1,1)$, $f(1) = (2,3)$, and $f(2) = (5,4)$. That is, asset 2 (resp., 1) responds relatively more when the issuer's type rises from 0 to 1 (resp., from 1 to 2). One can then verify that in the solution q^* to RLP, the issuer sells her entire portfolio when her type is 0, 2/3 shares of asset 1 and no shares of asset 2 when her type is 1, and no shares of asset 1 and 4/15 shares of asset 2 when her type is 2.

Intuitively, for each increase in her type, the issuer most efficiently distinguishes herself from the next lower type by retaining more of the asset whose valuation is raised proportionally more by the given type increase. In particular, the increase in her type from 1 to 2 leads her to sell more of asset 2, in contrast with the single-asset case where optimistic issuers always sell less. The next example illustrates the same principle in a more realistic setting with many issuer types.

> **EXAMPLE 2.** Suppose the issuer holds 1 share of each of 2 assets, with expected payouts of each (conditional on $t = 0,1,\ldots, 200$) given in Figure 1.[32] While the overall sensitivity to information for each asset is similar (a 10% increase in value over the range of t), the right panel shows that asset 2's return is more sensitive to increases in t for $t < 50$ and asset 1 is more sensitive when $t > 50$.

[32] Parameter assumptions are available by request.

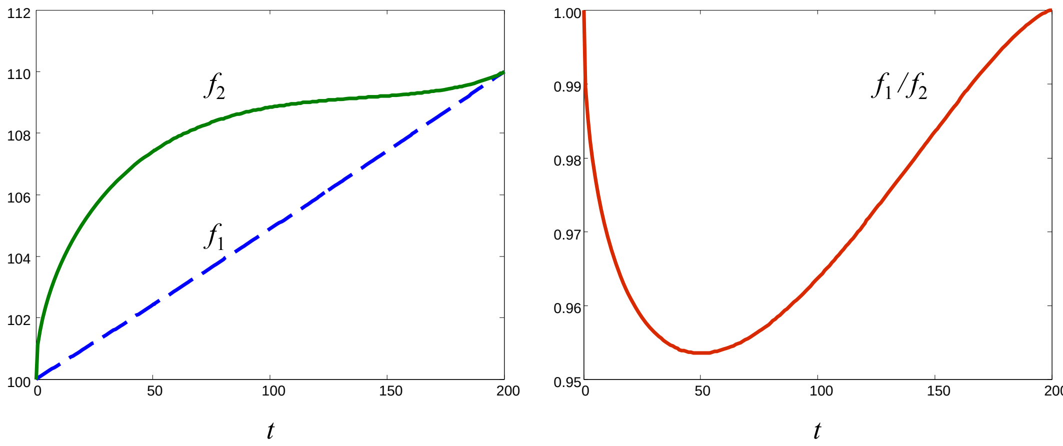

Figure 1: Expected Payoffs of Assets for EXAMPLE 2.
Asset 1's returns are less sensitive to information than asset 2's returns for low t, but are more sensitive for high t.

Figure 2 depicts the equilibrium strategies for $\delta = 0.9$ (given by the solid line with circles for t = 0, 10, … 200). The lowest type sells all shares of both assets. Initially, she signals type increases by selling fewer shares of asset 2, which is more sensitive than asset 1 is to her information when the issuer's type is low. Her approach changes when her type rises above 50: asset 1 now becomes more information-sensitive, so she signals better information by selling fewer shares of asset 1 and more shares of 2. Finally, when her type is so high that asset 1 is retained in its entirety, the issuer has no choice but to sell less of asset 2 as a signal (even though asset 2 is less informationally sensitive than asset 1 in this range).

Also depicted in Figure 2 is the equilibrium price function, as a contour plot showing the type investors infer given any quantity q; darker shades indicate lower types t, while white lines indicate the iso-price contours for t = 0, 10, … 200. On each such contour, investors' posterior beliefs and thus the prices of both assets are constant. As the contours are downward sloping, investors become more pessimistic as the issuer sells more shares of either asset. Moreover, for types t between 50 and 120, beliefs drop discontinuously for a small increase in either quantity. For example, starting from $q^*(60)$, a small increase in the quantity sold of either security will cause beliefs to drop discontinuously from 60 to below 20; similarly, a small increase from $q^*(90)$ would cause beliefs to drop to 0.

Discontinuities occur when the binding incentive constraint in RLP is non-local; for example, the type with the greatest incentive to mimic type 60 is type 18. The possibility that non-local constraints may bind distinguishes our setting from standard signaling models in which a single crossing property holds.

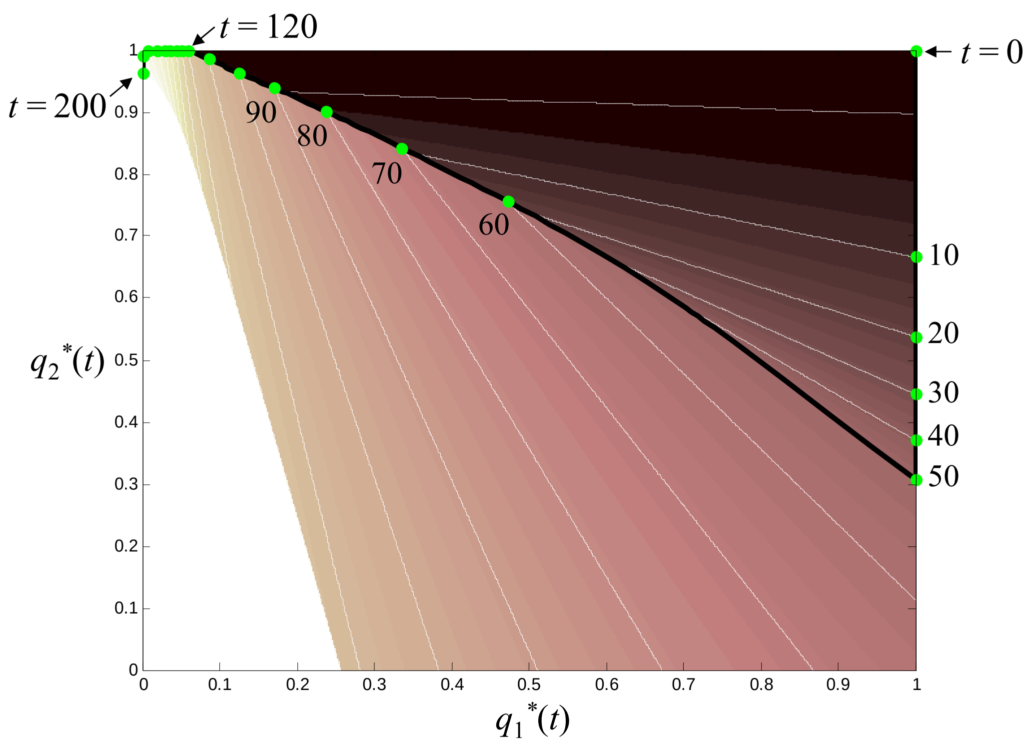

Figure 2: Equilibrium for EXAMPLE 2.
Circles show equilibrium quantities along the solid curve; shading indicates iso-price contours given the beliefs in (4). Note the discontinuity in prices along the upper right boundary of the equilibrium curve.

2.4. Informationally Ordered Assets

We saw in section 2.3 that greater optimism may sometimes lead an issuer to sell more shares of a given asset. This nonmonotonicity occurs because assets can alternate in their informational sensitivity as the issuer's optimism grows. In the remainder of the paper, we focus on settings in which assets' sensitivity to the common factor can be unconditionally ranked: if one asset is more sensitive than another for a given increase in optimism, then it is more sensitive for any such increase.

In this setting, the Asset Sale game has a simple outcome. The lowest type issuer sells all her assets as before. As her type rises, she retains her more informationally sensitive assets first. Intuitively, these assets are always more efficient signals of greater optimism. As a result, the model predicts that a firm will never sell an asset unless it also sells *all* of its less sensitive assets in their entirety, as predicted by Myers's (1984) Pecking Order hypothesis.

2.4.1. Information Sensitivity

We will say that one asset is more informationally sensitive than another if its expected value changes proportionally more for a given change in the issuer's type. To ensure that proportional changes are well-defined, we assume the conditional expected payoff of each asset is always positive:

> **ASSUMPTION B (POSITIVE EXPECTED PAYOFFS).** For all *i*, $f_i(0) > 0$.[33]

The formal definition is as follows.

> **INFORMATIONAL SENSITIVITY.** Asset *i* is *more informationally sensitive than asset j at type t* if, for any lower type *s*, $f_i(t)/f_i(s) > f_j(t)/f_j(s)$. If this holds for each type *t*, then asset *i* is *more informationally sensitive* than asset *j*.[34] The assets $1,\ldots,n$ display *increasing information sensitivity* (*IIS*) if each asset *i* is more informationally sensitive than any lower-indexed asset $j < i$.[35]

In this setting, the issuer prefers to retain her more informationally sensitive assets first as her optimism grows. Formally:

> **PROPOSITION 2.** Suppose asset *i* is more informationally sensitive at *t* than asset *j*. Then in any solution q^* to the RECURSIVE LINEAR PROGRAM, if $q_j^*(t) < a_j$ then

[33] Under this requirement, a security can have zero payouts in some states as long as, for any type *t*, there are states in which the security's payout is strictly positive.

[34] Equivalently, *i* is more informationally sensitive than *j* if $f_i(t)/f_j(t)$ is increasing in *t*.

[35] IIS is equivalent to the functions $f_i(t)$ being log-supermodular in (i,t).

$q_i^*(t)=0$: the issuer will not retain any shares of asset *j* unless she also retains all shares of asset *i*.

PROOF: Appendix. ◆

2.4.2. Hurdle Class Strategies

If we further assume IIS then, by PROPOSITION 2, the issuer will choose to sell all of her less informationally sensitive assets and retain all of her more informationally sensitive ones, with the exact cutoff, or *hurdle class*, determined by her type:

PROPOSITION 3. Let the issuer's assets display Increasing Information Sensitivity. Then an issuer of each type will choose a hurdle class quantity vector: a vector of the form $q^*(t)=(a_1,\ldots,a_{c-1},q_c(t),0,\ldots,0)$ for some hurdle class $c\in\{1,\ldots,n+1\}$ and some hurdle class quantity $q_c(t)\in[0,a_c)$.[36] Moreover, for any types $t > s$, $q^*(t)\le q^*(s)$, whence *t* chooses a weakly lower hurdle class than *s* does.

PROOF: Appendix. ◆

Intuitively, a low type has a stronger incentive to sell additional assets by choosing a higher hurdle class than a high type because, while the impact on revenue is the same, the opportunity cost of parting with the additional shares is smaller for the low type. Hence, a low type cannot choose a lower hurdle class than a high type does in equilibrium.

