

# Online Appendix to “Portfolio Liquidation and Security Design with Private Information”<sup>†</sup>

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## 1. Introduction

This document is the online appendix to “Portfolio Liquidation and Security Design with Private Information” by Peter M. DeMarzo, David M. Frankel, and Yu Jin (2017), henceforth “DFJ”. It contains the proofs of section 3 of DFJ, as well as the statements and proofs of technical results that we preview in section 3.5.2 of DFJ.

## 2. Technical Results

We first formally state the technical results that we discuss in section 3.5.2 of DFJ. We begin by specifying the continuous and discrete models. The continuous model, denoted “model  $\infty$ ”, is as follows. The issuer's type  $t \sim G$  has full support  $[0,1]$  and the final asset value  $Y$  has conditional and unconditional support  $[0, \bar{y}]$  and conditional distribution function  $H$  that satisfies the Hazard Rate Ordering property and Lipschitz- $H$  (defined in DFJ, section 3.5.2).

We now define a sequence of discrete models  $i = 1, 2, \dots$  that converge to model  $\infty$ . Let  $(N_i)_{i=1}^{\infty}$  and  $(N'_i)_{i=1}^{\infty}$  be any two increasing sequences of positive integers. In model  $i$ , let the gaps between adjacent types  $t$  and shocks  $Z$  be  $\Delta_i = 1/N_i$  and  $\Delta'_i = \bar{y}/N'_i$ , respectively. That is,  $t$  lies in  $S_i = \{0, \Delta_i, \dots, 1 - \Delta_i, 1\}$  and  $Z$  lies in  $S'_i = \{0, \Delta'_i, \dots, \bar{y} - \Delta'_i, \bar{y}\}$ . By

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construction, both gaps  $\Delta_i$  and  $\Delta'_i$  converge to zero as  $i$  goes to infinity. Let the conditional distribution of  $Y$  in model  $i$  be the restriction of the continuous distribution function  $H$  to types in  $S_i$  and shocks in  $S'_i$ . Similarly, the distribution of  $t$  in model  $i$  is the restriction of  $G$  to types  $t$  in  $S_i$ .<sup>67</sup>

Let  $E^i$  and  $E^\infty$  denote the expectations operators in models  $i$  and  $\infty$ , respectively. Let  $v^i(D, t) = E^i[\min(D, Y) | t]$  denote the expected payout of simple debt with face value  $D$  in model  $i$  given a type  $t \in S_i$ . Let  $v^\infty(D, t) = E^\infty[\min(D, Y) | t]$  denote the expected payout of the same security in model  $\infty$  given a type  $t \in [0, 1]$ .

Fix equilibria in models  $i$  and  $\infty$  in which the issuer's security is simple debt. Let  $D_t^i$  and  $D_t^\infty$  be the equilibrium face values of these securities for a given type  $t$ . The equilibrium price of this security in model  $i$ , denoted  $p^i(t)$ , is simply the security's expected payout  $v^i(D_t^i, t)$ . And the issuer's expected issuance profit, denoted  $u^i(t)$ , is simply the gains from trade  $(1 - \delta)v^i(D_t^i, t)$  as competition drives investors' payoffs to zero. Similarly, in the continuous model the price  $p^\infty(t)$  of the security equals the expected payout  $v^\infty(D_t^\infty, t)$  and the issuer's profit  $u^\infty(t)$  equals the expected gains from trade  $(1 - \delta)v^\infty(D_t^\infty, t)$ .

As argued in the footnote on p. 33 of DFJ, the face value function in the continuous model must satisfy the following problem.

**THE CONTINUOUS INITIAL VALUE PROBLEM (CIVP).**

$$D_0 = \bar{y} \text{ and, for each } t, \frac{dD_t^*}{dt} = \frac{1}{1 - \delta} \frac{\int_{y=0}^{D_t^*} H_2(y | t) dy}{1 - H(D_t^* | t)} < 0.$$

We now state our first result, which concerns existence and uniqueness.

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<sup>67</sup> That is, in model  $i$ , the probability that the type does not exceed some  $t$  in  $S_i$  is  $G(t)$ , while the probability, conditional on a type  $t$ , that the cash flow  $Y$  does not exceed some  $y$  in  $S'_i$  is  $H(y | t)$ .

**PROPOSITION 11.** Assume Hazard Rate Ordering and Lipschitz- $H$ .

1. There exists a unique function  $D^\infty$  that satisfies CIVP for  $v = v^\infty$ . This function is decreasing and differentiable in the type  $t$ , and takes values in  $(0, \bar{y}]$ . The associated price and profit functions,  $p^\infty$  and  $u^\infty$ , are decreasing and continuous in the type  $t$ .
2. For each discrete model  $i=1,2,\dots$ , there exists a unique, decreasing function  $D^i$  with  $D_0^i = \bar{y}$  and satisfying (11) with  $v = v^i$  for all  $t \in S_i$  and  $\Delta = \Delta_i$ .

An intuition is as follows. The Picard-Lindelöf theorem is the usual tool for proving the existence and uniqueness of the solution to a differential equation. Unfortunately, we cannot apply this theorem directly because the differential equation in CIVP is not Lipschitz continuous in  $D^\infty$ : it approaches negative infinity as  $D^\infty$  approaches  $\bar{y}$ .

We sidestep this difficulty in the following way. We define upper and lower bounds on  $D^\infty$  using a modification of CIVP that is Lipschitz continuous with constant  $k$ . We then show that as  $k$  grows, these upper and lower bounds approach the same limit, which satisfies CIVP and thus must be its unique solution  $D^\infty$ .

Having established the existence of a unique solution, we now show that the face value functions in the discrete model converge to the face value of the continuous model, uniformly in the issuer's type  $t$ . Since the discrete equilibria uniquely satisfy the intuitive criterion, this result gives a reason to focus on the continuous equilibrium that we have identified.

**PROPOSITION 12.** Assume HRO and Lipschitz- $H$ . For any  $\varepsilon > 0$  there is an  $i^*$  such that for all models  $i > i^*$  and all types  $t$  in  $[0,1]$ ,  $|D_t^i - D_t^\infty| < \varepsilon$ .<sup>68</sup>

The idea of the proof is as follows. For any model  $i = 1,2,\dots$  and constant  $k > 0$ , we first show that any solution  $D^i$  must lie between fixed upper and lower bounds  $\bar{D}_k^i$  and  $\underline{D}_k^i$ ,

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<sup>68</sup> Technically, the function  $D^i$  is defined only for types  $t$  in the discrete set  $S_i$ . We extend it to all types  $t$  in  $[0,1]$  by evaluating it at the highest type in  $S_i$  that does not exceed  $t$ .

where these bounds are Lipschitz continuous with constant  $k$ . Moreover, these bounds converge to the aforementioned upper and lower bounds on  $D^\infty$  as  $i$  grows. By the prior intuition, these bounds on  $D^\infty$  converge in turn to  $D^\infty$  as  $k$  grows. Thus, by taking  $i$  and  $k$  to infinity simultaneously,  $\bar{D}_k^i$  and  $\underline{D}_k^i$  - and thus  $D^i$  which lies between them - must converge to the unique solution  $D^\infty$  of model  $\infty$ .

Our next result gives conditions under which the convergence of the discrete model to the continuous model is uniform in various parameters. This property can be useful in applications in which the security design game is preceded by some interaction in which the issuer also chooses an optimal action, as in Frankel and Jin (2015). In such settings, the result can help establish that the issuer's optimal choices in the discrete model are well-approximated by her optimal choice in the continuous model.

Henceforth, we assume the distribution of types  $G$  is continuous with a strictly positive density  $g$  that satisfies the following technical condition:

**LIPSCHITZ-G (L-G).** There are constants  $k_3, k_4 \in (0, \infty)$  such that for all types  $t$  and  $t'$  in  $[0, 1]$ ,  $g(t) \leq k_3$  and  $|g(t) - g(t')| \leq k_4 |t - t'|$ .

We now show that the key functions of the model converge uniformly in the distributions  $G$  and  $H$ , the type  $t \in [0, 1]$ , and the cash flow parameter  $\bar{y}$ .<sup>69</sup> These key functions are the face value function  $D^i$ , price function  $p^i$ , and the conditional profit function  $u^i$ . We also show uniform convergence of the issuer's unconditional expected issuance profits  $Eu^i = E^i[u^i(t)]$  to its continuous analogue,  $Eu^\infty = E^\infty[u^\infty(t)]$ . Finally, let  $\Pi^i(t)$  denote the issuer's conditional *total* profits in model  $i$ : the sum of issuance profits  $u^i(t)$  and the conditional expected portfolio return  $E^i[Y | t]$ . Let  $E\Pi^i = E^i[\Pi^i(t)]$  denote unconditional expected total profits.<sup>70</sup> We show that these converge uniformly to their continuous counterparts,  $\Pi^\infty(t) = u^\infty(t) + E^\infty[Y | t]$  and  $E\Pi^\infty = E^\infty[\Pi^\infty(t)]$ .

<sup>69</sup> Uniformity in  $t$  does not apply to the expected profit functions  $Eu$  and  $E\Pi$ , as they do not depend on  $t$ .

<sup>70</sup> This last quantity is especially important in applications: if there is a pregame period, the issuer will act so as to maximize the sum of  $E\Pi^i$  (perhaps multiplied by a discount factor) and any pregame payoff.

**PROPOSITION 13.** Fix constants  $k_0, k_1, k_2, k_3, k_4$ , and  $\mathbf{y}$ , all in  $(0, \infty)$ . Let  $\mathbf{H}$  be the set of conditional distribution functions  $H(z|t)$  that satisfy Hazard Rate Ordering and Lipschitz- $H$  with constants  $k_0, k_1$ , and  $k_2$ . Let  $\mathbf{G}$  be the set of distribution functions  $G$  that satisfy Lipschitz- $G$  with constants  $k_3$  and  $k_4$ . For all  $\varepsilon > 0$  there is an  $i^*$  such that for all models  $i > i^*$ ,  $G$  in  $\mathbf{G}$ ,  $H$  in  $\mathbf{H}$ ,  $\bar{y}$  in  $[0, \mathbf{y}]$ , and  $t$  in  $[0, 1]$ ,  $|\omega^i(t) - \omega^\infty(t)|$  is less than  $\varepsilon$  for  $\omega$  equal to  $D, p, u$ , and  $\Pi$ ; and  $|\mathbb{E}\omega^i - \mathbb{E}\omega^\infty|$  is less than  $\varepsilon$  for  $\omega$  equal to  $u$  and  $\Pi$ .<sup>71</sup>

We next show a useful comparative statics property: the issuer chooses a higher face value when the rate of change function in (1) (which is negative) is smaller in absolute value. PROPOSITION 10 is essentially a special case of this result (and its proof relies on this result).

**PROPOSITION 14.** Let the discount factors  $\hat{\delta}$  and  $\tilde{\delta}$  lie in  $(0, 1)$ . Assume the conditional distribution functions  $\hat{H}$  and  $\tilde{H}$  satisfy HRO and Lipschitz- $H$ . Suppose that for any given  $D$  and  $t$ , the rate of change function  $\frac{1}{1-\delta} \int_{y=0}^D H_2(y|t) dy$  in (1) is no larger in absolute value when  $(\delta, H)$  equals  $(\hat{\delta}, \hat{H})$  than when it equals  $(\tilde{\delta}, \tilde{H})$ . Let the face value functions  $\hat{D}_t$  and  $\tilde{D}_t$  solve equation (1) for  $(\delta, H)$  equal to  $(\hat{\delta}, \hat{H})$  and  $(\tilde{\delta}, \tilde{H})$ , respectively. Then for all  $t$ ,  $\hat{D}_t \geq \tilde{D}_t$ : the issuer does not choose a lower face value under  $(\hat{\delta}, \hat{H})$  than under  $(\tilde{\delta}, \tilde{H})$ .

Finally, we show a homogeneity property that can simplify the analysis of models in which security design is embedded (e.g., Frankel and Jin (2015)).

**COROLLARY 15.** The face value functions  $D^i$  and  $D^\infty$ , the price functions  $p^i$  and  $p^\infty$ , the profit functions  $u^i$ ,  $\Pi^i$ ,  $u^\infty$ , and  $\Pi^\infty$ , and the issuer's expected profits

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<sup>71</sup> As in Proposition 14, we extend these functions to all types  $t$  in  $[0, 1]$  by evaluating them at the highest type in  $S_i$  that does not exceed  $t$ .

$Eu^i$ ,  $E\Pi^i$ ,  $Eu^\infty$ , and  $E\Pi^\infty$ , defined above, are all homogeneous of degree one in the cash flow parameter  $\bar{y}$ .

### 3. Proofs

We now give the omitted proofs from section 3 of DFJ, as well as the proofs of the results of section 2 of this online appendix.

**PROOF OF PROPOSITION 6.** We first define the Intuitive Criterion in the GSD game. Consider an equilibrium and any interim asset vector  $I$ . If type  $t$  sticks to her continuation strategy, she will get  $u(t|I) \stackrel{d}{=} U(I, P_t(I), \rho_t(I) | t, p)$ . If instead she chooses some other *ex-post* action  $(P, \rho)$ , for her to lose from this deviation it suffices that her equilibrium payoff  $u(t|I)$  exceeds her maximum payoff  $\rho - \delta E[W_p^I(Y) | t]$  from the deviation – or, equivalently, that

$$\rho < u(t|I) + \delta E[W_p^I(Y) | t]. \quad (13)$$

The right hand side of (13) is thus the minimum revenue that type  $t$  requires to be willing to deviate to  $(P, \rho)$ .

The Intuitive Criterion for the GSD game states that on seeing the deviation  $(P, \rho)$  following  $I$ , investors must put zero probability on type  $t$  if she is never willing to choose  $(P, \rho)$  following  $I$  but some other type might be: if condition (13) holds for  $t$  but fails for some other type  $s$ . That is:

**THE INTUITIVE CRITERION (GSD GAME).** An equilibrium of the GSD game with posterior belief function  $\mu$  and outcome  $u$  is *intuitive* if, on seeing any action  $(I, P, \rho)$ , investors' posterior probability  $\mu(t|I, P, \rho)$  is zero for any type  $t$  that satisfies (13) as long as there is some type  $s$  for which the inequality is reversed: for which  $\rho \geq u(s|I) + \delta E[W_p^I(Y) | s]$ .

Let  $E = (\hat{I}, P_t(\cdot), \rho_t(\cdot), p, \mu)$  be any equilibrium of the GSD game. We will show that the following equilibrium  $\tilde{E} = (\tilde{I}, \tilde{P}_t(\cdot), \tilde{\rho}_t(\cdot), \tilde{p}, \tilde{\mu})$  is also an equilibrium, and is intuitive if  $E$  is.

1. **Securitization.** In  $\tilde{E}$ , the issuer's interim asset vector  $\tilde{I} = (\tilde{I}^1)$  consists of a single security whose payout equals her cash flow:  $\tilde{I}^1(Y) = Y$ . She then chooses an *ex-post* action as follows for any given  $t$  and interim asset vector  $I$ . First, if she deviated in the prior stage (so that  $I \neq \tilde{I}$ ), she chooses the *ex-post* action  $(P_t(I), \rho_t(I))$  that she would choose in  $E$  after choosing  $I$ . Else she issues an *ex-post* action that consists of her equilibrium revenue cap  $\rho_t(\hat{I})$  in  $E$  together with an *ex-post* asset vector consisting of a single security whose payout equals the aggregate payout  $\sum_{j=1}^{\#(P_t(\hat{I}))} P_t^j(\hat{I})(\hat{I}(Y))$  of her equilibrium *ex-post* security vector  $P_t(\hat{I})$  in  $E$ .
2. **Beliefs.** Let  $\tilde{T}(P, \rho | I)$  denote the set of types whose strategies in  $\tilde{E}$  instruct them to choose the *ex-post* action  $(P, \rho)$  conditional on having chosen some interim asset vector  $I$ . Upon seeing an action  $(\tilde{I}, P, \rho)$  that is expected in  $\tilde{E}$ , the investors' posterior probability  $\mu(t | \tilde{I}, P, \rho)$  equals the probability that the issuer's type is  $t$  conditional on her type being in  $\tilde{T}(P, \rho | \tilde{I})$ .<sup>72</sup> That is, it equals  $g(t) \left[ \sum_{s \in \tilde{T}(P, \rho | \tilde{I})} g(s) \right]^{-1}$  if  $t \in \tilde{T}(P, \rho | \tilde{I})$  and zero otherwise. Upon seeing an action  $(I, P, \rho)$  that is unexpected in  $\tilde{E}$ ,  $\tilde{\mu}(t | I, P, \rho)$  equals  $\mu(t | I, P, \rho)$ : its value in  $E$ .

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<sup>72</sup> An action  $(\tilde{I}, P, \rho)$  is *expected* in an equilibrium if it is selected with positive probability in that equilibrium.

3. Pricing. Prices are given by equation (9) with  $\tilde{\mu}$  substituted for  $\mu$ : for any action  $(I, P, \rho)$ , the resulting price vector is  $\tilde{p}(I, P, \rho) = \sum_t E[P(I(Y)) | t] \tilde{\mu}(t | I, P, \rho)$ .

Pricing and Beliefs ensure that  $\tilde{E}$  satisfies Competitive Pricing and Rational Updating. As for Payoff Maximization, the payoff of an issuer of type  $t$  from taking an action  $(I, P, \rho)$  that is unexpected in  $\tilde{E}$  equals her payoff from taking this action in  $E$ , since, by Pricing and Beliefs, investors respond with the same price vector. As for expected actions in  $\tilde{E}$ , we rely on the following claim, whose proof appears below.

**CLAIM.** Let  $(\tilde{I}, P, \rho)$  be expected in  $\tilde{E}$ , and let  $\Sigma$  denote the set of actions  $(\hat{I}, P', \rho)$  that are taken in  $E$  by any type whose prescribed action in  $\tilde{E}$  is  $(\tilde{I}, P, \rho)$ .

Then an issuer of any given type gets the same payoff in  $\tilde{E}$  from taking the action  $(\tilde{I}, P, \rho)$  as she gets in  $E$  from taking any action in  $\Sigma$ .

By this Claim, no action in  $\tilde{E}$  affords an issuer of a given type more than her equilibrium payoff in  $E$ ; and taking her prescribed action in  $\tilde{E}$  gives her the same payoff that she gets, in equilibrium, in  $E$ . Thus she will take her prescribed action in  $\tilde{E}$ : Payoff Maximization holds, whence  $\tilde{E}$  is an equilibrium as claimed.

**PROOF OF CLAIM.** Let  $(\hat{I}, P', \rho)$  be any action in  $\Sigma$ . Note first that  $(\hat{I}, P', \rho)$  has the same aggregate payout function  $W(Y) = P^1(\tilde{I}^1(Y))$  as  $(\tilde{I}, P, \rho)$  and thus the same aggregate expected payout conditional on the issuer's type,  $E[W(Y) | t]$ . Hence the Claim holds if  $(\hat{I}, P', \rho)$  raises the same revenue in  $E$  as  $(\tilde{I}, P, \rho)$  raises in  $\tilde{E}$ . We first show that all actions in  $\Sigma$  raise the same revenue in  $E$ :

**LEMMA.** In  $E$ , following the equilibrium interim asset choice  $\hat{I}$ , if there are two pactions that are expected given  $\hat{I}$  and that have the same revenue cap and aggregate payout function, they must raise the same revenue.



**PROOF OF LEMMA.** Suppose not: there are types  $t$  and  $t'$  (possibly equal) who, after choosing  $\hat{I}$  and seeing their types, choose pactions  $(P, \rho)$  and  $(P', \rho)$ , respectively, that have the same revenue cap and aggregate payout function, such that  $(P, \rho)$  yields less revenue than  $(P', \rho)$ :  $\rho \wedge \sum_{j=1}^{\#(P)} p^j(I, P, \rho) < \rho \wedge \sum_{j=1}^{\#(P')} p^j(I, P', \rho)$ . Since the aggregate payout functions are identical, their conditional expectations are the same conditional on the issuer's type being  $t$ :  $E[W_P^{\hat{I}}(Y)|t] = E[W_{P'}^{\hat{I}}(Y)|t]$ . But then by (8), type  $t'$ 's payoff  $U(\hat{I}, P', \rho | t, p)$  from  $(P', \rho)$  exceeds her payoff  $U(\hat{I}, P, \rho | t, p)$  from  $(P, \rho)$  which she is therefore unwilling to choose - a contradiction. ♦

By this Lemma, each action in  $\Sigma$  raises the same issuance revenue  $R_\Sigma$  in  $E$ . And if, in  $E$ , the issuer chooses action  $(\hat{I}, P', \rho)$  in  $\Sigma$  then, by (7) and (9), investors' willingness to pay  $WTP_E(\hat{I}, P', \rho) = \sum_{j=1}^{\#(P')} p^j(\hat{I}, P', \rho)$  for her *ex-post* assets equals their estimate  $\sum_t E[W(Y)|t] \mu(t | \hat{I}, P', \rho)$  of the aggregate payout given what the action  $(\hat{I}, P', \rho)$  reveals about the issuer's type. The issuance revenue  $R_\Sigma$  in  $E$  is then the minimum of  $WTP_E(\hat{I}, P', \rho)$  and the cap  $\rho$ . Since, by the above Lemma, this issuance revenue is the same for any action  $(\hat{I}, P', \rho)$  in  $\Sigma$ , it follows that either (i)  $WTP_E(\hat{I}, P', \rho) > \rho$  for every such action or (ii)  $WTP_E(\hat{I}, P', \rho) \leq \rho$  for every such action. Moreover, in case (ii),  $WTP_E(\hat{I}, P', \rho)$  must take a common value  $WTP_E(\Sigma)$  for any action  $(\hat{I}, P', \rho)$  in  $\Sigma$ .

Likewise, if the issuer chooses action  $(\tilde{I}, P, \rho)$  in  $\tilde{E}$  then the amount  $\tilde{p}^1(\tilde{I}, P, \rho)$  that investors are willing to pay for her *ex-post* security is just their estimate

$\sum_t E[W(Y)|t] \tilde{\mu}(t|\tilde{I}, P, \rho)$  of this security's payout conditional on the action  $(\tilde{I}, P, \rho)$ ; the resulting issuance revenue is the minimum of  $\tilde{p}^1(\tilde{I}, P, \rho)$  and the cap  $\rho$ .

But for each  $t$ , by Beliefs,  $\tilde{\mu}(t|\tilde{I}, P, \rho)$  can be written as  $\frac{g(t)1_{t \in \tilde{T}(P, \rho|\tilde{I})}}{\sum_s g(s)1_{s \in \tilde{T}(P, \rho|\tilde{I})}}$ . And

the sets  $T(P', \rho|\hat{I})$  of types who, in  $E$ , take actions  $(\hat{I}, P', \rho) \in \Sigma$ , is a partition of the set  $\tilde{T}(P, \rho|\tilde{I})$  of types who take the action  $(\tilde{I}, P, \rho)$  in  $\tilde{E}$ . Thus we can rewrite  $\tilde{\mu}(t|\tilde{I}, P, \rho)$  as a weighted sum  $\sum_{(\hat{I}, P', \rho) \in \Sigma} \mu(t|\hat{I}, P', \rho) w_{(\hat{I}, P', \rho)}$  where the

weight  $w_{(\hat{I}, P', \rho)} = \frac{\sum_s g(s)1_{s \in T(P', \rho|\hat{I})}}{\sum_s g(s)1_{s \in \tilde{T}(P, \rho|\tilde{I})}}$  is the probability that the issuer would have

chosen  $(\hat{I}, P', \rho)$  in  $E$  conditional on her having chosen  $(\tilde{I}, P, \rho)$  in  $\tilde{E}$ . Hence we

can rewrite  $\tilde{p}^1(\tilde{I}, P, \rho)$  as  $\sum_{(\hat{I}, P', \rho) \in \Sigma} w_{(\hat{I}, P', \rho)} \sum_t E[W(Y)|t] \mu(t|\hat{I}, P', \rho)$  or,

equivalently, as  $\sum_{(\hat{I}, P', \rho) \in \Sigma} w_{(\hat{I}, P', \rho)} R_E(\hat{I}, P', \rho)$ . Since, the weights  $w_{(\hat{I}, P', \rho)}$  sum to one,

the revenue  $\rho \wedge R_{\tilde{E}}(\tilde{I}, P, \rho)$  raised by the action  $(\tilde{I}, P, \rho)$  in  $\tilde{E}$  equals  $\rho$  in case

(i) above (where the revenue raised in  $E$  by each  $(\hat{I}, P', \rho)$  in  $\Sigma$  is also  $\rho$ ) and the

common value  $WTP_E(\Sigma)$  in case (ii) (when the revenue raised in  $E$  by each

$(\hat{I}, P', \rho)$  in  $\Sigma$  is also  $WTP_E(\Sigma)$ ). This proves the Claim. ♦

It remains to show that if  $E$  is intuitive, so is  $\tilde{E}$ . Suppose  $E$  is intuitive. Let  $u(\cdot|I)$  and  $\tilde{u}(\cdot|I)$  denote the equilibrium payoffs in  $E$  and  $\tilde{E}$ , respectively, of an issuer of type  $t$  who chooses the interim asset vector  $I$  and then follows her continuation strategy in the given equilibrium. Consider any action  $(I, P, \rho)$  for which there are types  $t$  and  $s$  satisfying:

$$\rho < \tilde{u}(t|I) + \delta E[W_p'(Y)|t] \text{ and } \rho \geq \tilde{u}(s|I) + \delta E[W_p'(Y)|s]. \quad (14)$$

First assume  $(I, P, \rho)$  is unexpected in  $\tilde{E}$ . As previously noted, the payoff of an issuer of type  $t$  from taking the action  $(I, P, \rho)$  in  $\tilde{E}$  equals her payoff from taking this action in  $E$ . It follows that  $\tilde{u}(v|I) = u(v|I)$  for  $v = s, t$ . Hence  $\rho < u(t|I) + \delta E[W_p'(Y)|t]$  and  $\rho \geq u(s|I) + \delta E[W_p'(Y)|s]$  whence, since  $E$  is intuitive,  $\mu(t|I, P, \rho)$  is zero. But since  $(I, P, \rho)$  is unexpected in  $\tilde{E}$ , Beliefs implies that  $\mu(t|I, P, \rho)$  equals  $\tilde{\mu}(t|I, P, \rho)$  which thus also is zero as required.

Now suppose instead that  $(I, P, \rho)$  is expected in  $\tilde{E}$ :  $\tilde{T}(P, \rho|I)$  is nonempty. By (14),  $\tilde{u}(t|I)$  exceeds  $\rho - \delta E[W_p'(Y)|t]$  which is the highest payoff type  $t$  can expect from  $(I, P, \rho)$ . Hence  $t$  is not in  $\tilde{T}(P, \rho|I)$  whence, by Beliefs,  $\tilde{\mu}(t|I, P, \rho)$  is zero. We conclude that  $\tilde{E}$  is intuitive as claimed.  $\blacklozenge$

**PROOF OF PROPOSITION 7.** We first define the notions of equilibrium, intuitive beliefs, and fair pricing for an EPSD game. Each definition is the natural restriction of the analogous definition for a GSD game.

**EPSD EQUILIBRIUM.** A perfect Bayesian equilibrium of the EPSD game is a security design  $S_t \in M$  and price cap  $\bar{\rho}_t \in \mathfrak{R}_+$  for each type  $t$ , as well as a price function  $\hat{p}(S, \bar{\rho})$  and belief function  $\hat{\mu}(t|S, \bar{\rho})$ , with the following properties:

1. **Payoff Maximization:** for all  $t$ , the issuer's choice  $(S_t, \bar{\rho}_t)$  solves  $\max_{S, \bar{\rho}} \{ \hat{p}(S, \bar{\rho}) - \delta E[S(Y)|t] \}$  subject to  $S \in M$ .
2. **Competitive Pricing:** for any monotone security  $S \in M$  and price cap  $\bar{\rho}$ , the price function  $\hat{p}(S, \bar{\rho})$  equals  $\min \{ \bar{\rho}, \sum_t E[S(Y)|t] \hat{\mu}(t|S, \bar{\rho}) \}$ .
3. **Rational Updating:** the investors' belief function  $\hat{\mu}(t|S, \bar{\rho})$  follows Bayes's rule when applicable.

**FAIR PRICING (EPSD GAME).** An equilibrium  $\left((S_t, \bar{\rho}_t)_{t=0}^T, \hat{p}, \hat{\mu}\right)$  of the GSD game is *fairly priced* if, for each type  $t$ , the price of *ex-post* security  $S_t$  equals its expected value conditional on the issuer's type:  $\hat{p}(S_t, \bar{\rho}_t) = E[S_t(Y)|t]$ .

The *outcome* of the EPSD game is the function  $\hat{u}(t) = \hat{p}(S_t, \bar{\rho}_t) - \delta E[S_t(Y)|t]$  giving the securitization payoff of each type  $t$ .

**THE INTUITIVE CRITERION (EPSD GAME).** A perfect Bayesian equilibrium  $\left((S_t, \bar{\rho}_t)_{t=0}^T, \hat{p}(\cdot, \cdot), \hat{\mu}(\cdot | \cdot, \cdot)\right)$  of the EPSD game, with outcome  $\hat{u}(\cdot)$ , is *intuitive* if, for any security  $S \in M$  and revenue cap  $\bar{\rho} \in \mathfrak{R}_+$  for which  $\bar{\rho} \geq \hat{u}(t) + \delta E[S(Y)|t]$  for some type  $t$ ,  $\bar{\rho} < \hat{u}(s) + \delta E[S(Y)|s]$  implies  $\hat{\mu}(s | S, \bar{\rho}) = 0$ .

Define the following two sets.

1.  $I_{AS}$  is the set of intuitive Asset Sale Equilibria  $(q(\cdot), \bar{p}(\cdot), p(\cdot, \cdot), \mu(\cdot | \cdot, \cdot))$  of the asset sale game  $G_{AS}$  with portfolio  $(F^*, a)$  in which investor beliefs following any issuance choice  $(q, \bar{p})$  depend only on the quantity choices  $q$  and the issuer's maximum issuance revenue  $\bar{\rho} = \bar{p}q$ . Changing notation slightly, we now denote these beliefs as  $\mu(\cdot | q, \bar{p}q)$  rather than  $\mu(\cdot | q, \bar{p})$ .
2.  $I_{SD}$  is the set of intuitive equilibria  $\left((S_t, \bar{\rho}_t)_{t=0}^T, \hat{p}(\cdot, \cdot), \hat{\mu}(\cdot | \cdot, \cdot)\right)$  of the Ex Post Security Design game  $G_{SD}$ .

These two sets are equivalent in the following sense

**LEMMA.**

1. For any equilibrium  $e$  in  $I_{AS}$ , there is an equilibrium  $\hat{e}$  in  $I_{SD}$  with the same outcome. If type  $t$  sells the quantities  $q(t)$  in  $e$ , then this type issues the security  $S_t(Y) = q(t)F^*(Y)$  in  $\hat{e}$ .

2. For any equilibrium  $\hat{e}$  in  $I_{SD}$ , there is a set of equilibria  $e$  in  $I_{AS}$  with the same outcome. If type  $t$  issues the security  $S_t$  in  $\hat{e}$ , then this type sells the quantities  $q^{S_t}$  in any such equilibrium  $e$ .