Any hurdle class vector $(a_1,\ldots,a_{c-1},q_c,0,\ldots,0)$ for $q_c(t)\in[0,a_c)$ is uniquely identified by the total number $q_c+\sum_{i=0}^{c-1}a_c$ of shares that the issuer sells (of all assets combined). This number can be thought of as a one-dimensional signal. Thus, under IIS the issuer's signaling problem reduces to the familiar and tractable one-dimensional case.

One remaining complication is that for each type $t>0$, RLP includes $t-1$ incentive compatibility (IC) constraints: one for each lower type *s*. The next result shows that, in fact, the IC constraint for the next lower type $t-1$ is binding and, moreover, identifies a unique hurdle class vector for type *t*:

[36] When *c* equals *n* + 1, *q* equals *a*: the issuer sells her entire portfolio.

PROPOSITION 4. If the assets display Increasing Information Sensitivity, then $q^*(0) = a$ and, for each $t > 0$, $q^*(t)$ is the unique hurdle class vector that satisfies the incentive compatibility constraint for type $t-1$, which is

$$q^*(t)\left[f(t) - \delta f(t-1)\right] = (1-\delta) q^*(t-1) f(t-1). \quad (6)$$

PROOF: Appendix. ◆

2.5. Application: Prioritized Assets

To apply the preceding result, one must know whether one asset is more sensitive than another to an issuer's information. In practice, this condition may be hard to check. In this section we give a simple rule that can be used to make this determination: under a weak distributional condition, *more senior assets* are less informationally sensitive. Hence, by the prior result, they will be sold first, as predicted by Myers's (1984) pecking-order hypothesis.

Importantly, we develop this result using a novel, generalized notion of seniority that permits the ranking of assets that are not ordered by strict priority. A debt asset has strict priority over another asset if its owners must be paid its face value before the owners of the other asset are paid anything. While our notion agrees with strict priority in this case, it also applies when neither asset has strict priority.

Generally speaking, seniority refers to the claim of one asset versus another to a future cash flow that secures both assets. Hence, we assume that the payout of each asset $i \in \{1, \ldots, n\}$ of the issuer is given by some nondecreasing, nonnegative function $F_i(Y)$ of a common, stochastic cash flow $Y \in \Re_+$. The cash flow Y might might represent a firm's future operating profits or the aggregate repayments on a bank's loan portfolio.

The distributional assumption we need is as follows.

HAZARD RATE ORDERING (HRO). For any type *t*, the posterior support of Y coincides with the prior support. Moreover, for any types $t > s$, the ratio

$\Pr(Y \geq y \mid s) / \Pr(Y \geq y \mid t)$ is decreasing in *y* in the support of *Y* whenever the ratio is well-defined.[37]

While stronger than first-order stochastic dominance, HRO is weaker than the monotone likelihood ratio property, which is commonly assumed in signaling environments.[38] If the conditional hazard rate of *Y* given *t* is well defined, HRO states that this hazard rate is decreasing in the type *t*; however, HRO also applies when the hazard rate is undefined.[39]

As for seniority, let us say that asset *i* is senior to asset *j* if the latter asset has a relatively stronger claim on the cash flow when the cash flow is low: if the payout ratio $r(Y) = F_j(Y) / F_i(Y)$ is increasing in the cash flow *Y*. The precise definition, which permits the denominator to be zero and the ratio to be locally constant, is as follows.

PRIORITIZED ASSETS. Asset *i* is *senior* to asset *j* (and *j* is *junior* to *i*) if the payout $F_j(Y)$ to asset *j* can be written as the product of the payout $F_i(Y)$ of asset *i* and a nonnegative, nondecreasing function $r(Y)$ of the cash flow that takes more than one value with positive probability on that portion of the support of the cash flow where asset *i*'s payout is positive.[40] The assets $\{F_i\}$ are *prioritized* if any asset *i* is senior to any higher-indexed asset $j > i$.

Strict priority is implied by the above definition: if asset *i* has strict priority over asset *j*, the function *r*(*Y*) is zero while asset *i* is being repaid, grows while asset *j* is being repaid, and becomes constant once asset *j* has been repaid in full. However, the definition is more general: asset *i* may be senior to asset *j* even if the payouts to both rise with the cash flow, as long as the payout to asset *j* rises relatively faster.

[37] The only point *y* in the support of *Y* at which the ratio may be undefined is the the upper boundary of *Y*'s support, where $\Pr(Y \geq y \mid t)$ may equal zero.

[38] See Shaked and Shanthikumar (2007), p. 43, Thm. 1.C.1.

[39] The conditional hazard rate is $h(y \mid t) / \left[1 - H(y \mid t)\right]$ where *H* denotes the conditional distribution of the cash flow *Y* given the type *t*, and *h* is the corresponding density. This hazard rate is undefined if the conditional distribution is discrete or has atoms, while the condition in HRO is still well defined in this case.

[40] I.e., there is a constant $c > 0$ such that $\Pr(F_i(Y) > 0 \text{ and } r(Y) < c)$ and $\Pr(F_i(Y) > 0 \text{ and } r(Y) > c)$ are both positive. This requirement rules out trivial cases where securities differ only on measure zero outcomes.

Some examples appear in Figure 3. The securities are ranked in order, with F_1 the most senior and F_5 the most junior, with the only exception that F_4 is noncomparable with F_2 or F_3. Thus (F_1, F_4, F_5) and (F_1, F_2, F_3, F_5) are prioritized sets.

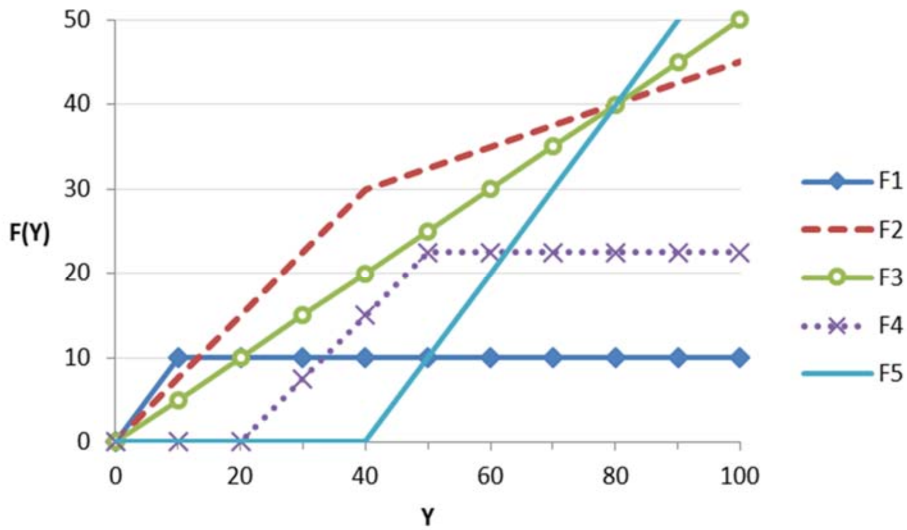

Figure 3: Examples of Prioritized Securities.
Securities are ranked with F_1 most senior and F_5 most junior, with the exception that F_4 is non-comparable with F_2 and F_3.

We can now relate a security's seniority to the order in which it will be sold: under HRO, junior securities will be retained first as an issuer becomes more optimistic. Or, equivalently, senior securities will be sold first:

> **PROPOSITION 5.** Suppose HRO holds and the securities are prioritized. Then the assets satisfy Increasing Informational Sensitivity, so their liquidation is as set out in PROPOSITION 3 and PROPOSITION 4. In particular, the issuer will not sell any shares of a given asset unless she also sells all more senior assets in their entirety.
>
> **PROOF:** Appendix. ◆

3. Security Design

Having solved the general portfolio liquidation problem, we now apply the above tools to the problem of security design. More precisely, we generalize the single *ex-ante* security case studied by DeMarzo and Duffie (1999) by permitting an issuer to design multiple securities before and/or after she receives her information. We first show that any intuitive equilibrium of this game is equivalent to one in which the issuer pools her initial assets, sees her information, and then designs and sells a single security whose payout equals the aggregate payout of the securities she sells in the original equilibrium.

To further sharpen these predictions, we then impose the above Hazard Rate Ordering property. We also assume *monotonicity*: the payout received by investors, as well as that retained by the issuer, is a nonnegative and nondecreasing function of the final cash flow.[41]

Under these two assumptions, we show that the issuer's optimal *ex-post* security is standard debt with a face value that is decreasing in her information. We establish this by proving a formal equivalence between *ex-post* security design and a strategy in which the issuer pools her initial assets, designs a maximal set of prioritized debt securities or "tranches" that are secured by the pool, sees her information, and then chooses how much of each tranche to sell as in the asset sale game.[42] By PROPOSITION 5, the issuer will sell those tranches whose seniority lies above a threshold that is increasing in her information. But this strategy is equivalent to selling a single standard debt security whose face value is decreasing in the issuer's information. Thus, the two strategies are equivalent and optimal.

In each such strategy, the issuer begins by pooling her initial assets. In DeMarzo (2005), in contrast, it is sometimes better not to pool. Why? In our setting, the initial assets partition a common underlying cash flow. Hence, each pre-existing asset can be exactly replicated by a new asset that is secured by the pool of initial assets. As pooling does not restrict the issuer, it cannot harm her. In DeMarzo (2005), in contrast, the initial assets are backed by distinct cash flows. This gives rise to two new effects. First, pooling prevents

[41] Monotonity is a common assumption in the security design literature; see, e.g., DeMarzo (2005), DeMarzo and Duffie (1999), Frankel and Jin (2015), Hart and Moore (1995), Matthews (2001), and Nachman and Noe (1994). It can be justified by supposing the issuer has free disposal over her cash flow Y and can also contribute cash to inflate it. Hence, if her payout were decreasing in Y, she would freely dispose of some cash in order to raise this payout. And if, alternatively, the payout to investors were falling in Y, the issuer would contribute cash to inflate Y, thus paying investors less and raising her payout by more than the amount contributed. Finally, nonnegativity is motivated by the common limited-liability feature of financial assets.

[42] The two strategies are equivalent because any monotone security can be mimicked by a suitable portfolio of tranches and vice-versa. Equivalence means that the portfolios sold by the issuer in the two strategies raise the same revenue from investors and, for any realization of the underlying cash flow, entail the same final aggregate payout to investors.

the issuer from signaling that one cash flow is high while another is low, which is harmful to her. Second, by diversifying away the residual risk in the cash flows, pooling also allows the issuer to issue debt with a lower default risk and thus to reap more of the gains from trade with investors.[43] Whether pooling is optimal depends on the tradeoff between these two forces, as shown in Theorem 5 in DeMarzo (2005).[44]

As they are somewhat technical, the proofs of this section's results appear in our online appendix (DeMarzo, Frankel, and Jin 2017).