**PROOF OF LEMMA.** Consider any equilibrium  $e$  in  $I_{AS}$ . Its outcome is  $u(t) = q(t)[p(q(t), \bar{p}(t)) - \delta f^*(t)]$  and investors' price function is  $p(q, \bar{p}) = \bar{p} \wedge \sum_t f^*(t) \mu(t|q, \bar{p}q)$ . We build an equivalent Security Design Equilibrium  $\hat{e}$  in  $I_{SD}$  as follows. Type  $t$  issues the monotone security  $S_t(y) = q(t)F^*(y)$  with maximum revenue  $\bar{\rho}_t$  equal to  $\bar{p}(t)q(t)$ . This implies, in particular, that  $q_i^{S_t} = q(t)[F^*(y_i) - F^*(y_{i-1})]/(y_i - y_{i-1}) = q_i(t)$ ; collecting terms,  $q^{S_t} = q(t)$ . Given any security design choice  $(S, \bar{\rho})$ , investors respond with beliefs  $\hat{\mu}(\cdot|S, \bar{\rho}) = \mu(\cdot|q^S, \bar{\rho})$  and associated price

$$\hat{p}(S, \bar{\rho}) = \max\{\bar{\rho}, \sum_t E[S(Y)|t] \hat{\mu}(t|S, \bar{\rho})\},$$

whence Competitive Pricing holds in  $\hat{e}$ . A type  $t$  issuer's payoff from the choice  $(S, \bar{\rho})$  is  $\hat{p}(S, \bar{\rho}) - \delta E[S(Y)|t] = \max\{\bar{\rho}, q^S \sum_t f^*(t) \mu(t|q^S, \bar{\rho})\} - \delta q^S f^*(t)$ , which is identical to her payoff  $q^S [p(q^S, \bar{p}) - \delta f^*(t)]$  from the issuance choice  $(q^S, \bar{p})$  in  $e$  where  $\bar{p}$  is any  $n$ -vector of asset price caps that gives the same maximum revenue:  $\bar{\rho} = \bar{p}q^S$ . Moreover, all such asset price cap vectors give the same payoff in  $e$  because – by assumption – investor beliefs and thus issuance revenue depend only on the quantity vector  $q$  and maximum revenue  $q\bar{p}$ . Hence, since  $(q(t), \bar{p}(t))$  maximizes the payoff of type  $t$  in  $e$ , it follows that  $(S_t, \bar{\rho}_t)$  maximizes the payoff of type  $t$  in  $\hat{e}$ : Payoff Maximization holds in  $\hat{e}$ . By restricting to choices made in equilibrium, the preceding also implies that the outcome  $\hat{u}(t) = \hat{p}(S_t, \bar{\rho}_t) - \delta E[S_t(Y)|t]$  of  $\hat{e}$  equals the outcome  $u(t) = q(t)[p(q(t), \bar{p}(t)) - \delta f^*(t)]$  of  $e$ . By construction,  $\hat{\mu}(\cdot|S_t, \bar{\rho}_t) = \mu(\cdot|q^{S_t}, \bar{\rho}_t) = \mu(\cdot|q(t), \bar{p}(t))$  and the latter is given by Bayes's Rule

whenever possible. Since the issuer's behavior is also the same in the two cases, Rational Updating holds:  $\hat{e}$  is a Security Design Equilibrium. Finally, consider any security design choice  $(S, \bar{\rho})$  in  $\hat{e}$  such that, for some type  $t$ ,

$$\bar{\rho} \geq \hat{u}(t) + \delta E[S(Y) | t] = u(t) + \delta q^S f^*(t).$$

Then following the issuance choice  $(q^S, \bar{\rho})$  in  $e$ , investors put zero weight on any type  $s$  for which  $\bar{\rho}$  is less than  $u(s) + \delta q^S f^*(s)$  which equals  $\hat{u}(s) + \delta E[S(Y) | s]$ . Hence  $\hat{\mu}(s | S, \bar{\rho}) = \mu(s | q^S, \bar{\rho}) = 0$ :  $\hat{e}$  is intuitive.

Consider any equilibrium  $\hat{e}$  in  $I_{SD}$ . Its outcome is  $\hat{u}(t) = \hat{p}(S_t, \bar{\rho}_t) - \delta E[S_t(Y) | t]$  and the investors' price function is  $\hat{p}(S, \bar{\rho}) = \max\{\bar{\rho}, \sum_t E[S(Y) | t] \hat{\mu}(t | S, \bar{\rho})\}$ . We build an equivalent Asset Sale Equilibrium  $e$  in  $I_{AS}$  as follows. (As the price cap vector will not be unique, this defines a set of equilibria  $e$ , as indicated in the statement of this proposition.) Type  $t$  chooses the quantity vector  $q(t) = q^{S_t}$  and any price cap vector  $\bar{p}(t)$  that satisfies  $\bar{p}(t)q(t) = \bar{\rho}_t$ . In response to any asset sale choice  $(q, \bar{p})$ , investors' posterior beliefs are  $\mu(\cdot | q, \bar{p}) = \hat{\mu}(\cdot | qF^*, \bar{p}q)$ .<sup>73</sup> The associated price vector is  $p(q, \bar{p}) = \bar{p} \wedge \sum_t f^*(t) \mu(t | q, \bar{p})$ , which implies Competitive Pricing in  $e$ . The price functions in  $e$  and  $\hat{e}$  are thus related by  $p(q, \bar{p})q = \hat{p}(qF^*, \bar{p}q)$ . Hence, a type  $t$  issuer's payoff  $q[p(q, \bar{p}) - \delta f^*(t)]$  in  $e$  from the issuance choice  $(q, \bar{p})$  equals her payoff  $\hat{p}(qF^*, \bar{p}q) - \delta E[qF^*(Y) | t]$  in  $\hat{e}$  from the security design  $(qF^*, \bar{p}q)$ . Thus, since  $(S_t, \bar{\rho}_t)$  maximizes the payoff of type  $t$  in  $\hat{e}$ , it follows that  $(q^{S_t}, \bar{p})$  maximizes the payoff of type  $t$  in  $e$ , where  $\bar{p}$  is any price cap vector that satisfies  $\bar{p}q^{S_t} = \bar{\rho}_t$ : Payoff Maximization holds in  $e$ . By restricting to choices made in equilibrium, the preceding

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<sup>73</sup> The security  $qF^*(Y)$  in  $G_{SD}$  promises the same realized payout to investors as the quantity vector  $q$  in  $G_{AS}$ .

also implies that the outcome  $u(t) = q(t) \left[ p(q(t), \bar{p}(t)) - \delta f^*(t) \right]$  of  $e$  equals the outcome  $\hat{u}(t) = \hat{p}(S_t, \rho_t) - \delta E[S_t(Y) | t]$  of  $\hat{e}$ . By construction,

$$\mu(\cdot | q(t), \bar{p}(t)) = \hat{\mu}(\cdot | q(t) F^*, \bar{p}(t) q(t)) = \hat{\mu}(\cdot | S_t, \bar{\rho}_t)$$

and the last is given by Bayes's Rule whenever possible. Since the issuer's behavior is also the same in the two settings, Rational Updating holds:  $e$  is an Asset Sale Equilibrium. Finally, consider any issuance choice  $(q, \bar{p})$  in  $e$  such that, for some type  $t$ ,  $\bar{p}q \geq u(t) + \delta q f^*(t) = \hat{u}(t) + \delta E[S(Y) | t]$  where  $S$  denotes the security  $qF^*$ . Then, following the corresponding issuance choice  $(S, \bar{p}q)$  in  $\hat{e}$ , investors put zero weight on any type  $s$  for which  $\bar{p}q < \hat{u}(s) + \delta E[S(Y) | s] = u(s) + \delta q f^*(s)$ . Hence,  $\mu(s | q, \bar{p}) = \hat{\mu}(s | qF^*, \bar{p}q) = 0$ :  $e$  is intuitive. ♦

Now let  $e_{AS}^*$  denote the intuitive equilibrium  $e^*$  of the Asset Sale game, specialized to the case in which the endowment  $(a, f)$  equals the endowment  $(a^*, f^*)$  of  $G_{AS}$ . We claim that  $e_{AS}^*$  lies in  $I_{AS}$ . Why? In  $e_{AS}^*$ , each type  $t$  can choose any price cap vector that exceeds  $f^*(t)$  and investors ignore these caps. Thus, beliefs on the equilibrium path do not depend on the price cap vector. Following any out-of-equilibrium choice  $(q, \bar{p})$ , investors' beliefs are given by (4) and (5), which are also independent of the price cap vector. So investor beliefs in  $e_{AS}^*$  do not depend on the price cap vector at all. Thus, as  $e_{AS}^*$  is intuitive by PROPOSITION 1, it must lie in  $I_{AS}$ .

By the above Lemma, then, there exists an equilibrium  $e_{SD}^*$  in  $I_{SD}$  with the same outcome as  $e_{AS}^*$ , in which each type  $t$  issues the security  $S_t^*(Y) = q^*(t) F^*(Y)$  where  $q^*(t)$  is chosen by type  $t$  in  $e_{AS}^*$  and solves RLP. Finally, FOSD together with the fact that  $F^*$  is monotone implies that ASSUMPTION A holds. Hence, by PROPOSITION 1, the quantity choice function in any equilibrium in  $I_{AS}$  must be a solution  $q^*(\cdot)$  to RLP. Thus, by the above Lemma, in any equilibrium in  $I_{SD}$ , the issuer's optimal security  $S_t^*(Y)$  must equal  $q^*(t) F^*(Y)$  where  $q^*(t)$  solves RLP. ♦

**PROOF OF PROPOSITION 8.** From PROPOSITION 7,  $S_t^*(Y) = q^*(t)F^*(Y)$ . By PROPOSITION 5, IIS holds so that from PROPOSITION 3,  $q_i^*(t) = 1$  for  $i < h(t)$  and  $q_i^*(t) = 0$  for  $i > h(t)$ . Therefore  $S_t^*(y_i) = q^*(t)F^*(y_i) = y_i$  for  $i < h(t)$  and

$$S_t^*(y_i) = q^*(t)F^*(y_i) = y_{h(t)-1} + q_{h(t)}^*(t)(y_{h(t)} - y_{h(t)-1})$$

for  $i \geq h(t)$ . Thus,  $S_t^*(Y) = \min(D_t^*, Y)$  where  $D_t^* = y_{h(t)-1} + q_{h(t)}^*(t)(y_{h(t)} - y_{h(t)-1})$ . Finally, since  $q^*$  and  $h$  are decreasing in  $t$  by PROPOSITION 3,  $D_t^*$  is decreasing in  $t$ . That  $D_t^*$  satisfies (10) follows from PROPOSITION 4. Finally, as the face value function is strictly decreasing, the equilibrium is fully revealing; hence,  $u^*(t) = (1 - \delta)v(D_t^*, t)$ . Off-equilibrium, suppose debt  $D \in (D_{t+1}^*, D_t^*)$  is issued. The price  $p$  that would make type  $t$  willing to deviate satisfies  $p = u^*(t) + \delta v(D, t) = (1 - \delta)v(D_t^*, t) + \delta v(D, t)$ . For any type  $s < t$ , the incentive constraint for  $s$  implies  $v(D_t^*, t) - \delta v(D_t^*, s) \leq u^*(s)$ . Combining these, we have  $p + \delta \underbrace{[v(D_t^*, t) - v(D, t) - (v(D_t^*, s) - v(D, s))]}_{>0} \leq u^*(s) + \delta v(D, s)$ , where the term

in square brackets is positive: the value of junior debt with face value  $D_t^* - D$  is higher for type  $t$  than for type  $s < t$ . Therefore, types  $s < t$  would not find it profitable to deviate given price  $p$ . The beliefs specified in (4) must therefore concentrate on  $t$ , and so the market price following a deviation to  $D$  is  $v(D, t)$ . ♦



PROOF OF PROPOSITION 10: Let the range of  $D^\infty$  be  $(\underline{D}, \bar{y}]$ . Define  $U(t, \hat{t}, D)$  to be the payoff  $E[\min(D, Y) | \hat{t}] - \delta E[\min(D, Y) | t]$  of an issuer with type  $t$  when she issues debt with face value  $D$  and the investors believe her type is  $\hat{t}$ . Then

$$\frac{U_D}{U_{\hat{t}}} = \frac{\Pr(Y > D | \hat{t}) - \delta \Pr(Y > D | t)}{\frac{\partial}{\partial \hat{t}} E[\min(D, Y) | \hat{t}]}$$

is nonincreasing in  $t$  by First-Order Stochastic Dominance (FOSD). Since by assumption  $D_t$  is decreasing in  $t$  and since  $U_2(t, \hat{t}, D) \geq 0$  by FOSD, part 1 of Theorem 6 in Mailath and von Thadden (2013) implies that an issuer of type  $t$  will not imitate any other type: she will not sell debt with face value  $D_t^\infty$  for any  $\hat{t} \neq t$ .

It remains to consider whether she will sell debt with a face value not in the range of  $D^\infty$ , as well as other types of monotone securities. We may assume that investors respond to any such deviation with the most pessimistic beliefs: that her type is zero.

First consider a deviation of a type  $t$  to debt with a face value  $D$  that is not in the range  $[\underline{D}, \bar{y}]$  of  $D^\infty$ . Assume this deviation makes type  $t$  strictly better off than sticking to the equilibrium. First, suppose  $D > \bar{y}$ . This security has the same payout for any  $Y$  as debt with face value  $\bar{y}$ , and both lead investors to believe that  $t = 0$ . By the preceding result, no type strictly prefers such a deviation. Now consider  $D < \underline{D}$ . Investor beliefs cannot be more optimistic than those that result from debt with face value  $\underline{D}$ , since these are the beliefs that the type is one. Moreover, there are gains from trade, so a higher face value is more profitable for the issuer (holding investor beliefs constant). Hence, any  $D < \underline{D}$  is worse for the issuer than  $D = \underline{D}$ .

Finally, we consider deviations of type  $t$  to a general monotone security  $\hat{S}$ . The issuer's securitization profit from  $\hat{S}$  equals the price  $E[\hat{S}(Y) | 0]$  assigned by investors less the discounted expected security payout  $\delta E[\hat{S}(Y) | t]$ . This profit can be written as

$$\int_{Y=0}^{\bar{y}} \hat{S}(Y) d[H(Y|0) - \delta H(Y|t)]$$

where  $H$  denotes the conditional distribution of  $Y$  given  $t$ . The security  $\hat{S}$  is monotone if and only if  $\hat{S}(0) = 0$  and, for all  $Y$ , the control variable  $c(Y) = \hat{S}'(Y)$  lies in  $[0, 1]$ . For all

$Y \in [0, \bar{y}]$  define the Hamiltonian

$$\mathcal{H} = \lambda(Y)c(Y) + \widehat{S}(Y) [H'(Y|0) - \delta H'(Y|t)] + \mu_0(Y)c(Y) + \mu_1(Y)[1 - c(Y)]$$

where  $\mu_0, \mu_1 \geq 0$  are Lagrange multipliers that capture the constraints on  $c$  and  $\lambda(Y)$  is a costate variable. By Pontryagin's maximization principle (Pontryagin *et al* 1962), the optimal control  $c$  must maximize the Hamiltonian  $\mathcal{H}$  while the multipliers  $\mu_0, \mu_1 \geq 0$  must minimize it. As  $\mathcal{H}$  is linear in  $c$ , this implies the first order condition

$$0 = \frac{\partial}{\partial c(Y)} \mathcal{H} = \lambda(Y) + \mu_0(Y) - \mu_1(Y) \quad (16)$$

as well as the Kuhn-Tucker (complementary slackness) conditions:

$$\mu_0(Y)c(Y) = 0 \text{ and } \mu_1(Y)[1 - c(Y)] = 0. \quad (17)$$

Additionally, for all  $Y$ , the costate equation

$$\lambda'(Y) = -\mathcal{H}_S = -H'(Y|0) + \delta H'(Y|t) \quad (18)$$

must be satisfied. Finally, as the final state  $\widehat{S}(\bar{y})$  is not fixed, the terminal costate must be zero:

$$\lambda(\bar{y}) = 0. \quad (19)$$

Solving (18) subject to (19), we obtain  $\lambda(Y) = [1 - H(Y|0)] \left[ 1 - \delta \frac{1-H(Y|t)}{1-H(Y|0)} \right]$ . By HRO,  $\frac{1-H(Y|t)}{1-H(Y|0)}$  is increasing in  $Y$  since  $t > 0$ . Hence, by (16), (17), and the nonnegativity of  $\mu_0$  and  $\mu_1$ , there exists an  $D \in \mathfrak{R}$  such that  $c(Y) = 1$  for all  $Y < D$  and  $c(Y) = 0$  for all  $Y > D$ : the optimal security is debt with face value  $D$  and thus, as shown above,  $D = D_t^\infty$ .