3.1. The General Security Design Game

In the Asset Sale game introduced in section 2, the issuer chooses how many shares to sell of each of a given set of assets. This limits her; for instance, she cannot borrow money using some of her assets as collateral. Doing so is equivalent to selling a debt security that is secured by her assets. In this section we discard this restriction: the issuer can design general securities (not only debt) that are secured by her assets. She can do so both before and after she sees her information. Thus, the model is a generalization of both DeMarzo and Duffie (1999), in which the issuer designs a single security before seeing her information, and an earlier version of this paper (DeMarzo 2003) in which she designs a single security *ex post*.

As in section 2.5, we assume the issuer is endowed with a set of initial assets that are secured by a common, stochastic cash flow Y.[45] Before learning her type, the issuer designs a set of interim securities whose payouts are functions of the payouts of the initial assets. After she sees her type, she then designs a set of *ex-post* securities whose payouts are functions of the payouts of the interim securities.[46]

[43] The second effect can also be seen in the present model, as a benefit to the issuer from a reduction in the conditional volatility of her underlying cash flow; see EXAMPLE 5 below.
[44] The proof of Theorem 5 in DeMarzo (2005) relies on an earlier version of the results of this section which appeared in the working-paper version of our paper, DeMarzo (2003).
[45] This means that the sum of initial asset payouts equals the realization Y of the cash flow. This assumption is equivalent to assuming that, in addition to her initial assets, the issuer can also use any residual cash flow as collateral for her securities. Hence, for all intents and purposes, this residual cash flow is an initial asset.
[46] Requiring the issuer to issue interim securities is not restrictive: any of them can be a pass-through security whose payout is identically equal to the payout of some initial asset. However, this flexibility is needed in order to reduce the general security design problem to the asset liquidation problem of our base model.

In this section, we show that any intuitive equilibrium of this game is equivalent to one in which the issuer pools her initial assets and sells a single *ex-post* security whose payout is a function of the pool's final value. In subsequent sections we show that under additional mild assumptions, this security is standard debt.

Formally, the general security design game is as follows.

> **THE GENERAL SECURITY DESIGN (GSD) GAME.** The issuer is endowed with a finite set $F(Y)=\left(F_i(Y)\right)_{i=1}^{n}$ of initial assets whose payout functions partition the common cash flow: $\sum_{i=1}^{n} F_i(Y)=Y$. Timing is as follows.
>
> 1. The issuer designs a finite set $I=\left(I^k\right)_{k=1}^{K}$ of *interim* securities; the payout of each such security *k* is some function $I^k\left(F(Y)\right)$ of her initial asset payouts which we write, for brevity, as $I^k(Y)$.[47] The interim assets satisfy limited liability: $\sum_{k=1}^{K} I^k(Y)\in[0,Y]$.
> 2. The issuer sees her type *t* and offers investors a finite vector $P=\left(P^j\right)_{j=1}^{J}$ of some number of *ex-post* securities, where the payout of *ex-post* security *j* is a function $P^j\left(I(Y)\right)$ of her interim assets' payouts. These payouts also satisfy limited liability: $\sum_{j=1}^{J} P^j\left(I(Y)\right)\in\left[0,\sum_{k=1}^{K} I^k(Y)\right]$. At the same time, the issuer commits to a global revenue cap $\rho\in\Re_+$: if her issuance revenue exceeds the cap, she keeps ρ and discards the rest.[48]
> 3. Investors assign a price p^j to each *ex-post* security *j*.

Let

$$W_P^I(Y)=\sum_{j=1}^{J} P^j\left(I(Y)\right) \tag{7}$$

[47] Formally, we define the new function $\hat{I}^k$ satisfying $\hat{I}^k(Y)=I^k\left(F(Y)\right)$ and then rename it to I^k.

[48] Like the price caps in the Asset Sale model, we use the revenue cap in order to rule out implausible equilibria. The cap never binds in equilibrium.

denote the aggregate payout promised to investors, given the interim security vector *I* and the *ex-post* security vector *P*. By limited liability, this payout lies in $\left[0,Y\right]$. If a type-*t* issuer chooses the interim security vector *I*, the *ex-post* security vector *P*, and the revenue cap ρ, her expected surplus from issuance equals $\left(\rho \wedge \sum_{j=1}^{J} p^j\right) - \delta E\left[W_P^I\left(Y\right) | t\right]$: her issuance revenue less the discounted expected value of the securities issued.

The following two examples illustrate the generality and flexibility of the GSD game. We begin with a two-security generalization of DeMarzo and Duffie (1999).

> **EXAMPLE 3**. The issuer may design interim assets consisting of two prioritized securities: a senior tranche $I^1\left(Y\right) = \min\left\{D, Y\right\}$ with face value *D* and a junior tranche $I^2\left(Y\right) = \min\left\{D', Y - I^1\left(Y\right)\right\}$ with face value D'. After discovering her type *t*, she may then sell a quantity z_t^i of each tranche $i = 1,2$. Formally, her two *ex-post* securities are then given by $P_t^i\left(I\left(Y\right)\right) = z_t^i I^i\left(Y\right)$ for $i = 1,2$.

The following is a two-security generalization of the one-security case considered by DeMarzo (2005) and Frankel and Jin (2015).

> **EXAMPLE 4.** The issuer may sell senior and junior debt *ex post* with type-contingent face values D_t and D_t' respectively. Formally, she designs a single *ex-ante* pass-through security $I^1\left(y\right) = y$, while her two *ex-post* securities are then given by $P_t^1\left(I\left(y\right)\right) = \min\left\{D_t, y\right\}$ and $P_t^2\left(I\left(y\right)\right) = \min\left\{D_t', y - P_t^1\left(I\left(y\right)\right)\right\}$.

We will refer to the action $\left(P,\rho\right)$ that the issuer chooses after learning her type as her *ex-post action*. A *strategy* for the issuer is a triplet $\left(I, P_\cdot\left(\bullet\right), \rho_\cdot\left(\bullet\right)\right)$, which specifies her interim asset vector *I* and, for any type *t* and interim asset vector I' (which equals *I* unless the issuer deviated), her *ex-post* action $\left(P_t\left(I'\right), \rho_t\left(I'\right)\right)$. A *pricing function* for investors is a vector $p = \left(p^j\right)_{j=1}^{J}$ where $p^j = p^j\left(I, P, \rho\right)$ is the price of *ex-post* security *j* when the issuer chooses the action $\left(I, P, \rho\right)$. The issuer's *conditional expected payoff* from this action, given her type *t* and the investors' pricing function *p*, is thus

$$U\left(I,P,\rho\,|\,t,p\right)=\left[\rho\wedge\sum_{j=1}^{J}p^{j}\left(I,P,\rho\right)\right]-\delta\,E\left[W_{P}^{I}(Y)\,|\,t\right]. \tag{8}$$

Similarly, her *unconditional expected payoff* from the strategy $\left(I,P_{\cdot}(\bullet),\rho_{\cdot}(\bullet)\right)$, given the investors' pricing function *p*, is $U\left(I,P_{\cdot}(\bullet),\rho_{\cdot}(\bullet)\,|\,p\right)=\sum_{t}g(t)U\left(I,P_{t}(I),\rho_{t}(I)\,|\,t,p\right)$, where $g(t)$ denotes the prior probability that her type is *t*.

EQUILIBRIUM: GSD GAME. An equilibrium of the General Security Design game consists of a strategy $\left(\overline{I},P_{\cdot}(\bullet),\rho_{\cdot}(\bullet)\right)$ for the issuer,[49] together with a price function *p* and a belief function μ, such that the following conditions hold.

1. Payoff Maximization: (a) the prescribed interim asset vector $\overline{I}$ is contained in $\arg\max_{I}\,U\left(I,P_{\cdot}(\bullet),\rho_{\cdot}(\bullet)\,|\,p\right)$ and (b) for each interim asset vector *I*, the *ex-post* action $\left(P_{t}(I),\rho_{t}(I)\right)$ chosen by each type *t* lies in $\arg\max_{(P,\rho)}\,U\left(I,P,\rho\,|\,t,p\right)$.

2. Rational Updating: let $T\left(P,\rho\,|\,I\right)$ denote the set of types who choose the *ex-post* action $\left(P,\rho\right)$ conditional on having chosen any given interim asset vector *I*. Upon seeing the action $\left(I,P,\rho\right)$, the investors' posterior probability $\mu(t\,|\,I,P,\rho)$ equals the probability that the issuer's type is *t* conditional on her type being in $T\left(P,\rho\,|\,I\right)$: it equals $g(t)\left[\sum_{s\in T(P,\rho|I)}g(s)\right]^{-1}$ if $t\in T\left(P,\rho\,|\,I\right)$ and zero otherwise.[50]

3. Competitive Pricing: for any action $\left(I,P,\rho\right)$ of the issuer, the price of each *ex-post* security equals its expected payout given investors' posterior beliefs:

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[49] This strategy instructs the issuer to choose the interim asset vector $\overline{I}$ and then, following any interim asset vector *I*, to issue the ex-post action $\left(P_{t}(I),\rho_{t}(I)\right)$ when her type is *t*.

[50] In particular, if an unexpected interim asset vector *I* is chosen, investors believe that it was simply a mistake that conveys no information about the issuer's type. This is in the spirit of the "no signaling what you don't know" condition of perfect Bayesian equilibrium (Fudenberg and Tirole 1991, p. 332): the issuer's interim asset vector *I* cannot signal her type *t* as she chooses *I* before learning *t*.

$$p\left(I,P,\rho\right)=\sum_t E[P\left(I\left(Y\right)\right)|\,t]\ \mu(t\,|\,I,P,\rho)\,. \tag{9}$$

The *outcome* $u\left(t\right)$ of the above equilibrium – a function that gives the expected payoff of each type t from playing her prescribed strategy - is just her payoff $U\left(\bar{I},P_t\left(\bar{I}\right),\rho_t\left(\bar{I}\right)|\,t,p\right)$ from issuing the prescribed interim asset vector $\bar{I}$, learning her type *t*, and then choosing her prescribed *ex-post* action $\left(P_t\left(\bar{I}\right),\rho_t\left(\bar{I}\right)\right)$.

3.2. Reduction to the Ex-Post Security Design Game

The GSD game must be simplified so as to be tractable. We do this in two steps. In this section, we show that one can focus on simpler strategies in which the issuer pools her initial assets and sells a single *ex-post* security that is secured by her initial asset pool. Formally, for any equilibrium *E* of the GSD game, let $S_t\left(Y\right)$ be the aggregate payout of the *ex-post* securities that a type *t* issuer sells in *E* when her underlying cash flow is *Y*. Then it is also an equilibrium of the GSD game for each type *t* issuer to pool her initial assets and sell a single *ex-post* security with payout function $S_t\left(Y\right)$.[51] Moreover, the new equilibrium is intuitive if the original one is:

PROPOSITION 6. Let $E=\left(\hat{I},P_\cdot\left(\bullet\right),\rho_\cdot\left(\bullet\right),p,\mu\right)$ be any equilibrium of the GSD game, in which the aggregate payout function of the *ex-post* securities of a type-*t* issuer is $S_t\left(Y\right)$. Then there is an equilibrium $\tilde{E}$ of the GSD game, with the same outcome, in which the issuer pools her initial assets and, on seeing her type *t*, issues a single *ex-post* security with payout function $S_t\left(Y\right)$. This security yields the same payoff and issuance revenue as a type-*t* issuer gets in *E*. And if the original equilibrium *E* is intuitive, then so is $\tilde{E}$.[52]

PROOF: Online appendix. ◆

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[51] Technically, the GSD framework requires us to write the *ex-post* security's payout as a function of the payouts of some interim securities. To do so, we can assume the issuer designs a single pass-through interim security whose payout equals the aggregate value *Y* of her initial assets.

[52] The definition of Intuitive Equilibrium in the context of the GSD game appears in the proof.

3.3. Solving the Ex-Post Security Design Game

By PROPOSITION 6, we may restrict to a setting in which the issuer pools her initial assets and then, on seeing her type, designs a single *ex-post* security whose payout is some function of the payout of the pool. We will refer to this restriction of the GSD Game as the *Ex-Post Security Design* (EPSD) game.

While the EPSD game is simpler than the GSD game, the issuer's security is still a multidimensional signal of her type. The multidimensional signaling problem has not been solved in general.[53] However, we can solve it in an important special case, using results from prior sections. As in section 2.5, we will assume the issuer's information satisfies the Hazard Rate Ordering property. Moreover, we will restrict the issuer to the following set of *monotone securities*:

$$M = \{S\text{: } S(y) \text{ and } y - S(y) \text{ are nonnegative and nondecreasing in } y \in \text{supp}(Y).\}.$$

We will also assume, for now, that the cash flow is discrete.[54] For convenience, we also normalize the lowest realization to zero:[55]

ASSUMPTION C (DISCRETE CASH FLOW). $Y \in \{y_0, \ldots, y_n\}$ where $y_0 = 0$.

Given her information t, if the issuer sells security $S \in M$ for the price p, her total payoff is $p + \delta E[Y - S(Y) \mid t]$ which can be written as the sum of the exogenous component $\delta E[Y \mid t]$, which we ignore, and her issuance profit $p - \delta E[S(Y) \mid t]$.

The key insight of this section is that under the above assumptions, the EPSD game can be solved as an instance of the Asset Sale game. Suppose that, *ex ante*, the issuer pools and

[53] Researchers have studied the two-dimensional signaling problem; see, e.g., Quinzii and Rochet (1985). We are not aware of solutions in three or more dimensions.

[54] The continuous case is studied below in section 3.5.

[55] We can recover the unnormalized case as follows. Suppose that there is an equilibrium of the normalized model, in which the issuer sells a security with payout $S_t(Y)$ for price p_t and the lowest realization of Y is zero. Now suppose her cash flow is instead the unnormalized $Y' = Y + y_0$ for some positive y_0. Since each type of issuer can commit to transfer the lowest cash flow y_0 to investors, the issuer will sell a risk-free bond with face value y_0 in order to reap the full gains from trade $(1-\delta)y_0$ from this transfer. As for the residual cash flow $Y = Y' - y_0$, the original equilibrium analysis applies: a type t issuer will sell a security with payout $S_t(Y)$ for the price p_t. Her issuance revenue is simply the sum $y_0 + p_t$ of the prices of these two securities.

tranches her initial assets into n interim securities. The payout $F_i^*(Y)$ of tranche $i = 1,...,n$ equals $(y_i - y_{i-1}) \times 1[Y \geq y_i]$ where the indicator function $1[Y \geq y_i]$ equals one if $Y \geq y_i$ and zero otherwise. That is, tranche i pays the incremental cash flow $y_i - y_{i-1}$ if the cash flow Y is at least y_i, and zero otherwise. The payouts of the n tranches clearly sum to the cash flow Y. Let $f^*(t) = E[F^*(Y) \mid t]$ denote the corresponding conditional expected payout vector.

After tranching her asset pool, the issuer sees her type t and chooses a quantity q_i of each tranche i to sell. This subgame can be thought of as an Asset Sale game with endowment (a^*, f^*) where a^* denotes a vector consisting of n ones. Crucially, there is a one-to-one correspondence between portfolios q in this Asset Sale game, on the one hand, and monotone securities S in the EPSD game, on the other. The correspondence is as follows.

1. For any monotone security S, let q^S denote the row n-vector whose ith component q_i^S equals the proportion $[S(y_i) - S(y_{i-1})]/(y_i - y_{i-1})$ of the cash flow increment that security S pays to investors when the cash flow rises from y_{i-1} to y_i. It is easy to see that the portfolio consisting of q_i^S shares of each tranche i has the same payout function as the security S. And since S is monotone, the increment $S(y_i) - S(y_{i-1})$ in the security's payout lies between zero and $y_i - y_{i-1}$, inclusive, whence each quantity q_i^S lies in $[0,1] = [0, a_i^*]$ and is thus feasible.

2. Conversely, let q be a portfolio of tranches in the given Asset Sale game. This portfolio has the same payout function as the single security $S^q(Y) = qF^*(Y)$. Clearly, $S^q(Y)$ is zero at $Y = 0$, is nondecreasing in the cash flow Y as the tranches have this property, and rises no more quickly than $a^*F^*(Y) = Y$ which has a slope of at most one in Y. Thus, S^q is a monotone security as claimed.[56]

By this correspondence and PROPOSITION 1, there is a unique intuitive outcome of the EPSD game, which can be computed using RLP:

[56] A similar construction appears in Hart and Moore (1995), although they use it for a different purpose.

PROPOSITION 7. Suppose the issuer is restricted to monotone securities $S \in M$ in the EPSD game and the issuer's information satisfies First-Order Stochastic Dominance: for any cutoff y and types $t > s$, $\Pr(Y \le y \mid s) \ge \Pr(Y \le y \mid t)$. Then in any intuitive equilibrium of the EPSD game, the issuer's optimal security $S_t^*(Y)$ equals $q^*(t)F^*(Y)$ where $q^*(t)$ solves the RECURSIVE LINEAR PROGRAM.

PROOF: Online appendix. ◆

3.4. The Optimality of Debt

While PROPOSITION 7 shows how to compute the issuer's unique optimal security, it does not reveal much about the security's structure. Under HRO, we can say much more: this security is standard debt. And in order to signal optimism, the issuer chooses a lower face value of this debt, thus retaining a larger portion of her cash flow.

A rough argument is as follows. One can easily verify that the tranches F^* are PRIORITIZED ASSETS (Section 2.5), where the seniority of tranche i is declining in the index i.[57] Hence, under HRO, PROPOSITION 3 implies that the issuer uses a hurdle class strategy: she sells all shares of tranches i with indices below some threshold c, no shares of tranches $i > c$, and some $q_c^*(t)$ shares of tranche c. Her corresponding monotone security in the EPSD game is thus

$$\begin{aligned} S^q(Y) = qF^*(Y) &= q_c^*(t)(y_c - y_{c-1}) \times 1[Y \ge y_c] + \sum_{i=1}^{c-1}(y_i - y_{i-1}) \times 1[Y \ge y_i] \\ &= \min(D, Y) \end{aligned}$$

where $D = y_{c-1} + q_c^*(t)(y_c - y_{c-1})$. But this is just a standard debt contract with face value D. By PROPOSITION 3, $q^*(t)$ is nonincreasing in t, whence higher-type issuers choose a weakly lower face value D.[58]

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[57] For any two tranches $i < j$, $F_j^*(Y) = F_i^*(Y)r(Y)$ where $r(Y)$ equals zero if $Y < y_j$ and $(y_j - y_{j-1})/(y_i - y_{i-1})$ otherwise. Thus, tranche i is senior to tranche j.

[58] We will show that this face value is, in fact, strictly decreasing in the issuer's type.

Prior results can now be used to determine the face value D for any type t. Let $v(D,t) = E[\min\{Y,D\} \mid t]$ denote the expected payout of debt with face value D conditional on the issuer's type t. Let D_t^* denote the equilibrium face value chosen by an investor of type t. If an investor of type t imitates a type t+1 investor by choosing face value D_{t+1}^*, investors must assign the value $v\left(D_{t+1}^*, t+1\right)$ to her security by fair pricing. Since her valuation of the security is $\delta v\left(D_{t+1}^*, t\right)$, her payoff from the deviation is the difference $v\left(D_{t+1}^*, t+1\right) - \delta v\left(D_{t+1}^*, t\right)$. On the other hand, if she chooses her equilibrium face value of D_t^*, investors will pay her $v\left(D_t^*, t\right)$ for a security that she values at $\delta v\left(D_t^*, t\right)$, so her payoff is $\left(1-\delta\right) v\left(D_t^*, t\right)$. Since, by PROPOSITION 4**,** each type $t = 0, \ldots, T-1$ is just willing not to imitate type $t+1$, the face value D_{t+1}^* of type $t+1$ is uniquely[59] given by

$$v\left(D_{t+1}^*, t+1\right) - \delta v\left(D_{t+1}^*, t\right) = \left(1-\delta\right) v\left(D_t^*, t\right). \tag{10}$$

Finally, the most pessimistic issuer (type zero) sells her entire cash flow, which is equivalent to choosing debt with face value D_0^* equal the maximum cash flow y_n. This yields the following characterization for the security design game:

> **PROPOSITION 8.** Suppose the issuer's information satisfies HRO and the issuer is restricted to monotone securities. Then in the unique intuitive equilibrium of the Ex-Post Security Design game, the optimal security design is standard debt with face value given by (10) with initial condition $D_0^* = y_n$. The face value D_t^* is positive and strictly decreasing in the issuer's type t.
>
> **PROOF:** Online appendix. ◆

3.5. Continuous Security Design

The preceding results assume discrete distributions of the issuer's type t and the cash flow Y. In the applications in DeMarzo (2005) and Frankel and Jin (2015), these distributions

[59] The left side of (10) is strictly increasing in $D \in (0, y_n]$. Because $v(D,t+1) > V(D,t)$ and $v(0,t) = 0$, a unique solution D_{t+1}^* to (10) exists, which lies in $\left(0, D_t^*\right)$.

are continuous. We will show that in the continuous model, the optimal face value of the issuer's debt is given in closed form by equation (1).