**Q.E.D.**PROPOSITION 10

**PROOF OF PROPOSITION 11:** First,  $1 - H(y|t)$  can be written as  $1 - H(y|1) + \int_{s=t}^1 H_2(y|s) ds$ , which is smaller for  $H = \widetilde{H}$  than for  $H = \widehat{H}$ . Hence, the rate of change function in (1) is no larger in absolute value when  $H$  equals  $\widehat{H}$  than when it equals  $\widetilde{H}$ . The result then follows from PROPOSITION 15. **Q.E.D.**PROPOSITION 11

PROOF OF PROPOSITIONS 12-15, AND COROLLARY 16: In this proof, we will write  $D^\infty(t)$  and  $D^i(t)$  in place of  $D_t^\infty$  and  $D_t^i$ . We will also treat  $\bar{y}$  as a parameter and the relative cash flow  $Z = Y/\bar{y}$  as an exogenous random variable with support  $[0, 1]$  and conditional distribution function  $\widehat{H}$  defined by  $\widehat{H}(z|t) = H(z\bar{y}|t)$  for  $z$  in  $[0, 1]$ . Lipschitz- $H$  then entails the following property.<sup>1</sup>

**Lipschitz- $\widehat{H}$ .** There are constants  $\widehat{k}_0, \widehat{k}_1, \widehat{k}_2 \in (0, \infty)$  such that for all  $z, t, t', t'' \in [0, 1]$ ,

$$\frac{\partial \widehat{H}(z|t)}{\partial z} \in (\widehat{k}_0, \widehat{k}_1), \quad -\frac{\partial \widehat{H}(z|t)}{\partial t} \in [\widehat{k}_0 z(1-z), \widehat{k}_1], \quad \left| \frac{\partial \widehat{H}(z|t')}{\partial z} - \frac{\partial \widehat{H}(z|t'')}{\partial z} \right| < \widehat{k}_2 |t' - t''|, \text{ and}$$

$$\left| \frac{\partial \widehat{H}(z|t')}{\partial t} \Big|_{t=t'} - \frac{\partial \widehat{H}(z|t'')}{\partial t} \Big|_{t=t''} \right| < \widehat{k}_2 |t' - t''|.$$

Henceforth we rename  $\widehat{H}$  to  $H$  and each  $\widehat{k}_n$  to  $k_n$  (for  $n = 0, 1, 2$ ), and refer to Lipschitz- $\widehat{H}$  as Lipschitz- $H$  or L- $H$ .

Recall that  $E^i$  and  $E^\infty$  denote the expectation operators in models  $i$  and  $\infty$ . For any  $D \in \mathfrak{X}$ , let

$$v^{H\bar{y}i}(D, t) = E^i[\min\{D, \bar{y}Z\} | t] = \sum_{c=1}^{1/\Delta'_i} \min\{D, \bar{y}c\Delta'_i\} [H(c\Delta'_i|t) - H((c-1)\Delta'_i|t)] \quad (20)$$

<sup>1</sup>Lipschitz- $H$  implies that there are constants  $k_0, k_1, k_2 \in (0, \infty)$  such that for all  $z, t, t', t'' \in [0, 1]$ ,

$$\frac{\partial \widehat{H}(z|t)}{\partial z} = \frac{\partial H(z\bar{y}|t)}{\partial z} = \frac{\partial H(z\bar{y}|t)}{\partial (z\bar{y})} \frac{d(z\bar{y})}{dz} \in (k_0\bar{y}, k_1\bar{y}),$$

$$-\frac{\partial \widehat{H}(z|t)}{\partial t} = -\frac{\partial H(z\bar{y}|t)}{\partial t} \in [k_0 y(\bar{y} - y), k_1] = [\bar{y}^2 k_0 z(1-z), k_1],$$

$$\left| \frac{\partial \widehat{H}(z|t')}{\partial z} - \frac{\partial \widehat{H}(z|t'')}{\partial z} \right| = \left| \frac{\partial H(z\bar{y}|t')}{\partial (z\bar{y})} - \frac{\partial H(z\bar{y}|t'')}{\partial (z\bar{y})} \right| \frac{d(z\bar{y})}{dz}$$

$$= \left| \frac{\partial H(y|t')}{\partial y} - \frac{\partial H(y|t'')}{\partial y} \right| \bar{y} < k_2 \bar{y} |t' - t''|,$$

and

$$\left| \frac{\partial \widehat{H}(z|t')}{\partial t} \Big|_{t=t'} - \frac{\partial \widehat{H}(z|t'')}{\partial t} \Big|_{t=t''} \right| = \left| \frac{\partial H(z\bar{y}|t')}{\partial t} \Big|_{t=t'} - \frac{\partial H(z\bar{y}|t'')}{\partial t} \Big|_{t=t''} \right|$$

$$= \left| \frac{\partial H(y|t')}{\partial t} \Big|_{t=t'} - \frac{\partial H(y|t'')}{\partial t} \Big|_{t=t''} \right| < k_2 |t' - t''|.$$

This implies Lipschitz- $\widehat{H}$  with constants  $\widehat{k}_0 = k_0 \min\{\bar{y}, \bar{y}^2\}$ ,  $\widehat{k}_1 = k_1 \max\{\bar{y}, 1\}$ , and  $\widehat{k}_2 = k_2 \max\{\bar{y}, 1\}$ .

and

$$v^{H\bar{y}}(D, t) = E^\infty [\min \{D, \bar{y}Z\} | t] = \int_{z=0}^1 \min \{D, \bar{y}z\} dH(z|t) \quad (21)$$

denote the expected payout of debt with face value  $D$  in models  $i$  and  $\infty$ , respectively, conditional on the issuer's type  $t$ . Integrating by parts yields

$$v^{H\bar{y}}(D, t) = D - \bar{y} \int_{z=0}^{D/\bar{y}} H(z|t) dz. \quad (22)$$

Given the above notation, the incentive compatibility condition in model  $i$  becomes:

**Discrete Initial Value Problem with Parameters  $H, \delta, \bar{y}$  ( $\mathbf{DP}^{H\delta\bar{y}i}$ ).** The condition

$$\begin{aligned} & v^{H\bar{y}i}(D^{H\delta\bar{y}i}(t + \Delta_i), t + \Delta_i) - \delta v^{H\bar{y}i}(D^{H\delta\bar{y}i}(t + \Delta_i), t) \\ &= (1 - \delta) v^{H\bar{y}i}(D^{H\delta\bar{y}i}(t), t) \end{aligned} \quad (23)$$

with  $D^{H\delta\bar{y}i} : S_i \rightarrow \mathfrak{R}$ , together with the initial value  $D^{H\delta\bar{y}i}(0) = \bar{y}$ .

Define the function

$$f^{H\delta\bar{y}}(D, t) = -\frac{1}{1 - \delta} \frac{v_2^{H\bar{y}}(D, t)}{v_1^{H\bar{y}}(D, t)} = \frac{\bar{y}}{1 - \delta} \frac{\int_{z=0}^{D/\bar{y}} \frac{\partial H(z|t)}{\partial t} dz}{1 - H\left(\frac{D}{\bar{y}}|t\right)} \leq 0, \quad (24)$$

where the second equality follows from (22). A restatement of CP in model  $\infty$  is as follows.

**Continuous Initial Value Problem with Parameters  $H, \delta, \bar{y}$  ( $\mathbf{CP}^{H\delta\bar{y}}$ ).** The differential equation

$$\frac{dD^{H\delta\bar{y}}}{dt} = f^{H\delta\bar{y}}(D^{H\delta\bar{y}}, t), \quad (25)$$

with  $D^{H\delta\bar{y}} : [0, 1] \rightarrow \mathfrak{R}$ , together with the initial value  $D^{H\delta\bar{y}}(0) = \bar{y}$ .

In model  $i$ , let  $p^{H\delta\bar{y}i}(t)$  denote  $v^{H\bar{y}i}(D^{H\delta\bar{y}i}(t), t)$ : the price and equilibrium expected payout of a standard debt contract, conditional on the type  $t$ . Let  $u^{H\delta\bar{y}i}(t)$  denote  $(1 - \delta)p^{H\delta\bar{y}i}(t)$ : the issuer's profit and expected gains from trade from such a contract conditional on the type  $t$ . Let  $Eu^{GH\delta\bar{y}i} = E^i [u^{H\delta\bar{y}i}(t)]$  denote the unconditional expected gains from trade and issuer's profit. Analogously, in the continuous model let

$p^{H\delta\bar{y}}(t)$  denote  $v^{H\bar{y}}(D^{H\delta\bar{y}}(t), t)$ , let  $u^{H\delta\bar{y}}(t)$  denote  $(1 - \delta)p^{H\delta\bar{y}}(t)$ , and let  $Eu^{GH\delta\bar{y}}$  denote  $E^\infty[u^{H\delta\bar{y}}(t)]$ . Let  $\Pi^{GH\delta\bar{y}i}(t) = u^{GH\delta\bar{y}i}(t) + E^i[\bar{y}Z|t]$  denote the issuer's conditional (on  $t$ ) expected total profits in model  $i$  and let  $E\Pi^{GH\delta\bar{y}i} = E^i[\Pi^{GH\delta\bar{y}i}(t)]$  denote her unconditional expected total profits. In the continuous model, define the analogous quantities  $\Pi^{GH\delta\bar{y}}(t) = u^{GH\delta\bar{y}}(t) + E^\infty[\bar{y}Z|t]$  and  $E\Pi^{GH\delta\bar{y}} = E^\infty[\Pi^{GH\delta\bar{y}}(t)]$ . As noted, we extend the functions in model  $i$  to any type  $t \in [0, 1]$  by evaluating them at the highest type in  $S_i$  that does not exceed  $t$ .

With this notation, Propositions 12-15, as well as Corollary 16, are implied by the following result.

**THEOREM 1** *Fix constants  $k_0, k_1, k_2, k_3, k_4$ , and  $\mathbf{y}$ , all in  $(0, \infty)$ . Let  $\mathcal{G}$  be the set of distribution functions  $G$  that satisfy Lipschitz- $G$  with constants  $k_3$  and  $k_4$ . Let  $\mathcal{H}$  be the set of conditional distribution functions  $H$  that satisfy Lipschitz- $H$  with constants  $k_0, k_1$ , and  $k_2$ , and Hazard Rate Ordering. For any distribution function  $G$  in  $\mathcal{G}$ , conditional distribution function  $H$  in  $\mathcal{H}$ , parameter  $\bar{y}$  in  $(0, \mathbf{y}]$ , and discount factor  $\delta$  in  $(0, 1)$ :*

1. *There exists a unique function  $D^{H\delta\bar{y}}$  that satisfies  $CP^{H\delta\bar{y}}$ . This function is decreasing and differentiable, and takes values in  $(0, \bar{y}]$ . The associated price and profit functions,  $p^{H\delta\bar{y}}$  and  $u^{H\delta\bar{y}}$ , are continuous and decreasing in the type  $t$  as well.*
2. *For each discrete model  $i = 1, 2, \dots$ , there exists a unique, decreasing function  $D^{H\delta\bar{y}i}$  that satisfies  $DP^{H\delta\bar{y}i}$ .*
3. *The sequences of face value functions, price functions, conditional and unconditional expected securitization profit functions, and conditional and unconditional expected total profit functions in the discrete models converge to their continuous counterparts as  $i$  grows, uniformly in the distributions  $G$  and  $H$ , the parameter  $\mathbf{y}$ , and (except in the case of the unconditional expected profit functions which do not depend on the type) the type  $t \in [0, 1]$ . More precisely:*

- *For all  $\varepsilon > 0$  there is an  $i^*$  such that for all models  $i > i^*$ ,  $G$  in  $\mathcal{G}$ ,  $H$  in  $\mathcal{H}$ ,  $\bar{y}$*

in  $(0, \mathbf{y}]$ , and  $t \in [0, 1]$ ,  $\left| \omega^{H\delta_{\bar{y}i}}(t) - \omega^{H\delta_{\bar{y}}}(t) \right| < \varepsilon$  for each  $\omega = D, p, u, \Pi$ , and  $\left| E\omega^{H\delta_{\bar{y}i}} - E\omega^{H\delta_{\bar{y}}} \right| < \varepsilon$  for each  $\omega = u, \Pi$ .

4. All of the functions defined above are homogeneous of degree one in the parameter  $\bar{y}$ :  $\omega^{H\delta_{\bar{y}}} = \bar{y}\omega^{H\delta_1}$  and  $\omega^{H\delta_{\bar{y}i}} = \bar{y}\omega^{H\delta_{1i}}$  for each  $\omega = D, p, u, Eu, \Pi, E\Pi$ .
5. Let  $\tilde{H}, \hat{H} \in \mathcal{H}$  and  $\tilde{\delta}, \hat{\delta} \in (0, 1)$  satisfy  $f^{\tilde{H}\tilde{\delta}_{\bar{y}}} \leq f^{\hat{H}\hat{\delta}_{\bar{y}}} \leq 0$ . Then  $D^{\tilde{H}\tilde{\delta}_{\bar{y}}}(t) \leq D^{\hat{H}\hat{\delta}_{\bar{y}}}(t)$  for all  $t \in [0, 1]$ .

We now prove Theorem 1. Without loss of generality, we restrict to face values  $D$  that do not exceed the maximum final asset value  $\bar{y}$ .<sup>2</sup> First, we define an integration by parts formula for the function  $v^{H\bar{y}i}$  defined in (20).

CLAIM 2 For any face value  $D \in [0, \bar{y}]$  and type  $t$ ,

$$v^{H\bar{y}i}(D, t) = D - \bar{y} \int_{z=0}^{\frac{D}{\bar{y}}} H \left( \Delta'_i \left[ \frac{z}{\Delta'_i} \right] | t \right) dz. \quad (26)$$

PROOF OF CLAIM 2. One can easily verify (using  $H(1|t) = 1$  and  $H(0|t) = 0$ ) that for  $D \in [0, \bar{y}]$ ,  $v^{H\bar{y}i}(D, t) = \min\{D, \bar{y}\} - \sum_{c=1}^{\lfloor \frac{D}{\bar{y}\Delta'_i} \rfloor - 1} H(c\Delta'_i|t) [\min\{D, \bar{y}(c+1)\Delta'_i\} - \min\{D, \bar{y}c\Delta'_i\}]$  by (20). As  $c$  is an integer,  $c \leq x$  iff  $c \leq \lfloor x \rfloor$ . Thus,

$$\min\{D, \bar{y}(c+1)\Delta'_i\} - \min\{D, \bar{y}c\Delta'_i\} = \begin{cases} \bar{y}\Delta'_i & \text{if } c \leq \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor - 1 \\ D - \bar{y}c\Delta'_i & \text{if } c = \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \\ 0 & \text{if } c > \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \end{cases},$$

whence, as  $D \leq \bar{y}$ ,

$$v^{H\bar{y}i}(D, t) = D - \bar{y}\Delta'_i \left[ \sum_{c=1}^{\left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor - 1} H(c\Delta'_i|t) + H \left( \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i | t \right) \left( \frac{D}{\bar{y}\Delta'_i} - \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \right) 1 \left( \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \leq \frac{1}{\Delta'_i} - 1 \right) \right]. \quad (27)$$

---

<sup>2</sup>Choosing a higher face value is equivalent to choosing  $\bar{y}$  since the underlying assets cannot be worth more than  $\bar{y}$ .