3.5.1. Characterizing the Face Value

Recall that $v(D,t) = E[\min\{Y,D\}\,|\,t]$ denotes the conditional expected payout of simple debt with face value *D*. If the increment between types is Δ rather than one, the incentive compatibility condition (10) becomes

$$v\left(D_{t+\Delta}, t+\Delta\right) - \delta v\left(D_{t+\Delta}, t\right) = \left(1-\delta\right) v\left(D_t, t\right). \tag{11}$$

Taking the derivative of (11) with respect to the increment Δ at $\Delta = 0$, and letting $\pi(D,t)$ be the conditional probability $\Pr\left(Y < D\,|\,t\right)$ of default, we obtain equation (1) from the introduction.[60] Specifically, by PROPOSITION 8, the most pessimistic issuer sells her entire cash flow, and higher types choose a lower face value and borrow less to signal their optimism:

$$D_0 = \bar{y} \quad \text{and} \quad \frac{dD_t}{dt} = -\frac{v_2(D_t,t)}{\left(1-\delta\right)\left(1-\pi(D_t,t)\right)} < 0, \tag{12}$$

where $\bar{y}$ is the maximum value of the cash flow *Y*.

If this initial value problem has a unique solution D_t^∞, then it is a separating equilibrium of the continuous model:[61]

> **PROPOSITION 9.** Assume HRO and the existence of a unique, continuous solution D_t^∞ to (12). Then there is a separating equilibrium of the continuous model of the security design game in which, given her type *t*, the issuer announces a security whose payout function is $S^*(Y) = \min\left(D_t^\infty, Y\right)$.
>
> **PROOF:** Online appendix. ◆

[60] The derivative is $(1-\delta)v_1\left(D_t,t\right)\frac{dD_t}{dt} + v_2\left(D_t,t\right) = 0$, which yields (1) since $v_1\left(D,t\right) = 1 - \pi\left(D,t\right)$.

[61] Existence and uniqueness are discussed below in section 3.5.2.

We now illustrate the continuous solution with a numerical example, depicted in Figure 4 below.

EXAMPLE 5. Suppose the cash flows Y are lognormally distributed conditional on the issuer's information. Let the issuer's information equal the mean of ln(Y), so that $Y = 100e^{t-\sigma^2/2+\sigma Z}$ where Z is standard normal.[62] Suppose $t \in [0, .25]$ and $\delta = 0.95$. Figure 4 shows the issuer's optimal face value as a function of t for different volatilities σ. As shown, the face value falls in t, while it is nonmonotone in the volatility σ.

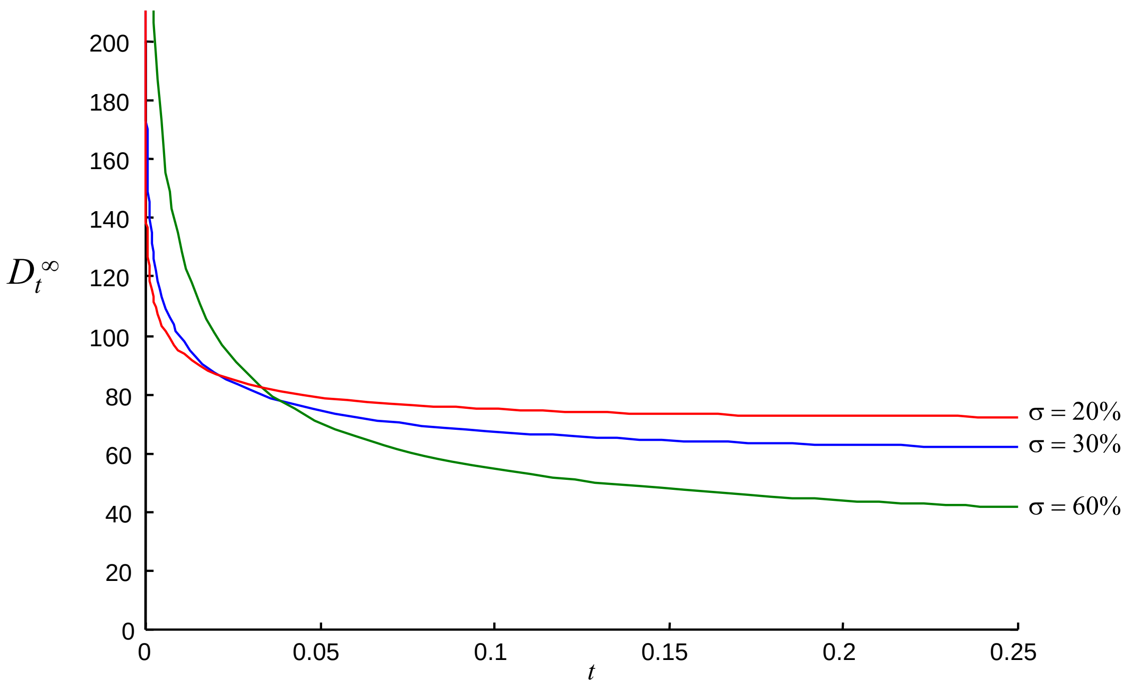

Figure 4: Optimal Face Value of Debt as a Function of Issuer Type and Asset Volatility

Since the quality of the debt depends upon *both* face value and volatility, it is more meaningful to compare the issuance revenue $E[\min(D_t^\infty, Y) \mid t]$ in these same three cases. This comparison appears in Figure 5. As shown, issuance revenue is decreasing in the volatility of the cash flows. Intuitively, when the volatility σ is large, the "cash flow surprise" $Y - E[Y \mid t]$ is more likely to take on large negative values which cause the

[62] The convexity correction $-\sigma^2/2$ implies that the conditional expected value $E[Y \mid t]$ equals $100e^t$ and thus is unaffected by changes in the volatility σ.

security to default. This makes the issuer's information more critical in evaluating the expected payout of her security. Thus, convincing investors that her type is higher causes a larger increase in the security's price when volatility is high. In order to deter imitation by lower types, the issuer must therefore send a more costly signal: she must give up more issuance revenue. Hence issuance revenue declines more quickly in the issuer's type when volatility σ is higher in Figure 6.

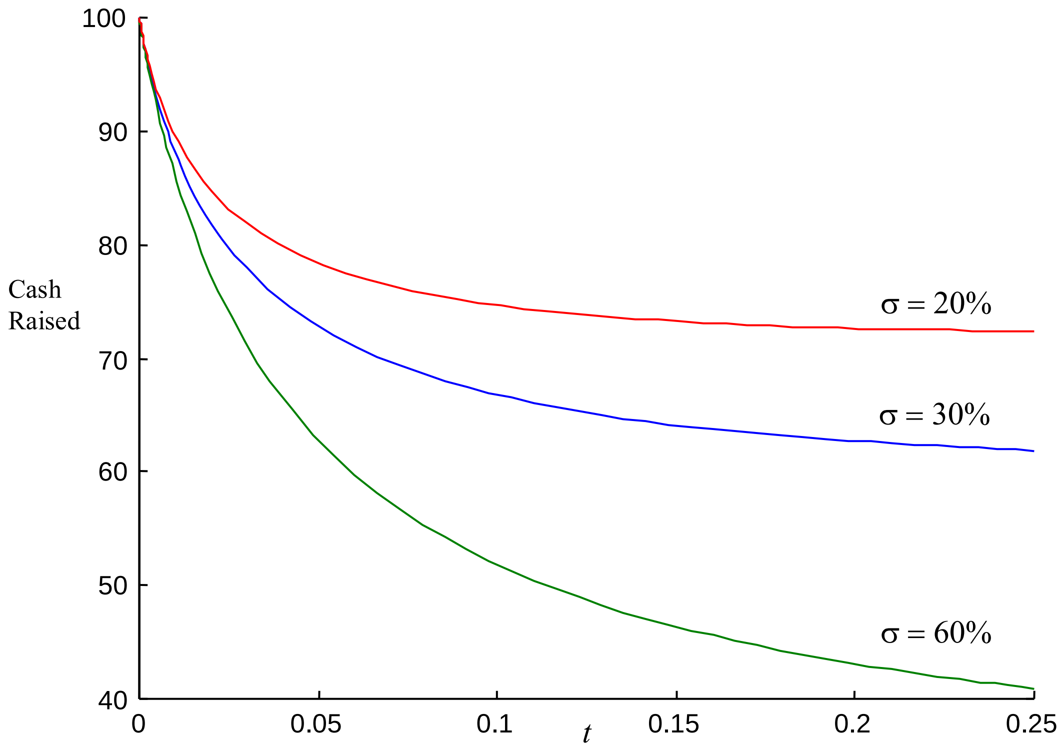

Figure 5: Issuance Revenue as a Function of Issuer Type and Asset Volatility

After a suitable rescaling, the curves in Figure 6 can be reinterpreted as the issuer's conditional expected payoffs.[63] Thus, the incentive to sell assets with minimal informational sensitivity leads the issuer to prefer a less volatile underlying cash flow. This "implied risk aversion" holds even though all participants are risk neutral and creates the incentive for pooling in DeMarzo (2005).

[63] In this rescaling, the numbers on the vertical axis are multiplied by the difference $1-\delta$ in discount factors.

3.5.2. Existence, Uniqueness, and Convergence (Technical)

PROPOSITION 9 assumes the existence of a unique solution to equation (1). In our online appendix, we verify this property. We assume the issuer's type $t \sim G$ has full support [0,1] and the final cash flow Y has conditional and unconditional support $[0, \bar{y}]$ and conditional distribution function $H(\cdot \mid t)$. The function H satisfies the Hazard Rate Ordering property and the following regularity condition:

> **LIPSCHITZ-*H* (L-*H*).** The conditional distribution function H is continuously differentiable, has no atoms, and has some minimal sensitivity to its arguments: there are constants k_0, $k_1 \in (0,\infty)$ such that for all t in [0,1] and y in $[0, \bar{y}]$, the derivative $H_1(y \mid t)$ is in (k_0,k_1) and $-H_2(y \mid t)$ lies in $[k_0 y(\bar{y} - y), k_1)$.[64,65] Further, both partial derivatives of H are Lipschitz continuous in the type t: there is a constant k_2 in $(0,\infty)$ such that for all y in $[0, \bar{y}]$ and t', t'' in [0,1], both $\left|H_1(y \mid t') - H_1(y \mid t'')\right|$ and $\left|H_2(y \mid t') - H_2(y \mid t'')\right|$ are less than $k_2 \left|t' - t''\right|$.

We show that, in this setting, there exists a unique solution to equation (1), which is the limit of the solutions to the discrete problem (11) as the gap Δ between types shrinks to zero. Hence, the unique solution to equation (1) is a good approximation of the unique intuitive equilibrium in the discrete model. This strengthens the case for the solution to using equation (1) to study the continuous model.

In addition, a researcher may wish to embed the security design problem in a setting in which the joint distribution of types and shocks depends on the actions chosen by the issuer and other players in a prior "pregame" period (e.g., Frankel and Jin 2015). In our online appendix, we also show that as the gaps between types and shocks shrink to zero, the gap between profits in the discrete and continuous model also shrinks to zero, uniformly in the

[64] A higher type t is good news about the shock and thus lowers $H(y|t)$. Accordingly, we state the bounds in terms of $-H_2(y \mid t)$ which is nonnegative.

[65] As $H(0 \mid t)$ and $H(\bar{y} \mid t)$ are identically zero and one, respectively, we cannot require that $H_2(y \mid t)$ be sensitive to the type t for each y. However, we can require that as y moves away from its highest and lowest values, this sensitivity rises at least linearly in y. This is ensured by the factor $y(\bar{y} - y)$ in the lower bound on $-H_2(y \mid t)$.

joint distribution of types and shocks (and thus in the pregame action profile). This result may be used in applications to show that that the issuer's optimal action in the discrete case converges to that of the continuous model.[66] This robustness property further strengthens the case for using the solution to (1) rather than any alternative equilibria.

3.6. Empirical Predictions

Begley and Purnanandam (2017) find that when the equity (retained) tranche of a pool of residential mortgage-backed securities makes up a larger proportion of the pool's face value, the loans in the pool have lower subsequent delinquency rates conditional on observables and the securities that are sold fetch higher prices conditional on their credit ratings. This is implied by equation (1): the issuer's face value D_t is lower (and so her equity tranche is higher) when her type t is higher (and so she is more optimistic about the repayment probabilities of the underlying loan pool).

Begley and Purnanandam also find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation loans. This is also predicted by equation (1), by the following result.

PROPOSITION 10. Assume HRO and Lipschitz-*H* holds for conditional distribution functions $\hat{H}$ and $\tilde{H}$. Suppose that $\tilde{H}(y|1) \geq \hat{H}(y|1)$ and $\tilde{H}_2(y|t) \leq \hat{H}_2(y|t)$ for all $t \in [0,1]$ and $y \in [0,\overline{y}]$. Let the face value functions $\hat{D}_t$ and $\tilde{D}_t$ solve equation (1) for H equal to $\hat{H}$ and $\tilde{H}$, respectively. Then $\hat{D}_t \geq \tilde{D}_t$ for all t: the issuer does not choose a lower face value under $\hat{H}$ than under $\tilde{H}$.

PROOF: Online appendix. ◆

PROPOSITION 10 relates to Begley and Purnanandam as follows. Suppose a bank plans to issue loans with aggregate face value $\overline{y}$. Prior to lending, the bank chooses whether to issue no-documentation or full-documentation loans. It then issues the loans and privately observes a local macroeconomic shock t that affects the distribution of the aggregate

[66] The convergence of the issuer's optimal action will generally require additional assumptions, such as the strict quasiconcavity of the issuer's expected total profits as a function of the issuer's pregame action.

repayment $Y \in [0, \bar{y}]$. Assume, plausibly, that the distribution of the macro shock *t* does not depend on the outcome of the prior coin flip.

Let $\tilde{H}(\cdot \mid t)$ and $\hat{H}(\cdot \mid t)$ denote the distributions of the aggregate repayment *Y* conditional on the macro shock *t*, in the no- and full-documentation cases respectively. The main effect of requiring documentation is to prevent loan applicants from exaggerating their financial strength. Thus, even if the shock *t* takes its highest value of one, the probability that the repayment *Y* falls below any fixed threshold will be higher when there are more no-documentation loans in the pool. This is the first inequality assumed in PROPOSITION 10. Moreover, as no-documentation borrowers tend to be more financially fragile conditional on observables, the repayment *Y* will be more sensitive to the shock *t*. Thus, the probability that *Y* falls below any given threshold will be more sensitive to the macro shock *t* in the case of no-documentation loans. This is captured by the second inequality assumed in PROPOSITION 10.

PROPOSITION 10 then predicts that if the bank chooses to issue no-documentation loans, the face value of its security will be lower for any given shock *t*: it will retain a higher portion of its loans' face value, as Begley and Purnanandam find. An intuition appears above in section 1.

4. Relaxing Monotonicity

ASSUMPTION A includes a monotonicity property: each asset's expected value is nondecreasing in the issuer's type. In our online appendix we show that, for generic parameters, this property can be omitted in the case of two assets and two types. Intuitively, one type must expect her portfolio to have a higher total payout than the other. Swapping indices if needed, we can assume this is type 2. Moreover, an increase in the issuer's type from 1 to 2 must raise the expected value of one asset proportionally more than that of the other. Again swapping indices if needed, we can assume this is asset 2. Hence, Increasing Informational Sensitivity (IIS) holds automatically in the 2x2 case: an increase in the issuer's type from 1 to 2 raises the expected value of asset 2 proportionally more than that of asset 1. We show, further, that IIS can be used in place of monotonicity to prove that there is a unique intuitive equilibrium in the 2x2 case.

As in our base model under IIS, the equilibrium displays the Pecking Order property: the issuer will not retain any shares of asset 1 unless she retains asset 2 in its entirety. However, a stronger prediction emerges in the special case in which asset 1 is worth *less* to type 2 than to type 1 - a case that ASSUMPTION A rules out. In this situation, the issuer uses only asset 2 to signal her type: she never retains any shares of asset 1.

To illustrate, consider debt and equity securities, and suppose that both the mean and volatility of the cash flow are higher for type 2 than for type 1. With a large enough difference in volatility, debt will be worth less to type 2 than to type 1. In this case, both types sell the debt in its entirety: the issuer retains only equity in order to signal her type. Intuitively, type 1 gains more from retaining debt than type 2 does. Hence, if type 2 were to retain some debt, then she would have to retain even more equity in order to signal credibly that she is not type 1. Such inefficient signaling is ruled out by the Intuitive Criterion. Although we do not pursue this point further, the feature that securities whose value declines with type will not be retained should extend to more general settings.

5. Final Remarks

In this paper we study the problem of an informed issuer who wishes to sell securities to raise cash. Using the Intuitive Criterion, we identify the unique equilibrium when the issuer has a fixed set of assets whose expected payouts are nondecreasing in her information. We show by example that the usual monotonicity property can fail: a more optimistic issuer may sell more shares of a given asset. This occurs when the assets alternate in their relative sensitivity to the issuer's information.

In the rest of the paper we assume the assets can be ordered according to their informational sensitivity. In this case, the issuer sells her least informationally sensitive assets first. If her information satisfies the Hazard Rate Ordering property and her assets securities can be ranked by seniority (using a novel and weak definition), senior assets are less informationally sensitive. In this case, the issuer will retain all securities with seniority below a threshold, where this threshold increases with her optimism.

We also consider a general setting in which, before and/or after seeing her information, the issuer can design new assets that are secured by her initial assets. Under the Hazard Rate

Ordering, she has two equivalent, optimal strategies. First, she may pool her initial assets, see her information, and issue a single debt security whose face value is decreasing in her degree of optimism. Second, she may pool her initial assets, design a maximal set of prioritized tranches, see her information, and sell those tranches whose seniority exceeds a threshold that is increasing in her information. The latter behavior, in particular, mirrors the way loan pools are commonly securitized. When cash flows are continuously distributed, the level of debt issued (or, equivalently, the most junior tranche sold) is given by a simple differential equation.

The model has empirical predictions that have been confirmed by Begley and Purnanandam (2017) in the case of residential mortgage-backed securities. First, controlling for observables (including the proportion of no-documentation loans), loans in securitization deals with larger equity (retained) tranches have lower delinquency rates. Second, loan pools with more no-documentation loans have larger equity tranches.

We focus throughout our paper on a market setting in which the issuer can signal quality through quantity choices. Williams (2015) instead adapts the competitive search framework of Guerrieri, Shimer, and Wright (2010) to demonstrate that it is also possible to achieve a similar equilibrium outcome in which issuers signal quality by choosing the liquidity of the market in which they choose to trade, and shows that debt is the optimal monotone security in that context as well. An interesting extension might be to consider equilibrium liquidity choices when the issuer can sell multiple securities as in our paper. Another possibility, in a dynamic context, might be to allow the issuer to signal by delaying its trades as in Varas (2014).

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7. Appendix

This appendix contains the omitted proofs of the results in section 2. Omitted proofs from section 3 appear in our online appendix: DeMarzo, Frankel, and Jin (2017).

PROOF OF PROPOSITION 1. We first show by induction that each $u^*(t)$ is well defined (exists and is unique) and positive. First, $u^*(0)$ is well-defined and positive by ASSUMPTION A. Now suppose $u^*(s)$ is well-defined and positive for $s < t$. Then for type t, the set $A_t = \{q \in A: \text{ for all } s < t, u^*(s) \geq q[f(t) - \delta f(s)]\}$ is nonempty as it includes the zero vector; it is compact as it is defined using weak inequalities. Hence, $u^*(t)$ and $q^*(t)$ exist by the Extreme Value Theorem. Also, since there exists $\varepsilon > 0$ sufficiently small so that εa lies in A_t, so $u^*(t) \geq (1-\delta)\varepsilon a f(t) \geq (1-\delta)\varepsilon a f(0) > 0$: it is positive as well.

Why is u^* nonincreasing? By construction, the payoff $u^*(s)$ that RLP assigns to any type s is not less than the payoff $q^*(t)[f(t) - \delta f(s)]$ that this type would get by imitating any type $t > s$ in any fairly priced equilibrium. This imitation payoff, in turn, cannot be less than the payoff $u^*(t) = (1-\delta) q^*(t) f(t)$ assigned to type t since the sales revenue $q^*(t) f(t)$ is the same and, by ASSUMPTION A, the opportunity cost of selling the assets is no higher for s than for t: $\delta q^*(t) f(s) \leq \delta q^*(t) f(t)$. Intuitively, while type t sells for a higher price, in order to separate from lower types it must raise less cash, reducing its attainable surplus.