If  $z \in [c\Delta'_i, (c+1)\Delta'_i)$ , then  $\lfloor z/\Delta'_i \rfloor = c$ . So

$$\begin{aligned} \sum_{c=1}^{\lfloor D/(\bar{y}\Delta'_i) \rfloor - 1} H(c\Delta'_i|t) &= \sum_{c=1}^{\lfloor D/(\bar{y}\Delta'_i) \rfloor - 1} \frac{1}{\Delta'_i} \int_{z=c\Delta'_i}^{(c+1)\Delta'_i} H\left(\Delta'_i \left\lfloor \frac{z}{\Delta'_i} \right\rfloor |t\right) dz \\ &= \frac{1}{\Delta'_i} \int_{z=\Delta'_i}^{\Delta'_i \lfloor \frac{D}{\bar{y}\Delta'_i} \rfloor} H\left(\Delta'_i \left\lfloor \frac{z}{\Delta'_i} \right\rfloor |t\right) \mathbf{1}\left(\left\lfloor \frac{z}{\Delta'_i} \right\rfloor < \frac{1}{\Delta'_i}\right) dz \end{aligned} \quad (28)$$

since  $\mathbf{1}\left(\left\lfloor \frac{z}{\Delta'_i} \right\rfloor < \frac{1}{\Delta'_i}\right)$  equals one for all  $z < \Delta'_i \lfloor \frac{D}{\bar{y}\Delta'_i} \rfloor$ . Moreover,

$$\begin{aligned} &H\left(\left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i |t\right) \left(\frac{D}{\bar{y}\Delta'_i} - \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor\right) \mathbf{1}\left(\left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \leq \frac{1}{\Delta'_i} - 1\right) \\ &= \frac{1}{\Delta'_i} \int_{z=\Delta'_i}^{\frac{D}{\bar{y}}} H\left(\left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i |t\right) \mathbf{1}\left(\left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \leq \frac{1}{\Delta'_i} - 1\right) dz \end{aligned} \quad (29)$$

$$= \frac{1}{\Delta'_i} \int_{z=\Delta'_i}^{\frac{D}{\bar{y}}} H\left(\left\lfloor \frac{z}{\Delta'_i} \right\rfloor \Delta'_i |t\right) \mathbf{1}\left(\left\lfloor \frac{z}{\Delta'_i} \right\rfloor < \frac{1}{\Delta'_i}\right) dz \quad (30)$$

where the last equality holds for two reasons. First,  $\left\lfloor \frac{z}{\Delta'_i} \right\rfloor = \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor$  in the interval of integration in line (29). Second, since  $\left\lfloor \frac{z}{\Delta'_i} \right\rfloor$  and  $\frac{1}{\Delta'_i}$  are both integers,  $\left\lfloor \frac{z}{\Delta'_i} \right\rfloor < \frac{1}{\Delta'_i}$  if and only if  $\left\lfloor \frac{z}{\Delta'_i} \right\rfloor \leq \frac{1}{\Delta'_i} - 1$ . Combining (27), (28), and (30), we obtain

$$v^{H\bar{y}i}(D, t) = D - \bar{y} \int_{z=\Delta'_i}^{\frac{D}{\bar{y}}} H\left(\Delta'_i \left\lfloor \frac{z}{\Delta'_i} \right\rfloor |t\right) \mathbf{1}\left(\left\lfloor \frac{z}{\Delta'_i} \right\rfloor < \frac{1}{\Delta'_i}\right) dz.$$

As  $\mathbf{1}\left(\left\lfloor \frac{z}{\Delta'_i} \right\rfloor < \frac{1}{\Delta'_i}\right)$  equals one except possibly at the upper endpoint of the integral, it can be omitted. Finally, the lower limit of integration can be reduced to zero since  $H(0|t) = 0$ .

**Q.E.D.** Claim 2

We next show that the first derivatives with respect to  $D$  and  $t$  of the expected payout  $v^{H\bar{y}i}(D, t)$  in model  $i$  are bounded above and below (parts 1 and 2) and converge uniformly to the respective first derivatives of the expected payout  $v^{H\bar{y}}(D, t)$  in the continuous case (parts 3 and 4):

**CLAIM 3** *Assume L-H. Define*

$$\Omega^{H\bar{y}i}(t', t'', D', D'') = v^{H\bar{y}i}(D', t') - v^{H\bar{y}i}(D'', t'') - [v^{H\bar{y}}(D', t') - v^{H\bar{y}}(D'', t'')].$$

1. For all  $D \in (\bar{y}\Delta'_i, \bar{y}]$ , and all  $t', t''$  in  $S_i$  such that  $t' > t''$ ,

$$\max \left\{ 0, k_0 \frac{(D')^2 (3\bar{y} - 2D')}{6\bar{y}^2} \right\} < \frac{v^{H\bar{y}i}(D, t') - v^{H\bar{y}i}(D, t'')}{t' - t''} < \bar{y}k_1 \min \left\{ \frac{D}{\bar{y}}, 1 - \Delta'_i \right\},$$

where  $D' = D - 2\bar{y}\Delta'_i$ .

2. For all  $D', D''$  in  $[0, \bar{y}]$  such that  $D' > D''$  and for all  $t \in S_i$ ,

$$\begin{aligned} k_1 \left( 1 - \frac{D''}{\bar{y}} + \Delta_i \right) &> \frac{v^{H\bar{y}i}(D', t) - v^{H\bar{y}i}(D'', t)}{D' - D''} \\ &> k_0 \left( 1 - \Delta'_i \left\lceil \frac{D'}{\bar{y}\Delta'_i} \right\rceil + \Delta'_i \right) \geq k_0 \left( 1 - \frac{D'}{\bar{y}} \right). \end{aligned}$$

3. For all  $\varepsilon > 0$  there exists an  $i^*$  such that if  $i > i^*$ , then for all  $G \in \mathcal{G}$ ,  $H \in \mathcal{H}$ ,  $\bar{y} \in (0, \mathbf{y}]$ ,  $D$  in  $[0, \bar{y}]$  and  $t', t''$  in  $S_i$  such that  $t' > t''$ ,  $|\Omega^{H\bar{y}i}(t', t'', D, D)| < \varepsilon(t' - t'')$ .

4. For all  $\varepsilon > 0$  there exists an  $i^*$  such that if  $i > i^*$ , then for all  $G \in \mathcal{G}$ ,  $H \in \mathcal{H}$ ,  $\bar{y} \in (0, \mathbf{y}]$ ,  $t \in S_i$ , and  $D', D''$  in  $[0, \bar{y}]$  such that  $D' > D''$ ,  $|\Omega^{H\bar{y}i}(t, t, D', D'')| < \varepsilon(D' - D'')$ .

PROOF OF CLAIM 3. Part 1. By (26),

$$v^{H\bar{y}i}(D, t') - v^{H\bar{y}i}(D, t'') = \bar{y} \int_{z=0}^{\frac{D}{\bar{y}}} \left[ H \left( \Delta'_i \left\lfloor \frac{z}{\Delta'_i} \right\rfloor |t'' \right) - H \left( \Delta'_i \left\lfloor \frac{z}{\Delta'_i} \right\rfloor |t' \right) \right] dz$$

which by L-H is less than  $\bar{y}k_1(t' - t'') \left( \frac{D}{\bar{y}} - \Delta'_i \right)$  and at least

$$\bar{y}k_0(t' - t'') (\Delta'_i)^2 \int_{z=0}^{\frac{D}{\bar{y}}} \left\lfloor \frac{z}{\Delta'_i} \right\rfloor \left( \frac{1}{\Delta'_i} - \left\lfloor \frac{z}{\Delta'_i} \right\rfloor \right) dz, \quad (31)$$

which is zero if  $D \leq \bar{y}\Delta'_i$  and positive otherwise. Let  $c = \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor$  where  $D' = D - 2\bar{y}\Delta'_i$ .

Recall  $N'_i = 1/\Delta'_i$ . Hence, for  $D > \bar{y}\Delta'_i$ , the integral in (31) is at least

$$\begin{aligned} \int_{z=0}^{c\Delta'_i} \left\lfloor \frac{z}{\Delta'_i} \right\rfloor \left( N'_i - \left\lfloor \frac{z}{\Delta'_i} \right\rfloor \right) dz &= \Delta'_i \sum_{n=1}^{c-1} n (N'_i - n) = \Delta'_i \frac{c(c-1)(3N'_i - 2c + 1)}{6} \\ &\geq \Delta'_i \frac{\left( \frac{D'}{\bar{y}\Delta'_i} + 1 \right) \frac{D'}{\bar{y}\Delta'_i} \left( 3N'_i - 2\frac{D'}{\bar{y}\Delta'_i} - 1 \right)}{6} \text{ as } c \geq \frac{D'}{\bar{y}\Delta'_i} + 1 \\ &> \frac{(D')^2 (3\bar{y} - 2D')}{6\bar{y}^3 (\Delta'_i)^2} \text{ as } D' \leq \bar{y} (1 - 2\Delta'_i). \end{aligned}$$



This proves the result.

Part 2. By L-H and (20),

$$\begin{aligned} & v^{H\bar{y}i}(D', t) - v^{H\bar{y}i}(D'', t) \\ &= \sum_{c=1}^{1/\Delta'_i} [\min\{D', \bar{y}c\Delta'_i\} - \min\{D'', \bar{y}c\Delta'_i\}] [H(c\Delta'_i|t) - H((c-1)\Delta'_i|t)] \end{aligned}$$

but

$$\begin{aligned} \min\{D', \bar{y}c\Delta'_i\} - \min\{D'', \bar{y}c\Delta'_i\} &= \begin{cases} 0 & \text{if } c \leq \frac{D''}{\bar{y}\Delta'_i} \\ \bar{y}c\Delta'_i - D'' & \text{if } \frac{D''}{\bar{y}\Delta'_i} < c < \frac{D'}{\bar{y}\Delta'_i} \\ D' - D'' & \text{if } c \geq \frac{D'}{\bar{y}\Delta'_i} \end{cases} \\ &= \begin{cases} 0 & \text{if } c \leq \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor \\ \bar{y}c\Delta'_i - D'' & \text{if } \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor < c < \left\lfloor \frac{D'}{\bar{y}\Delta'_i} \right\rfloor \\ D' - D'' & \text{if } c \geq \left\lfloor \frac{D'}{\bar{y}\Delta'_i} \right\rfloor \end{cases} \end{aligned}$$

so

$$\begin{aligned} v^{H\bar{y}i}(D', t) - v^{H\bar{y}i}(D'', t) &\geq \sum_{c=\left\lfloor \frac{D'}{\bar{y}\Delta'_i} \right\rfloor}^{1/\Delta'_i} (D' - D'') [H(c\Delta'_i|t) - H((c-1)\Delta'_i|t)] \\ &> (D' - D'') k_0 \Delta'_i \left( \frac{1}{\Delta'_i} - \left\lfloor \frac{D'}{\bar{y}\Delta'_i} \right\rfloor + 1 \right) \\ &\geq (D' - D'') k_0 (1 - D'/\bar{y}). \end{aligned}$$

Moreover,  $v^{H\bar{y}i}(D', t) - v^{H\bar{y}i}(D'', t)$  is at most

$$\begin{aligned} & \sum_{c=\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor}^{1/\Delta'_i} (D' - D'') [H(c\Delta'_i|t) - H((c-1)\Delta'_i|t)] \\ &< (D' - D'') k_1 \Delta'_i \left( \frac{1}{\Delta'_i} - \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor + 1 \right) \leq (D' - D'') k_1 (1 - D''/\bar{y} + \Delta_i). \end{aligned}$$

Part 3. Let  $i$  be large enough that  $\mathbf{y}k_2\Delta'_i < \varepsilon$ . Then

$$\begin{aligned} |\Delta(t', t'', D, D)| &= \left| \sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} [\min(D, \bar{y}c\Delta'_i) - \min(D, \bar{y}z)] d[H(z|t') - H(z|t'')] \right| \\ &\leq \sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} |\min(D, \bar{y}c\Delta'_i) - \min(D, \bar{y}z)| d[H(z|t') - H(z|t'')] \\ &\leq \sum_{c=1}^{1/\Delta'_i} \bar{y}k_2(\Delta'_i)^2 |t' - t''| = \bar{y}k_2\Delta'_i |t' - t''| < \varepsilon |t' - t''| \end{aligned}$$

since  $|\min(D, \bar{y}c\Delta'_i) - \min(D, \bar{y}z)| < \bar{y}\Delta'_i$  and by L-H,  $\left| \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} d[H(z|t') - H(z|t'')] \right| \leq k_2\Delta'_i |t' - t''|$ .

Part 4. As  $\bar{y} > 0$ ,  $|\Omega^{H\bar{y}i}(t, t, D', D'')| = \left| \sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \eta(\bar{y}c\Delta'_i, \bar{y}z, D'', D') dH(z|t) \right|$   
where  $\eta(\zeta', \zeta'', \zeta_0, \zeta_1) = \max\{\zeta_0, \min\{\zeta_1, \zeta'\}\} - \max\{\zeta_0, \min\{\zeta_1, \zeta''\}\}$ .

REMARK 4  $\eta(\zeta', \zeta'', \zeta_0, \zeta_1)$  lies in  $[0, \zeta' - \zeta'']$  if  $\zeta e' \geq \zeta e''$  and is zero if either

$$\max\{\zeta', \zeta''\} \leq \zeta e_0$$

or  $\min\{\zeta', \zeta''\} \geq \zeta e_1$ .

Since  $z$  lies in  $[(c-1)\Delta'_i, c\Delta'_i]$ , the integrand  $\eta(\bar{y}c\Delta'_i, \bar{y}z, D'', D')$  is zero if either  $c \leq \frac{D''}{\bar{y}\Delta'_i}$  or  $c \geq \frac{D'}{\bar{y}\Delta'_i} + 1$ . Hence,

$$\begin{aligned} &\sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \eta(\bar{y}c\Delta'_i, \bar{y}z, D'', D') dH(z|t) \\ &= \sum_{c=\left\lceil \frac{D''}{\bar{y}\Delta'_i} \right\rceil}^{\min\left\{1, \left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor\right\}} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \eta(\bar{y}c\Delta'_i, \bar{y}z, D'', D') dH(z|t). \end{aligned} \quad (32)$$

Since  $c\Delta'_i \geq z$  in each integral, the integrands  $\eta(\bar{y}c\Delta'_i, \bar{y}z, D'', D')$  are all nonnegative, so we may dispense with the absolute value signs. The first summand, which corresponds to

$c = \lceil D''/\bar{y}\Delta'_i \rceil$ , is

$$\begin{aligned}
& \int_{z=\left(\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - 1\right)\Delta'_i}^{\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i} \eta\left(\bar{y}\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i, \bar{y}z, D'', D'\right) dH(z|t) \\
&= \left[ \begin{aligned} & \eta\left(\bar{y}\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i, D'', D'', D'\right) \left[ H\left(\frac{D''}{\bar{y}}|t\right) - H\left(\left(\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - 1\right)\Delta'_i|t\right) \right] \\ & + \int_{z=D''/\bar{y}}^{\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i} \eta\left(\bar{y}\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor \Delta'_i, \bar{y}z, D'', D'\right) dH(z|t) \end{aligned} \right] \\
&\leq \left[ \begin{aligned} & \bar{y}k_1(\Delta'_i)^2 \left( \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - \frac{D''}{\bar{y}\Delta'_i} \right) \left[ \frac{D''}{\bar{y}\Delta'_i} - \left( \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - 1 \right) \right] \\ & + \bar{y}k_1(\Delta'_i)^2 \left( \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - \frac{D''}{\bar{y}\Delta'_i} \right)^2 \end{aligned} \right] = \bar{y}k_1(\Delta'_i)^2 \left( \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - \frac{D''}{\bar{y}\Delta'_i} \right),
\end{aligned}$$

by L-H. There are now two cases.