We next show by induction that for any type t, $u^*(t)$ is an upper bound on the issuer's payoff and on social welfare in a fairly priced equilibrium. Investors must break even in any fairly priced equilibrium. This implies that the issuer's payoff equals social welfare. It also implies our base case: the payoff of an issuer of type 0 is bounded above by the potential gains from trade $u^*(0)$ for this type. As for types $t > 0$, if type t's equilibrium quantity vector q does not lie in A_t, then there is a type $s < t$ which, by choosing q, gets a payoff $q[f(t) - \delta f(s)]$ (by fair pricing for type t), which exceeds the upper bound $u^*(s)$ on her equilibrium payoff: type s will deviate. Hence type t must choose a quantity vector in A_t, whence $u^*(t)$ is an upper bound on t's payoff as well by fair pricing.[68]

From the prior result, any fairly priced equilibrium with outcome u^* must be optimal for the issuer and is efficient within the set of fairly priced equilibria. We now verify that the profile $\left(q^*(\cdot), \overline{p}^*(\cdot), p^*(\cdot,\cdot), \mu^*(\cdot \mid \cdot,\cdot)\right)$ defined in the text is a fairly priced equilibrium with outcome u. This will imply that it is an intuitive equilibrium since, as shown in the text, beliefs μ^* are intuitive.

By construction, the beliefs function $\mu^*\left(\cdot \mid \cdot,\cdot\right)$ is given by Bayes's rule on the equilibrium path, whence Rational Updating holds in the definition of an Asset Sale Equilibrium. Since the price function $p^*\left(\cdot,\cdot\right)$ is the result of substituting μ^* for μ in (2), Competitive Pricing holds. To verify that the profile is an equilibrium, it thus remains to verify Payoff Maximization. We do so below.

We next verify that the given strategies and beliefs satisfy fair pricing and yield the payoff function $u^*\left(\cdot\right)$ for the issuer. For any possible choice $\left(q, \overline{p}\right)$ of the issuer, let $T(q, \overline{p})$ denote the set $\left\{t \mid (q, \overline{p}) \text{ equals } (q^*(t), \overline{p}^*(t))\right\}$ of types who make this choice in equilibrium. If $T(q, \overline{p})$ is a singleton $\{t\}$, then $p^*(q, \overline{p}) = f(t)$ by (2) as the price caps $\overline{p}^*(t)$ are nonbinding by construction. Else let the types $s < t$ lie in $T(q, \overline{p})$, whence $q^*(s) = q^*(t) = q$ so, by RLP, $u^*(s) = (1-\delta)qf(s) \geq q(f(t)-\delta f(s))$ or, equivalently, $q(f(t)-f(s)) \leq 0$. Since $f(t) \geq f(s)$ by ASSUMPTION A, if $q_i > 0$ then $f_i(t) = f_i(s)$. Since this is true for all types in $T(q, \overline{p})$, $q_i > 0$ implies $p_i^*(q, \overline{p}) = f_i(t)$ for all $t \in T(q, \overline{p})$ by (2): the equilibrium is fairly priced. Thus $q^*(t)p^*(q^*(t), \overline{p}^*(t)) = q^*(t)f(t)$, whence the payoff of a type-t issuer is $u^*(t) = (1-\delta)q^*(t)f(t)$ as claimed.

[68] The definition of u^* does not rule out pooling. If $q^*\left(t\right)f\left(s\right)$ equals $q^*\left(t\right)f\left(t\right)$, then type s could pool with type t in equilibrium: $q^*\left(s\right)$ might equal $q^*\left(t\right)$.

We now turn to Payoff Maximization. We first show that no type wishes to mimic any other type. For suppose type s mimics type t. Since the equilibrium is fairly priced, type s does not gain from this deviation as long as the following inequality holds.

$$IC(s,t): \qquad q^*(t)\,(f(t) - \delta f(s)) \leq u^*(s) = (1-\delta)\, q^*(s)\, f(s).$$

To verify $IC(s,t)$ for all s, t, it suffices to show that if $IC(s,s')$ holds for all types s and s' that are strictly less than t (which is clearly so when $t = 0$), then $IC(s,s')$ also holds for all $s, s' \leq t$. For $s' < t$, $IC(s',t)$ must hold by RLP, while $IC(t,s')$ holds by the following lemma with $p = f(s')$ and $q = q^*(s')$; the lemma's assumptions hold since $f(s') \leq f(t)$ by ASSUMPTION A and since $IC(s,s')$ holds for all $s < t$ by induction.

LEMMA. Let $p \in \Re^n_+$ and let $q \in \Re^n_+$ satisfy $q \leq a$. Suppose $q(p-f(t)) \leq 0$ and, for all $s < t$, $q(p-\delta f(s)) \leq u^*(s)$. Then $q(p-\delta f(t)) \leq u^*(t)$.

PROOF OF LEMMA: First, $q(p-\delta f(t)) = q(p-f(t))+(1-\delta)qf(t) \leq (1-\delta)qf(t)$. Now if q lies in A_t (defined in RLP), then $(1-\delta)qf(t) \leq u^*(t)$ so we are done. Else let α be the largest scalar such that αq lies in A_t. Then $\alpha < 1$ and, for some $s < t$, $\alpha q(f(t)-\delta f(s)) = u^*(s) \geq q(p-\delta f(s))$. Rearranging, and using the fact that $\alpha < 1$ and the definition of u^*, we obtain

$$q(p-\delta f(t)) \leq (1-\delta)\alpha qf(t)+\delta(1-\alpha)q(f(s)-f(t)) \leq (1-\delta)\alpha qf(t) \leq u^*(t)$$

as claimed. ◆

Accordingly, no type will imitate any other type. We now show that no type will deviate to any choice $(q,\bar{p})$ that lies off the equilibrium path: for which $T(q,\bar{p}) = \varnothing$. As shown in the text, if type $\tau^*(q)$ does not gain from deviating to $(q,\bar{p})$, then no other type gains from deviating to $(q,\bar{p})$. It thus suffices to show that type $\tau^* = \tau^*(q)$ will not deviate to $(q,\bar{p})$. Given beliefs μ^*, the price vector following the deviation is $p^*(q,\bar{p}) = \bar{p} \wedge f(\tau^*) \leq f(\tau^*)$ by (2). Thus, for type τ^* not to gain by deviating to $(q,\bar{p})$, it suffices that $(1-\delta)qf(\tau^*) \leq u^*(\tau^*)$. By RLP, to prove this inequality it suffices to verify that q is in A_{τ^*}. For suppose not: q is not in A_{τ^*}. Let α be the largest scalar such that

αq is in A_{τ^*}. Then $\alpha < 1$ and by continuity we must have $\alpha q\,[f(\tau^*)-\delta f(s)] = u^*(s)$ for some $s < \tau^*$. Therefore, by (5),

$$\alpha q[f(\tau^*)-\delta f(s)] = u^*(s) = u^*(s)+\delta qf(s)-\delta qf(s) > u^*(\tau^*)+\delta qf(\tau^*)-\delta qf(s).$$

(The inequality is strict by (5) and the fact that $s < \tau^*$.) This can be rearranged to yield

$$u^*(\tau^*) < (1-\delta)\alpha qf(\tau^*) + \delta q(1-\alpha)(f(s)-f(\tau^*)) \le (1-\delta)\alpha qf(\tau^*),$$

where the last inequality follows since $\alpha < 1$ and $f(s) \le f(\tau^*)$. But this implies that αq, which lies in A_{τ^*}, is strictly better than $q^*(\tau^*)$ is for type $\tau^*(q)$ – a contradiction. Having now verified Payoff Maximization, we conclude that the profile $(q^*, \overline{p}^*, p^*, \mu^*)$ is an Asset Sale Equilibrium.

To finish the proof of PROPOSITION 1, it remains to show that every intuitive equilibrium of the asset sale game is fairly priced, has the same outcome $u = u^*$, and has an asset sale function $q^*(\cdot)$ that solves RLP. We showed above that $(q^*, \overline{p}^*, p^*, \mu^*)$ is an equilibrium. Now consider any other intuitive equilibrium with outcome u. First we show that it is fairly priced. Suppose that with positive probability, type t makes asset sale decision $(q, \overline{p})$. Assume investors respond with price vector p. First say $qp < qf(t)$: the issue is underpriced. Let s be the largest type such that $qf(s) < qf(t)$.[69] Then define $\lambda \in [0,1)$ such that

$$\frac{q(p-\delta f(t))}{q(f(t)-\delta f(t))} < \lambda < \frac{q(p-\delta f(s))}{q(f(t)-\delta f(s))}. \tag{13}$$

(Such a λ must exist as the second ratio is increasing in $-qf(s)$.) Consider the feasible deviation $(\lambda q, f(t))$ for type t. First, by ASSUMPTION A, for all $s' \le s$, $qf(s') \le qf(s) < qf(t)$ and so from (13), $\lambda q[f(t) - \delta f(s')] < q[p - \delta f(s')] \le u(s')$, where the last inequality follows from the incentive constraint for type s'. Thus, no type $s' \le s$ has an incentive to deviate to $(\lambda q, f(t))$.

[69] If there is no such type s, then for all types t', $f_i(t) \ge f_i(t')$ whenever q_i is positive. Thus, if type t deviates to $(q, f(t))$, investors will respond with a price vector $p' \ge f_i(t)$, which is better than p for t.

On the other hand, from (13), $\lambda q[f(t) - \delta f(t)] > q[p - \delta f(t)] = u(t)$, so type t could gain from the deviation if the realized price p equals $f(t)$. But then the Intuitive Criterion implies that $\mu(s' \mid \lambda q, f(t)) = 0$ for all $s' \leq s$. That is, investor beliefs must put weight only on types t' such that $qf(t') \geq qf(t)$. By ASSUMPTION A, for these types $f_i(t') \geq f_i(t)$ for all securities i such that $q_i > 0$. Hence, for each such security i,

$$p_i(\lambda q, f(t)) = f_i(t) \wedge \sum_{t'} f_i(t') \, \mu(t' \mid \lambda q, f(t)) = f_i(t).$$

Accordingly, type t gains from the deviation to $(\lambda q, f(t))$ - a contradiction. Thus, it must be the case that $qp \geq qf(t)$ for all types t that make sale decision $(q, \overline{p})$ in equilibrium; that is, there is no equilibrium underpricing. But from (2),

$$qp = q\left[\overline{p} \wedge \sum_{t'} f(t)\mu(t \mid q, \overline{p})\right] \leq \sum_{t'} qf(t)\,\mu(t \mid q, \overline{p}).$$

That is, qp is bounded above by a convex combination of $qf(t)$ for all types t that make sale decision $(q, \overline{p})$ in equilibrium. But as just shown, qp is also bounded below by $qf(t)$ for any such t. Thus, qp must equal $qf(t)$ for all types t that ever choose $(q, \overline{p})$: without underpricing there can be no overpricing.