Case 1:  $\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \leq \frac{1}{\Delta'_i}$ . The last summand in (32), which corresponds to  $c = \lfloor D'/\bar{y}\Delta'_i + 1 \rfloor$ , is<sup>3</sup>

$$\begin{aligned}
& \int_{z=\left(\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor - 1\right)\Delta'_i}^{\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \Delta'_i} \eta\left(\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \Delta'_i, \bar{y}z, D'', D'\right) dH(z|t) \\
&= \int_{z=\left(\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor - 1\right)\Delta'_i}^{D'/\bar{y}} \eta\left(\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \Delta'_i, \bar{y}z, D'', D'\right) dH(z|t) \\
&+ \eta\left(\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \Delta'_i, D', D'', D'\right) \left[ H\left(\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \Delta'_i|t\right) - H\left(\frac{D'}{\bar{y}}|t\right) \right] \quad (33)
\end{aligned}$$

$$\leq \bar{y}k_1(\Delta'_i)^2 \left[ \frac{D'}{\bar{y}\Delta'_i} + 1 - \left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \right]. \quad (34)$$

The remainder of the sum in (32) is

$$\sum_{c=\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor + 1}^{\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor - 1} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \eta(\bar{y}c\Delta'_i, \bar{y}z, D'', D') dH(z|t) \leq \bar{y}k_1(\Delta'_i)^2 \left( \left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor - \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - 1 \right).$$

Collecting terms,  $|\Omega^{H\bar{y}i}(t, t, D', D'')| \leq k_1\Delta'_i(D' - D'')$ . Now take  $i^*$  large enough that  $k_1\Delta'_{i^*} < \varepsilon$ .

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<sup>3</sup>By Remark 4, line (33) is zero as  $\bar{y}\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor \Delta'_i - D' = \bar{y}\Delta'_i \left( \left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor - \frac{D'}{\bar{y}\Delta'_i} \right) > 0$ . The inequality in line (34) then follows from Lipschitz-H.

Case 2:  $\left\lfloor \frac{D'}{\bar{y}\Delta'_i} + 1 \right\rfloor > \frac{1}{\Delta'_i}$  or, equivalently,  $\left\lfloor \frac{D'}{\bar{y}\Delta'_i} \right\rfloor > \frac{1}{\Delta'_i} - 1$ . The final sum on the right hand side of (32) then corresponds to  $c = 1/\Delta'_i$ . Moreover,

$$\sum_{c=\left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor + 1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \eta(\bar{y}c\Delta'_i, \bar{y}z, D'', D') dH(z|t) \leq \bar{y}k_1 (\Delta'_i)^2 \left[ \frac{1}{\Delta'_i} - \left\lfloor \frac{D''}{\bar{y}\Delta'_i} \right\rfloor - 1 \right].$$

Thus,

$$\begin{aligned} |\Omega^{H\bar{y}i}(t, t, D', D'')| &\leq \bar{y}k_1 (\Delta'_i)^2 \left[ \frac{1}{\Delta'_i} - 1 - \frac{D''}{\bar{y}\Delta'_i} \right] < \bar{y}k_1 (\Delta'_i)^2 \left[ \left\lfloor \frac{D'}{\bar{y}\Delta'_i} \right\rfloor - \frac{D''}{\bar{y}\Delta'_i} \right] \\ &\leq \bar{y}k_1 (\Delta'_i)^2 \left[ \frac{D'}{\bar{y}\Delta'_i} - \frac{D''}{\bar{y}\Delta'_i} \right] = k_1 \Delta'_i (D' - D'') \end{aligned}$$

as before. Q.E.D. Claim 3

For all  $D, D' \in [0, \bar{y}]$  and all  $t \in S_i \setminus \{1\}$ , define the difference quotients of  $v^{H\bar{y}i}(D, t)$  with respect to  $D$  and  $t$ :

$$\Delta_1^{H\bar{y}i}(D, D', t) = \frac{v^{H\bar{y}i}(D, t) - v^{H\bar{y}i}(D', t)}{D - D'} \text{ and} \quad (35)$$

$$\Delta_2^{H\bar{y}i}(D, t) = \frac{v^{H\bar{y}i}(D, t + \Delta_i) - v^{H\bar{y}i}(D, t)}{\Delta_i}. \quad (36)$$

By parts 1 and 2 of Claim 3, if  $D < D'$ , then

$$\Delta_1^{H\bar{y}i}(D, D', t) \in \left( k_0 \left( 1 - \Delta'_i \left\lfloor \frac{D'}{\bar{y}\Delta'_i} \right\rfloor + \Delta'_i \right), k_1 \left( 1 - \frac{D}{\bar{y}} + \Delta_i \right) \right) \subset (0, \infty) \quad (37)$$

while if  $D > \bar{y}\Delta'_i$ ,

$$\Delta_2^{H\bar{y}i}(D, t) \in (0, k_1 \min \{D, \bar{y}(1 - \Delta'_i)\}). \quad (38)$$

By (22), for any  $D \in [0, \bar{y}]$ , the partial derivatives of  $v^{H\bar{y}}(D, t)$  are given by

$$v_2^{H\bar{y}}(D, t) = -\bar{y} \int_{z=0}^{D/\bar{y}} \frac{\partial H(z|t)}{\partial t} dz \text{ and} \quad (39)$$

$$v_1^{H\bar{y}}(D, t) = 1 - H\left(\frac{D}{\bar{y}}|t\right) = \int_{z=D/\bar{y}}^1 \frac{\partial H(z|t)}{\partial z} dz. \quad (40)$$

For all  $D \in [0, \bar{y}]$  and  $t \in [0, 1]$ ,

$$\frac{\partial v_2^{H\bar{y}}(D, t)}{\partial D} = -\frac{\partial}{\partial t} H\left(\frac{D}{\bar{y}}|t\right) \in \left[ k_0 \frac{D}{\bar{y}} \left( 1 - \frac{D}{\bar{y}} \right), k_1 \right) \quad (41)$$

by (39) and L-H;

$$\left| \frac{\partial v_2^{H\bar{y}}(D,t)}{\partial t} \right| \leq k_2 D \quad (42)$$

by (39) and L-H;

$$\frac{\partial v_1^{H\bar{y}}(D,t)}{\partial D} = -\frac{\partial}{\partial D} H\left(\frac{D}{\bar{y}}|t\right) \in \left(-\frac{k_1}{\bar{y}}, -\frac{k_0}{\bar{y}}\right) \quad (43)$$

by (40) and L-H;

$$\frac{k_0 D^2 [3\bar{y} - 2D]}{6\bar{y}^2} < v_2^{H\bar{y}}(D,t) < k_1 D \quad (44)$$

by (39) and L-H;

$$k_0 \left(1 - \frac{D}{\bar{y}}\right) < v_1^{H\bar{y}}(D,t) < k_1 \left(1 - \frac{D}{\bar{y}}\right) \quad (45)$$

by (40) and L-H; for all  $t' \in [0, 1]$ ,

$$\left| v_1^{H\bar{y}}(D,t) - v_1^{H\bar{y}}(D,t') \right| = \left| H\left(\frac{D}{\bar{y}}|t\right) - H\left(\frac{D}{\bar{y}}|t'\right) \right| \leq k_1 |t - t'| \quad (46)$$

by (40) and L-H; for all  $D' \in [0, \bar{y}]$ ,

$$\left| v_1^{H\bar{y}}(D,t) - v_1^{H\bar{y}}(D',t) \right| = \left| H\left(\frac{D}{\bar{y}}|t\right) - H\left(\frac{D'}{\bar{y}}|t\right) \right| < \frac{k_1}{\bar{y}} |D - D'| \quad (47)$$

by L-H. By (44) and (45),

$$0 \leq \frac{k_0 D^2 (3\bar{y} - 2D)}{6\bar{y}k_1(\bar{y} - D)} < \frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} < \frac{k_1 \bar{y}}{k_0} \left(\frac{D}{\bar{y} - D}\right), \quad (48)$$

and

$$\frac{6\bar{y}k_1(\bar{y} - D)}{k_0 D^2 (3\bar{y} - 2D)} > \frac{v_1^{H\bar{y}}(D,t)}{v_2^{H\bar{y}}(D,t)} > \frac{k_0}{k_1 \bar{y}} \left(\frac{\bar{y} - D}{D}\right). \quad (49)$$

By (41), (43), (44), (45), and (49),

$$\frac{\partial}{\partial D} \left( \frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} \right) = \frac{\frac{\partial}{\partial D} v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} - \frac{v_2^{H\bar{y}}(D,t) \frac{\partial}{\partial D} v_1^{H\bar{y}}(D,t)}{\left[v_1^{H\bar{y}}(D,t)\right]^2} \in \left[ \gamma_1^{\bar{y}}(D), \gamma_2^{\bar{y}}(D) \right] \quad (50)$$

where

$$\gamma_1^{\bar{y}}(D) = \frac{k_0 D}{k_1 \bar{y}} \left[ 1 + \frac{Dk_0(3\bar{y} - 2D)}{6k_1(\bar{y} - D)^2} \right] \geq 0 \quad (51)$$

and

$$\gamma_2^{\bar{y}}(D) = \frac{k_1 \bar{y}}{k_0(\bar{y} - D)} \left[ 1 + \frac{Dk_1}{k_0(\bar{y} - D)} \right] > 0. \quad (52)$$

Note that  $\gamma_1^{\bar{y}}(D)$  lies in  $(0, \infty)$  if  $D \in (0, \bar{y})$ , is zero if  $D = 0$ , and is  $\infty$  if  $D = \bar{y}$ . Moreover,  $\gamma_2^{\bar{y}}(D)$  is increasing in  $D$ , lies in  $(0, \infty)$  if  $D \in [0, \bar{y})$ , and is  $\infty$  if  $D = \bar{y}$ . Finally, for  $0 \leq a \leq b$ ,

$$\max_{D \in [a, b]} \gamma_2^{\bar{y}}(D) \leq \frac{k_1 \bar{y}}{k_0(\bar{y} - b)} \left[ 1 + \frac{bk_1}{k_0(\bar{y} - b)} \right], \quad (53)$$

which is positive and, if  $b < \bar{y}$ , finite.

For any real number  $\ell$ , let  $(\ell, \infty]$  and  $[\ell, \infty]$  denote the sets  $(\ell, \infty) \cup \{\infty\}$  and  $[\ell, \infty) \cup \{\infty\}$ , respectively. Recall that  $f^{H\delta\bar{y}}(D, t)$  is defined in (24). For  $\underline{t} \in [0, 1]$  and  $a \in (0, \bar{y}]$ , and  $k \in (0, \infty]$ , we define the following modification of  $\text{CP}^{H\delta\bar{y}}$ :

CONTINUOUS INITIAL VALUE PROBLEM WITH PARAMETERS  $H, \delta, \bar{y}, \underline{t}, a, k$  ( $\text{CP}_{\underline{t}ak}^{H\delta\bar{y}}$ ).

The differential equation

$$\frac{dD_{\underline{t}ak}^{H\delta\bar{y}}}{dt} = \max \left\{ f^{H\delta\bar{y}} \left( D_{\underline{t}ak}^{H\delta\bar{y}}(t), t \right), -k \right\} \quad (54)$$

with  $D_{\underline{t}ak}^{H\delta\bar{y}} : [\underline{t}, 1] \rightarrow \mathfrak{R}$ , together with the initial value  $D_{\underline{t}ak}^{H\delta\bar{y}}(\underline{t}) = a$ .

Clearly, any  $D_{0\bar{y}\infty}^{H\delta\bar{y}}$  that solves  $\text{CP}_{0\bar{y}\infty}^{H\delta\bar{y}}$  must also be a solution  $D^{H\delta\bar{y}}$  to  $\text{CP}^{H\delta\bar{y}}$  and vice-versa.

CLAIM 5 Consider any  $\underline{t} \in [0, 1]$ ,  $H$  in  $\mathcal{H}$ ,  $a \in (0, \bar{y}]$ , and  $k \in (0, \infty]$ .

1. If either  $a < \bar{y}$  or  $k < \infty$  (or both), then there exists a unique solution to  $\text{CP}_{\underline{t}ak}^{H\delta\bar{y}}$ , which is decreasing and differentiable in  $t$  and takes values in  $(0, a]$ .
2. Let  $\tilde{\delta}, \hat{\delta} \in (0, 1)$  and  $\tilde{H}, \hat{H} \in \mathcal{H}$  satisfy  $f^{\tilde{H}\tilde{\delta}\bar{y}} \leq f^{\hat{H}\hat{\delta}\bar{y}}$ , and let  $a' \in (0, a]$  and  $k' \in [k, \infty]$ . Suppose there exist (possibly nonunique) solutions  $D_{\underline{t}ak}^{\hat{H}\hat{\delta}\bar{y}}$  and  $D_{\underline{t}a'k'}^{\tilde{H}\tilde{\delta}\bar{y}}$  to  $\text{CP}_{\underline{t}ak}^{\hat{H}\hat{\delta}\bar{y}}$  and  $\text{CP}_{\underline{t}a'k'}^{\tilde{H}\tilde{\delta}\bar{y}}$ , respectively. Then  $D_{\underline{t}ak}^{\hat{H}\hat{\delta}\bar{y}}(t) \geq D_{\underline{t}a'k'}^{\tilde{H}\tilde{\delta}\bar{y}}(t)$  for all  $t \in [\underline{t}, 1]$ .
3. If either  $a < \bar{y}$  or  $k < \infty$  (or both), then the function  $D_{\underline{t}ak}^{H\delta\bar{y}}$  is Lipschitz continuous in  $t$  with Lipschitz constant  $\min \{k, k_a\}$  where

$$k_a = \frac{k_1 \bar{y} a}{(1 - \delta) k_0 (\bar{y} - a)}. \quad (55)$$

4. If  $a < \bar{y}$ , then for all  $t \in [\underline{t}, 1]$ ,  $D_{\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \in [0, \bar{y} - a]$ .