Further, if t, t' make sale decision $(q, \overline{p})$ in equilibrium and $t > t'$, then $f(t) \geq f(t')$ by ASSUMPTION A. Since $qf(t) = qf(t')$, it must be that $f_i(t) = f_i(t')$ if $q_i > 0$. Hence, $p_i(q, \overline{p}) = f_i(t)$ if $q_i > 0$, whence the equilibrium is fairly priced.

It remains to show that in any intuitive equilibrium, the outcome u equals u^* and the equilibrium asset sale function $q(\cdot)$ solves RLP.

Let $q^*(\cdot)$ solve RLP. As shown above, any intuitive equilibrium is fairly priced, whence $u \leq u^*$. Let t be the smallest type such that either (a) $u(t) < u^*(t)$ or (b) $q(t)$ does not solve RLP. As $u(s) = u^*(s)$ for all $s < t$, the payoff $u(t)$ equals its maximum value of $u^*(t)$ if and only if $q(t)$ solves RLP. Hence, conditions (a) and (b) are equivalent: they *both* must hold for type t. Since $u(t) < u^*(t)$, it follows that $u(t) + \delta q^*(t)f(t)$ is less than $u^*(t) + \delta q^*(t)f(t)$, which equals $q^*(t)f(t)$ by RLP. Also by RLP, for all $s < t$, $q^*(t)f(t)$ does not exceed

$u^*(s) + \delta q^*(t)f(s)$, which equals $u(s) + \delta q^*(t)f(s)$ by hypothesis. Hence, there exists a price cap vector $\overline{p}$ close to but less than $f(t)$ such that for any type $s < t$,

$$u(t) + \delta q^*(t) f(t) < q^*(t)\overline{p} < q^*(t) f(t) \le u(s) + \delta q^*(t) f(s).$$

But then intuitive beliefs put no weight on types $s < t$ if $(q^*(t), \overline{p})$ is observed. Hence, $p(q^*(t), \overline{p}) = \overline{p}$. But then because $u(t) < q^*(t)(\overline{p} - \delta f(t))$, type t could gain by deviating to $(q^*(t), \overline{p})$ – a contradiction. It follows that there is no such minimum type t: for all types t, $u(t) = u^*(t)$ as claimed and $q(\cdot)$ solves RLP. ◆

PROOF OF PROPOSITION 2: Since $q_j^*(0) = a_j$ for all j, we may assume $t > 0$. Omitting the constraint $0 \le q \le a$ in RLP and terms that are independent of q, the Lagrangian is $qf(t) - \sum_{s<t} \lambda(s) q\left[f(t) - \delta f(s)\right]$. The derivative with respect to q_i can then be written as $\delta \sum_{s<t} \lambda(s) f_i(s) - f_i(t) k$ where $k = \sum_{s<t} \lambda(s) - 1$. Thus, $q_j^*(t) < a_j$ implies $\sum_{s<t} \lambda(s)\left[f_j(s)/f_j(t)\right] \le k/\delta$. But the Lagrange multipliers $\lambda(s)$ are all nonnegative and, as $t > 0$, positive for at least one $s < t$.[70] Since $f_i(s)/f_i(t) < f_j(s)/f_j(t)$, it follows that $\sum_{s<t} \lambda(s)\left[f_i(s)/f_i(t)\right] < k/\delta$, and thus $q_i^*(t) = 0$ as claimed. ◆

PROOF OF PROPOSITION 3: The fact that $q^*(t)$ is of the given form follows from IIS and PROPOSITION 2. The existence of a hurdle class implies that q^* is ordered: either $q^*(t) \le q^*(s)$ or $q^*(t) \ge q^*(s)$. Furthermore, PROPOSITION 1 implies $u^*(t) \le u^*(s)$ and thus $q^*(t) f(t) \le q^*(s) f(s)$. Since $f(t) \ge f(s)$ by ASSUMPTION A, it follows that $q^*(t) \ge q^*(s)$ as claimed. ◆

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[70] If instead none of the incentive compatibility constraints bind, then the equilibrium payoff $u^*(s) = (1-\delta) q^*(s) f(s)$ of each type $s < t$ must exceed $q^*(t)\left[f(t) - \delta f(s)\right]$ which in turn is not less than $(1-\delta) q^*(t) f(s)$ by ASSUMPTION A. But this can hold only if $q_\ell^*(t) < q_\ell^*(s)$ for at least one asset ℓ. Hence it is possible to raise $q_\ell^*(t)$ and thus type t's payoff $u^*(t)$ without violating any constraints – which contradicts the optimality of $q^*(t)$ in RLP.

PROOF OF PROPOSITION 4. First, $q^*(0) = a$ by RLP and ASSUMPTION B. For $t > 0$, since $f(t) - \delta f(t-1) > 0$ by ASSUMPTION A and ASSUMPTION B, a unique hurdle class vector satisfies equation (6), which is the incentive constraint in RLP for $s = t-1$. It remains to show that this constraint binds when $s = t-1$. Suppose instead that

$$q^*(t)[f(t) - \delta f(t-1)] < u^*(t-1) = (1-\delta)q^*(t-1)f(t-1).$$

From the definition of $q^*(t-1)$ in RLP, for any $s < t-1$, $q^*(t-1)\ (\ f(t-1) - \delta\ f(s)\ ) \le u^*(s)$. Combining these two yields:

$$q^*(t)[f(t) - \delta f(t-1)] - (1-\delta)q^*(t-1)f(t-1) + q^*(t-1)\ (\ f(t-1) - \delta\ f(s)\ ) < u^*(s)$$

or equivalently, $q^*(t)\ (\ f(t) - \delta\ f(s)\ ) + \delta\ (\ q^*(t-1) - q^*(t)\ )\ (\ f(t-1) - f(s)\ ) < u^*(s)$. From PROPOSITION 1, $q^*(t-1) \ge q^*(t)$, and from ASSUMPTION A, $f(t-1) \ge f(s)$. Thus,

$$q^*(t)\ (\ f(t) - \delta\ f(s)\ ) < u^*(s),$$

which implies that none of the incentive constraints in RLP bind for $q^*(t)$. This contradicts the optimality of $q^*(t)$ unless $q^*(t)\, f(t) = a\, f(t)$. But by the initial supposition,

$$q^*(t)\, f(t) < u^*(t-1) + \delta\ q^*(t)\, f(t-1) \le a\, f(t-1) \le a\, f(t).$$

Hence, the incentive constraint for $t-1$ must bind. ◆

PROOF OF PROPOSITION 5. The proposition follows from PROPOSITION 1 as long as we can show that if j is junior to i then it is more informationally sensitive; that for any $t > s$,

$$f_j(t) / f_j(s) > f_i(t) / f_i(s)\,.$$

Fix types $t > s$. As both informational sensitivity and priority are invariant to an arbitrary rescaling of each security's payout, w.l.o.g. we can let $f_j(t) = f_i(t) = 1$. Hence, we need to show that for $s < t$, $f_j(s) < f_i(s)$.

For convenience, let Y_t denote a random variable whose unconditional distribution equals the posterior distribution of the cash flow Y given the type t: $\Pr\left(Y_t \le z\right) = \Pr\left(Y \le z \mid t\right)$. Because asset j is junior to i, $F_j(Y) = r(Y)F_i(Y)$ where, by the above normalization,

$$0 = f_j(t) - f_i(t) = E[F_j(Y_t) - F_i(Y_t)] = E[(r(Y_t) - 1)F_i(Y_t)]$$
$$= E[(r(Y_t) - 1)F_i(Y_t) \mid F_i(Y_t) > 0].$$

Thus, because r is nondegenerate on the set $F_i(Y_t) > 0$, it must lie strictly below 1 and strictly above 1 with positive probability on this set.

Next we make use of the following lemma.

LEMMA: Suppose Y satisfies HRO. Then there exists a random variable Z with the same support as, and independent of, Y_t, such that Y_s and $Y_t \wedge Z$ have the same distributions.

PROOF: Let $\overline{y}$ denote the essential supremum of Y: $\overline{y} = \inf\{y : \Pr(Y \geq y) = 0\}$. Fix $s < t$, and let $R(y) = \dfrac{\Pr(Y_s \geq y)}{\Pr(Y_t \geq y)}$ for $y < \overline{y}$ and zero for $y > \overline{y}$, with $R(\overline{y})$ defined so that R is left-continuous. Moreover, $R(0) = 1$ and R is strictly decreasing on the support of Y and constant elsewhere. Thus we can define a new, independent random variable Z with distribution $Pr(Z \geq y) = R(y)$ that has the same support as Y.[71] Finally,

$$\Pr(Y_t \wedge Z \geq y) = \Pr(Y_t \geq y, Z \geq y) = \Pr(Y_t \geq y)\Pr(Z \geq y) = \Pr(Y_t \geq y)\frac{\Pr(Y_s \geq y)}{\Pr(Y_t \geq y)}$$

which equals $\Pr(Y_s \geq y)$ as claimed. ◆

Using the lemma, together with the monotonicity of the securities, we have

$$f_i(s) = E\left[F_i(Y) \mid s\right] = E\left[F_i(Y_s)\right] = E\left[F_i(Y_t \wedge Z)\right],$$

and similarly for f_j. If $r(z) < 1$, then

$$E\left[F_j(Y_t \wedge z)\right] = E\left[r(Y_t \wedge z)F_i(Y_t \wedge z)\right] \leq E\left[F_i(Y_t \wedge z)\right],$$

where the inequality is strict unless $F_i(z) = 0$. Conversely, if $r(z) \geq 1$ then

[71] See e.g. Theorem 12.4 of Billingsley (1986).

$$
\begin{aligned}
E\left[F_j(Y_t \wedge z)\right] &= E\left[F_j(Y_t) - \left(F_j(Y_t) - F_j(z)\right)^+\right] \\
&= 1 - E\left[\left(r(Y_t)F_i(Y_t) - r(z)F_i(z)\right)^+\right] \\
&\leq 1 - E\left[\left(r(z)F_i(Y_t) - r(z)F_i(z)\right)^+\right] \\
&\leq 1 - E\left[\left(F_i(Y_t) - F_i(z)\right)^+\right] = E\left[F_i(Y_t \wedge z)\right].
\end{aligned}
$$

Thus, $f_j(s) = E\left[F_j(Y_t \wedge Z)\right] < E\left[F_i(Y_t \wedge Z)\right] = f_i(s)$ where the inequality is strict because there is a positive probability that $F_i(Z) > 0$ and $r(Z) < 1$. ◆