PROOF OF CLAIM 5. Part 1. We first show that  $\max \left\{ f^{H\delta\bar{y}}(D, t), -k \right\}$  is (a) continuous in  $t \in [0, 1]$  and (b) Lipschitz continuous in  $D \in [0, a]$ . Since  $\max$  is a Lipschitz-continuous function, it suffices to show that  $f^{H\delta\bar{y}}(D, t)$  has properties (a) and (b) whenever  $f^{H\delta\bar{y}}(D, t) \geq -k$ . If  $k = \infty$ , then  $D \leq a < \bar{y}$ . If instead  $k < \infty$  and  $f^{H\delta\bar{y}}(D, t) \geq -k$  then by (48),  $\frac{k_0 D^2 (3\bar{y} - 2D)}{6\bar{y}k_1(\bar{y} - D)} < (1 - \delta)k$ ; rearranging,  $\frac{D^2}{\bar{y} - D} < \frac{6\bar{y}k_1(1 - \delta)k}{k_0(3\bar{y} - 2D)} \leq \frac{6k_1(1 - \delta)k}{k_0}$  (since  $D \leq \bar{y}$ ). Since  $\frac{D^2}{\bar{y} - D}$  is continuous and increasing in  $D$  and approaches  $\infty$  as  $D \uparrow \bar{y}$ , there is a constant  $a' < \bar{y}$  such that  $D \leq a'$  for any  $D$  and  $t$  satisfying  $f^{H\delta\bar{y}}(D, t) \geq -k$ . Collecting cases,  $D \leq b = \max \{a, a'\} < \bar{y}$ . Hence  $1 - H\left(\frac{D}{\bar{y}}|t\right) \geq 1 - H\left(\frac{b}{\bar{y}}|t\right) > 0$ ; as  $H$  is continuously differentiable in  $t$ ,  $f^{H\delta\bar{y}}(D, t)$  is continuous in  $t$  and thus satisfies (a). And by (50), (51), and (53),  $\left| \frac{\partial}{\partial D} f^{H\delta\bar{y}}(D, t) \right|$  is at most  $\frac{k_1 \bar{y}}{k_0(\bar{y} - b)(1 - \delta)} \left[ 1 + \frac{bk_1}{k_0(\bar{y} - b)} \right]$ , whence  $f^{H\delta\bar{y}}(D, t)$  satisfies (b). By the Picard-Lindelöf theorem, there thus exists a unique solution  $D_{\underline{t}ak}^{H\delta\bar{y}}$  to  $\text{CP}_{\underline{t}ak}^{H\delta\bar{y}}$ . The solution  $D_{\underline{t}ak}^{H\delta\bar{y}}$  is differentiable in  $t$  since the right hand side of (54) is finite. Finally,  $f^{H\delta\bar{y}}(D, t) < 0$  for all  $D \in (0, \bar{y}]$  and  $f^{H\delta\bar{y}}(0, t) = 0$ . Hence,  $D_{\underline{t}ak}^{H\delta\bar{y}}(t)$  is decreasing in  $t$  until and unless it hits zero, where it remains. Thus, by (48),  $D_{\underline{t}ak}^{H\delta\bar{y}}(t) \leq a$  for all  $t \in [\underline{t}, 1]$ . Finally,  $D_{\underline{t}ak}^{H\delta\bar{y}}(t) > 0$  by the following lemma. Let  $D_{\underline{t}ak}^{H\delta\bar{y}}(t)$  first reach its minimum value of  $\underline{D} \geq 0$  at  $t = \bar{t} > \underline{t}$ .

LEMMA 6 *The minimum face value  $\underline{D}$  is nonzero.*

PROOF OF LEMMA 6: For  $t \in [\underline{t}, \bar{t}]$ , the function  $D_{\underline{t}ak}^{H\delta\bar{y}}(t)$  has a strictly decreasing inverse  $t_{\underline{t}ak}^{H\delta\bar{y}}$  that satisfies the following inverse problem:

INVERSE CONTINUOUS INITIAL VALUE PROBLEM WITH PARAMETERS

$H, \bar{y}, \underline{t}, a, k$  (ICP $_{\underline{t}ak}^{H\delta\bar{y}}$ ). The differential equation

$$\frac{dt_{\underline{t}ak}^{H\delta\bar{y}}}{dD} = -\max \left\{ (1 - \delta) \frac{v_1^{H\bar{y}} \left( D, t_{\underline{t}ak}^{H\delta\bar{y}}(D) \right)}{v_2^{H\bar{y}} \left( D, t_{\underline{t}ak}^{H\delta\bar{y}}(D) \right)}, \frac{1}{k} \right\} \quad (56)$$

with  $t_{\underline{t}ak}^{H\delta\bar{y}} : [\underline{D}, a] \rightarrow [\underline{t}, \bar{t}]$ , together with the terminal value  $t_{\underline{t}ak}^{H\delta\bar{y}}(a) = \underline{t}$ .

By (49),  $\frac{dt_{tak}^{H\delta\bar{y}}}{dD} \leq -(1-\delta) \frac{k_0}{k_1\bar{y}} \left( \frac{\bar{y}-D}{D} \right)$ . Hence, as  $t_{tak}^{H\delta\bar{y}}(a) = \underline{t}$ ,

$$\begin{aligned} \bar{t} &= t_{tak}^{H\delta\bar{y}}(\underline{D}) = t_{tak}^{H\delta\bar{y}}(a) - \int_{D=\underline{D}}^a \frac{dt_{tak}^{H\delta\bar{y}}}{dD} dD \\ &\geq \underline{t} + \frac{(1-\delta)k_0}{k_4\bar{y}} \int_{D=\underline{D}}^{\bar{y}} \left( \frac{\bar{y}-D}{D} \right) dD = \underline{t} + \frac{(1-\delta)k_0}{k_4} F\left(\frac{\underline{D}}{\bar{y}}\right), \end{aligned}$$

where  $F(x)$  denotes  $x - \ln x - 1$ , which is finite and differentiable for all finite  $x > 0$ .  $F$  is decreasing in  $x \in (0, 1)$ :  $F'(x) = 1 - 1/x < 0$ . Thus,  $F\left(\frac{\underline{D}}{\bar{y}}\right) < \frac{k_4}{(1-\delta)k_0} (\bar{t} - \underline{t})$ , whence  $\underline{D} > \bar{y}F^{-1}\left(\frac{k_4}{(1-\delta)k_0} (\bar{t} - \underline{t})\right)$ . Moreover,  $F(1) = 0$  and  $\lim_{x \downarrow 0} F(x) = \infty$ , so the inverse  $F^{-1}$  is decreasing in  $x \in (0, \infty)$  and satisfies  $F^{-1}(0) = 1$  and  $\lim_{x \rightarrow \infty} F^{-1}(x) = 0$ . Hence, as  $\bar{t} - \underline{t} > 0$ ,  $\underline{D} > 0$ . Q.E.D. Lemma 6

Part 2. Suppose not. Since  $D_{tak}^{\hat{H}\hat{\delta}\bar{y}}(\underline{t}) = a \geq a' = D_{ta'k'}^{\hat{H}\hat{\delta}\bar{y}}(\underline{t})$  and both solutions are continuous in  $t$ , there must be  $\underline{t} \leq t_0 < t_1 \leq 1$  such that  $D_{tak}^{\hat{H}\hat{\delta}\bar{y}}(t_0) = D_{ta'k'}^{\hat{H}\hat{\delta}\bar{y}}(t_0)$  and, for all  $t \in (t_0, t_1]$ ,  $D_{tak}^{\hat{H}\hat{\delta}\bar{y}}(t) < D_{ta'k'}^{\hat{H}\hat{\delta}\bar{y}}(t)$ . Hence, by (54),

$$0 < D_{ta'k'}^{\hat{H}\hat{\delta}\bar{y}}(t_1) - D_{tak}^{\hat{H}\hat{\delta}\bar{y}}(t_1) = \int_{t=t_0}^{t_1} \left[ \max \left\{ f^{\hat{H}\hat{\delta}\bar{y}} \left( D_{ta'k'}^{\hat{H}\hat{\delta}\bar{y}}(t), t \right), -k' \right\} \right. \\ \left. - \max \left\{ f^{\hat{H}\hat{\delta}\bar{y}} \left( D_{tak}^{\hat{H}\hat{\delta}\bar{y}}(t), t \right), -k \right\} \right] dt$$

which is impossible: since  $f^{H\delta\bar{y}}(D, t)$  is decreasing in  $D$  and  $k' \geq k$ , the integrand is non-positive for all  $t$  in  $(t_0, t_1]$ .

Part 3 follows from (48) and the fact that  $D_{tak}^{H\delta\bar{y}} \leq a$ .

Part 4. By part 2,  $\Gamma_a(t) \stackrel{d}{=} D_{\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{ta^\infty}^{H\delta\bar{y}}(t) \geq 0$ . By (24),

$$\Gamma'_a(t) = \max \left\{ \begin{aligned} &\left[ f^{H\delta\bar{y}} \left( D_{\bar{y}k_a}^{H\delta\bar{y}}(t), t \right) - f^{H\delta\bar{y}} \left( D_{ta^\infty}^{H\delta\bar{y}}(t), t \right) \right], \\ &-k_a - f^{H\delta\bar{y}} \left( D_{ta^\infty}^{H\delta\bar{y}}(t), t \right) \end{aligned} \right\}.$$

Both entries in the max are nonpositive by (48), (24), and the fact that  $D_{ta^\infty}^{H\delta\bar{y}}(t) \leq a$ . Accordingly, for all  $t \in [\underline{t}, 1]$ ,  $\left| D_{\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{ta^\infty}^{H\delta\bar{y}}(t) \right| \leq \Gamma_a(\underline{t}) = \bar{y} - a$ . Q.E.D. Claim 5

Before addressing the discrete case, we prove some useful bounds:



CLAIM 7 Let  $w, w', \zeta e, \zeta e'$  be in  $(0, \bar{y}]$  and satisfy  $w' \geq w$ ,  $\zeta e' \geq \zeta e$ ,  $w > \zeta e$ , and  $w' > \zeta e'$ .

Then

$$0 \leq \Delta_2^{H\bar{y}i}(\zeta e', t) - \Delta_2^{H\bar{y}i}(\zeta e, t) \leq k_1(\zeta e' - \zeta e). \quad (57)$$

Moreover, if  $\min\{w, w', \zeta, \zeta'\} > \bar{y}\Delta'_i$  then

$$0 \geq \Delta_1^{H\bar{y}i}(\zeta e', w', t) - \Delta_1^{H\bar{y}i}(\zeta e, w, t) \geq -k_1 \left[ \frac{\max\{w' - w, \zeta e' - \zeta e\}}{\bar{y}} + \Delta'_i \right] \quad (58)$$

and

$$\frac{\Delta_2^{H\bar{y}i}(\zeta e, t)}{\Delta_1^{H\bar{y}i}(\zeta e, w, t)} \in \left( 0, \frac{k_1 \zeta e}{k_0 \left( 1 - \Delta'_i \left[ \frac{w}{\bar{y}\Delta'_i} \right] + \Delta'_i \right)} \right) \subset (0, \infty). \quad (59)$$

PROOF OF CLAIM 7. By (26), for any  $\zeta e \in [0, \bar{y}]$  and  $t \in S_i \setminus \{1\}$ ,

$$\begin{aligned} & v^{H\bar{y}i}(\zeta e, t + \Delta_i) - v^{H\bar{y}i}(\zeta e, t) \\ &= \bar{y} \int_{z=0}^{\frac{\zeta e}{\bar{y}}} \left[ H \left( \Delta'_i \left[ \frac{z}{\Delta'_i} \right] \middle| t \right) - H \left( \Delta'_i \left[ \frac{z}{\Delta'_i} \right] \middle| t + \Delta_i \right) \right] dz. \end{aligned} \quad (60)$$

As the integrand is nonnegative, (60) is nondecreasing in  $\zeta e$ , so  $\Delta_2^{H\bar{y}i}(\zeta', t) - \Delta_2^{H\bar{y}i}(\zeta, t) \geq 0$ . By L-H, for any  $z \in [0, \bar{y}]$ ,  $H(z|t) - H(z|t + \Delta_i) < k_1 \Delta_i$ . Equation (57) then follows from (36) and (60).

By (26),

$$\begin{aligned} \Delta_1^{H\bar{y}i}(\zeta e', w', t) - \Delta_1^{H\bar{y}i}(\zeta e, w, t) &= \frac{v^{H\bar{y}i}(\zeta e', t) - v^{H\bar{y}i}(w', t)}{\zeta e' - w'} - \frac{v^{H\bar{y}i}(\zeta e, t) - v^{H\bar{y}i}(w, t)}{\zeta e - w} \\ &= \frac{\bar{y}}{w - \zeta e} \int_{z=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}} H \left( \Delta'_i \left[ \frac{z}{\Delta'_i} \right] \middle| t \right) dz - \frac{\bar{y}}{w' - \zeta e'} \int_{z=\frac{\zeta e'}{\bar{y}}}^{\frac{w'}{\bar{y}}} H \left( \Delta'_i \left[ \frac{z}{\Delta'_i} \right] \middle| t \right) dz. \end{aligned}$$

Define the change of variables  $z' = \frac{\zeta}{\bar{y}} + \left( \frac{w - \zeta}{w' - \zeta'} \right) \left( z - \frac{\zeta'}{\bar{y}} \right)$ . When  $z = \frac{\zeta'}{\bar{y}}$ ,  $z' = \frac{\zeta}{\bar{y}}$ , and when  $z = \frac{w}{\bar{y}}$ ,  $z' = \frac{w'}{\bar{y}}$ . Moreover,  $dz = \frac{w' - \zeta'}{w - \zeta} dz'$  and  $z = \frac{\zeta'}{\bar{y}} + \left( \frac{w' - \zeta'}{w - \zeta} \right) \left( z' - \frac{\zeta'}{\bar{y}} \right)$  which we denote  $\psi(z')$ . So  $\int_{z=\frac{\zeta'}{\bar{y}}}^{\frac{w}{\bar{y}}} H \left( \Delta'_i \left[ \frac{z}{\Delta'_i} \right] \middle| t \right) dz = \int_{z'=\frac{\zeta}{\bar{y}}}^{\frac{w'}{\bar{y}}} H \left( \Delta'_i \left[ \frac{\psi(z')}{\Delta'_i} \right] \middle| t \right) \frac{w' - \zeta'}{w - \zeta} dz'$ . Renaming  $z'$  to  $z$  and simplifying,

$$\Delta_1^{H\bar{y}i}(\zeta e', w', t) - \Delta_1^{H\bar{y}i}(\zeta e, w, t) = \frac{\bar{y}}{w - \zeta e} \int_{z=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}} \left[ H \left( \Delta'_i \left[ \frac{z}{\Delta'_i} \right] \middle| t \right) - H \left( \Delta'_i \left[ \frac{\psi(z)}{\Delta'_i} \right] \middle| t \right) \right] dz. \quad (61)$$

We can write

$$z - \psi(z) = \frac{1}{w - \zeta e} \left( \left( z - \frac{\zeta e'}{\bar{y}} \right) [w - \zeta e] - \left( z - \frac{\zeta e}{\bar{y}} \right) [w' - \zeta e'] \right). \quad (62)$$

As the right hand side is linear in  $z$ , it reaches its maximum and minimum at the endpoints of the interval of integration. At the lower endpoint (at  $z = \frac{\zeta}{\bar{y}}$ ), the right hand side of (62) equals  $\frac{\zeta - \zeta'}{\bar{y}}$ , while at the upper endpoint (at  $z = \frac{w}{\bar{y}}$ ), it equals  $\frac{w - w'}{\bar{y}}$ . Thus,  $-\bar{w} \leq z - \psi(z) \leq -\underline{w}$  where  $\underline{w} = \bar{y}^{-1} \min \{w' - w, \zeta' - \zeta\}$  and  $\bar{w} = \bar{y}^{-1} \max \{w' - w, \zeta' - \zeta\}$ . As  $\underline{w}$  and  $\bar{w}$  are both nonnegative,  $z \leq \psi(z)$ , which by (61) establishes the first inequality in (58). Finally, by (61) and L-H,

$$\begin{aligned} & \Delta_1^{H\bar{y}i}(\zeta e', w', t) - \Delta_1^{H\bar{y}i}(\zeta e, w, t) \\ & \geq \frac{\bar{y}}{w - \zeta e} \int_{z' = \frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}} \left[ H \left( \Delta'_i \left[ \frac{\psi(z) - \bar{w}}{\Delta'_i} \right] \middle| t \right) - H \left( \Delta'_i \left[ \frac{\psi(z)}{\Delta'_i} \right] \middle| t \right) \right] dz \\ & \geq \frac{\bar{y}k_1}{w - \zeta e} \int_{z' = \frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}} \left[ \Delta'_i \left[ \frac{\psi(z) - \bar{w}}{\Delta'_i} \right] - \Delta'_i \left[ \frac{\psi(z)}{\Delta'_i} \right] \right] dz \\ & \geq -\frac{\bar{y}k_1}{w - \zeta e} \int_{z' = \frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}} [\bar{w} + \Delta'_i] dz = -k_1 [\bar{w} + \Delta'_i]. \end{aligned}$$

This establishes the second inequality in (58). Finally, (59) follows from (37) and (38).

Q.E.D. Claim 7

CLAIM 8 Fix  $H$ ,  $\bar{y}$ , and  $i$ . Define

$$\phi(D, t) = v^{H\bar{y}i}(D, t + \Delta_i) - \delta v^{H\bar{y}i}(D, t). \quad (63)$$

For any  $D \in (\bar{y}\Delta'_i, \bar{y}]$  and any  $t \in S_i$  there exists a unique solution  $D^* = D^*(D)$ , which lies in  $(\bar{y}\Delta'_i, D)$ , to

$$\phi(D^*, t) = (1 - \delta) v^{H\bar{y}i}(D, t). \quad (64)$$

Moreover,  $D^*(D)$  is increasing in  $D$  and  $D - D^*(D)$  is nondecreasing in  $D$ . Finally,

$$\frac{\partial \phi}{\partial D} \geq k_0(1 - \delta) \left( 1 - \frac{D}{\bar{y}} \right). \quad (65)$$

PROOF OF CLAIM 8. We first show three properties.

1.  $\phi(\bar{y}\Delta'_i, t) < (1 - \delta)v^{H\bar{y}i}(D, t)$ . Proof: by (26), for all  $t$  in  $S_i$ ,  $v^{H\bar{y}i}(\bar{y}\Delta'_i, t) = \bar{y}\Delta'_i$ . Hence,  $\phi(\bar{y}\Delta'_i, t) = (1 - \delta)\bar{y}\Delta'_i$ . Moreover,  $v^{H\bar{y}i}(D, t)$  is strictly increasing in  $D \in [0, \bar{y}]$  by part 2 of Claim 3, so  $v^{H\bar{y}i}(D, t) > \bar{y}\Delta'_i$ . The result then follows from (63) and (64).
2.  $\phi(D, t) > (1 - \delta)v^{H\bar{y}i}(D, t)$ . Proof: for  $D \in (\bar{y}\Delta'_i, \bar{y}]$ ,  $v^{H\bar{y}i}(D, t)$  is increasing in  $t$  by part 1 of Claim 3. The result then follows from (63) and (64).
3.  $\phi$  is continuous and increasing in  $D \in [0, \bar{y}]$ . Proof:  $v^{H\bar{y}i}(D, t)$  is continuous in  $D$  by (20), so  $\phi(D, t)$  also is continuous in  $D$ . By (26),

$$\phi(D, t) = (1 - \delta)D - \bar{y} \left[ \int_{z=0}^{\frac{D}{\bar{y}}} \left[ H\left(\Delta'_i \left\lfloor \frac{z}{\Delta'_i} \right\rfloor | t + \Delta_i\right) - \delta H\left(\Delta'_i \left\lfloor \frac{z}{\Delta'_i} \right\rfloor | t\right) \right] dz \right].$$

By L-H,  $1 - H(z|t) > k_0(1 - z)$  for all  $z \in [0, 1]$  (and thus, substituting  $z = 0$ ,  $k_0 < 1$ ), so letting  $z_0 = \Delta'_i \left\lfloor \frac{D}{\bar{y}\Delta'_i} \right\rfloor \leq \frac{D}{\bar{y}}$ ,

$$\begin{aligned} \frac{\partial \phi}{\partial D} &= 1 - \delta - [H(z_0|t + \Delta_i) - \delta H(z_0|t)] \geq (1 - \delta)[1 - H(z_0|t)] \\ &\geq (1 - \delta)k_0(1 - z_0) \geq k_0(1 - \delta) \left(1 - \frac{D}{\bar{y}}\right). \end{aligned}$$

This establishes the result as well as equation (65).

Facts 1-3 imply that for any  $D \in (\bar{y}\Delta'_i, \bar{y}]$ , there exists a unique  $D^* = D^*(D)$  satisfying (64), and that it lies in  $(\bar{y}\Delta'_i, D)$ . Moreover,  $D^*(D)$  is increasing in  $D$ .

Finally, let  $\widehat{D}_0 > D$  and let  $\widehat{D}^* = D^*(\widehat{D}_0)$ . To show that  $D - D^*(D)$  is nondecreasing in  $D$ , we must show that  $\widehat{D}_0 - \widehat{D}^* \geq D - D^*$  or, equivalently, that  $\widehat{D}_0 - D \geq \widehat{D}^* - D^*$ . By (63),

$$\phi(D, t) = v^{H\bar{y}i}(D, t + \Delta_i) - \delta v^{H\bar{y}i}(D, t) = (1 - \delta)v^{H\bar{y}i}(D, t) + v^{H\bar{y}i}(D, t + \Delta_i) - v^{H\bar{y}i}(D, t).$$

Hence, by (36), (57), and (64), and since  $\widehat{D}^* > D^*$ ,

$$\begin{aligned} (1 - \delta) \left[ v^{H\bar{y}i}(\widehat{D}_0, t) - v^{H\bar{y}i}(D, t) \right] &= \phi(\widehat{D}^*, t) - \phi(D^*, t) \\ &\geq (1 - \delta) \left[ v^{H\bar{y}i}(\widehat{D}^*, t) - v^{H\bar{y}i}(D^*, t) \right], \end{aligned}$$

whence by part 2 of Claim 3,

$$\frac{\widehat{D}^* - D^*}{\widehat{D}_0 - D} \leq \frac{\frac{v^{H\bar{y}i}(\widehat{D}_0, t) - v^{H\bar{y}i}(D, t)}{\widehat{D}_0 - D}}{\frac{v^{H\bar{y}i}(\widehat{D}^*, t) - v^{H\bar{y}i}(D^*, t)}{\widehat{D}^* - D^*}},$$

which is in  $(0, 1]$  by (35) and (58). Thus,  $\widehat{D}^* - D^* \leq \widehat{D}_0 - D$  as claimed. Q.E.D. Claim 8

For any real number  $\ell$ , let  $(\ell, \infty]$  and  $[\ell, \infty]$  denote the sets  $(\ell, \infty) \cup \{\infty\}$  and  $[\ell, \infty) \cup \{\infty\}$ , respectively. For any constants  $\underline{t} \in S_i$ ,  $a \in (0, \bar{y}]$ , and  $k \in (0, \infty]$ , consider the following initial value problem, where  $S_i^{\underline{t}}$  denotes the set of types  $t \geq \underline{t}$  in  $S_i$ :

DISCRETE INITIAL VALUE PROBLEM WITH PARAMETERS  $H, \bar{y}, \underline{t}, a, k$  ( $DP_{\underline{t}ak}^{H\delta\bar{y}i}$ ).

The condition

$$D_{\underline{t}ak}^{H\delta\bar{y}i}(t + \Delta_i) = \max \left\{ D_{\underline{t}ak}^{H\delta\bar{y}i*}(t + \Delta_i), D_{\underline{t}ak}^{H\delta\bar{y}i}(t) - k\Delta_i \right\} \quad (66)$$

with  $D_{\underline{t}ak}^{H\delta\bar{y}i} : S_i^{\underline{t}} \rightarrow \mathfrak{R}$ , where  $D_{\underline{t}ak}^{H\delta\bar{y}i*}(t + \Delta_i)$  is the (by Claim 8) unique solution  $D^* \in (\bar{y}\Delta'_i, D_{\underline{t}ak}^{H\delta\bar{y}i}(t))$  to

$$v^{H\bar{y}i}(D^*, t + \Delta_i) - \delta v^{H\bar{y}i}(D^*, t) = (1 - \delta) v^{H\bar{y}i}(D_{\underline{t}ak}^{H\bar{y}i}(t), t), \quad (67)$$

together with the initial value  $D_{\underline{t}ak}^{H\delta\bar{y}i}(\underline{t}) = a > \bar{y}\Delta'_i$ .

Clearly, any  $D_{0\bar{y}\infty}^{H\delta\bar{y}i}$  that solves  $DP_{0\bar{y}\infty}^{H\delta\bar{y}i}$  must also be a solution  $D^{H\delta\bar{y}i}$  to  $DP^{H\delta\bar{y}i}$  and vice-versa.

CLAIM 9 For any  $\underline{t} \in S_i$ ,  $a \in (\bar{y}\Delta'_i, \bar{y}]$ , and  $k \in (0, \infty]$ :

1. There exists a unique solution  $D_{\underline{t}ak}^{H\delta\bar{y}i}$  to  $DP_{\underline{t}ak}^{H\delta\bar{y}i}$ . This function is decreasing in  $t \in S_i^{\underline{t}}$  and takes values in  $(\bar{y}\Delta'_i, a]$ .
2. Let  $a' \in (\bar{y}\Delta'_i, a]$  and  $k' \in [k, \infty]$ . Then  $D_{\underline{t}ak}^{H\delta\bar{y}i}(t) \geq D_{\underline{t}a'k'}^{H\delta\bar{y}i}(t)$  for all types  $t \in S_i^{\underline{t}}$ .
3. Let  $k_a^i = \frac{k_1\bar{y}(\bar{y}+a)}{k_0(1-\delta)(\bar{y}(1-\Delta'_i)-a)}$ . For all types  $t \in S_i^{\underline{t}}$ ,  $0 > D_{\underline{t}ak}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}ak}^{H\delta\bar{y}i}(t) \geq -\Delta_i k_a$  and hence  $0 > D_{\underline{t}ak}^{H\delta\bar{y}i}(t + \Delta_i) - D_{\underline{t}ak}^{H\delta\bar{y}i}(t) \geq -\Delta_i \min\{k, k_a\}$ .

4. For all  $t \in S_i^t$ ,  $D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) \in [0, \bar{y} - a]$ .

PROOF OF CLAIM 9. Part 1. Follows from Claim 8.

Part 2. Clearly,  $D_{\underline{t}ak}^{H\delta\bar{y}i}(\underline{t}) - D_{\underline{t}a'k'}^{H\delta\bar{y}i}(\underline{t}) = a - a' \geq 0$ . And if, for some  $t \in S_i^t \setminus \{1\}$ ,  $D_{\underline{t}ak}^{H\delta\bar{y}i}(t) \geq D_{\underline{t}a'k'}^{H\delta\bar{y}i}(t)$ , then  $D_{\underline{t}ak}^{H\delta\bar{y}i}(t) - k\Delta_i \geq D_{\underline{t}a'k'}^{H\delta\bar{y}i}(t) - k'\Delta_i$  and, by Claim 8,  $D_{\underline{t}ak}^{H\delta\bar{y}i*}(t + \Delta_i) \geq D_{\underline{t}a'k'}^{H\delta\bar{y}i*}(t + \Delta_i)$ , so  $D_{\underline{t}ak}^{H\delta\bar{y}i}(t + \Delta_i) \geq D_{\underline{t}a'k'}^{H\delta\bar{y}i}(t + \Delta_i)$ .

Part 3. Let  $D' = D_{\underline{t}ak}^{H\delta\bar{y}i}(t)$ ,  $D'' = D_{\underline{t}ak}^{H\delta\bar{y}i}(t + \Delta_i)$ , and  $D^* = D_{\underline{t}ak}^{H\delta\bar{y}i*}(t + \Delta_i)$ . By Claim 8,  $D^* - D' < 0$  and  $\min\{D', D'', D^*\} > \bar{y}\Delta_i'$ . We will show that  $D^* - D' \geq -\Delta_i k_a$  which, by (66), implies  $0 > D'' - D' \geq -\Delta_i \min\{k, k_a\}$ . The result is trivial when  $a = \bar{y}$  since  $k_{\bar{y}} = \infty$ . Suppose  $a < \bar{y}$ . By (36), (38), (63), and (64),

$$\begin{aligned} \phi(D', t) - \phi(D^*, t) &= \phi(D', t) - (1 - \delta)v^{H\bar{y}i}(D', t) \\ &= v^{H\bar{y}i}(D', t + \Delta_i) - v^{H\bar{y}i}(D', t) \leq k_1 D' \Delta_i \leq k_1 a \Delta_i. \end{aligned}$$

But by (65),  $\phi(D', t) - \phi(D^*, t) \geq [D' - D^*]k_0(1 - \delta)\left(1 - \frac{a}{\bar{y}}\right)$ . Combining the two inequalities and using (55) yields the result.

Part 4. Fix  $\underline{t} \in [0, 1]$  and  $a \in (0, \bar{y})$ , whence  $k_a \in (0, \infty)$ . By part 2,  $D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) \geq D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t)$  for all types  $t \in S_i^t$ . Let  $\Gamma_a(t) = D_{\underline{t}\bar{y}k_a}^{H\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) \geq 0$ . As  $D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t + \Delta_i) = D_{\underline{t}a\infty}^{H\delta\bar{y}i*}(t + \Delta_i)$ ,

$$\Gamma_a(t + \Delta_i) - \Gamma_a(t) = \max \left\{ \begin{array}{l} D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - \left[ D_{\underline{t}a\infty}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) \right], \\ -k_a \Delta_i - \left[ D_{\underline{t}a\infty}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) \right] \end{array} \right\}$$

for any  $t < 1$  in  $S_i^t$  by (66). By Claim 8,  $D' - D^*(D')$  is nondecreasing in  $D'$ , so

$$D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - \left[ D_{\underline{t}a\infty}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) \right] \leq 0,$$

and by part 3,  $D_{\underline{t}a\infty}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) \geq -\Delta_i k_a$ . Hence,  $\Gamma_a(t + \Delta_i) \in [0, \Gamma_a(t)]$ , so for all  $t \in S_i^t$ ,  $\left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) \right| \leq \Gamma_a(\underline{t}) = \bar{y} - a$ . Q.E.D. Claim 9

As already noted, we extend any function defined on  $S_i$  to any  $t \in [0, 1]$  by evaluating it at  $\tau_t^i = \Delta_i \lfloor t/\Delta_i \rfloor$ .

CLAIM 10 Fix  $\underline{t} \in [0, 1)$ ,  $a \in (\bar{y}\Delta'_i, \bar{y}]$ , and  $k \in (0, \infty]$  such that either  $a < \bar{y}$  or  $k < \infty$  (or both). For any  $a_0, a_1 \in (0, a]$  and any type  $t \in [\underline{t}, 1]$ ,  $D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t)$  converges to zero as  $i \rightarrow \infty$  and  $|a_0 - a_1| \rightarrow 0$  (in either order), uniformly in  $H \in \mathcal{H}$ ,  $\bar{y} \in (0, \mathbf{y}]$ , and  $t$ .

PROOF OF CLAIM 10. By (67), for any  $t \in [0, 1]$ ,

$$(1 - \delta)\Delta_1^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i \right) \Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t) + \Delta_2^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), \tau_t^i \right) = 0 \quad (68)$$

where we define

$$\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t) = \frac{D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i) - D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t)}{\Delta_i}. \quad (69)$$

Equation (68) can be rewritten

$$\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t) = -\frac{1}{1 - \delta} \frac{\Delta_2^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), \tau_t^i \right)}{\Delta_1^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i \right)},$$

whence  $D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i) = D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - \frac{1}{1 - \delta} \frac{\Delta_2^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), \tau_t^i \right)}{\Delta_1^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i \right)} \Delta_i$  and thus, by (66),

$$D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i) = D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - \Delta_i \min \left\{ \frac{1}{1 - \delta} \frac{\Delta_2^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), \tau_t^i \right)}{\Delta_1^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i \right)}, k \right\},$$

and so

$$\begin{aligned} \Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) &\stackrel{d}{=} \frac{D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i) - D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t)}{\Delta_i} \\ &= -\min \left\{ \frac{1}{1 - \delta} \frac{\Delta_2^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), \tau_t^i \right)}{\Delta_1^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i \right)}, k \right\}. \end{aligned}$$

LEMMA 11 In the above formula for  $\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t)$ , we may replace  $D_{\underline{t}a_0k}^{H\delta\bar{y}i*}$  by  $D_{\underline{t}a_0k}^{H\delta\bar{y}i}$ . That is,

$$\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) = -\min \left\{ \frac{1}{1 - \delta} \frac{\Delta_2^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i), \tau_t^i \right)}{\Delta_1^{H\bar{y}i} \left( D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i \right)}, k \right\}. \quad (70)$$

PROOF OF LEMMA 11. Let  $w = w' = D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t)$ . Let  $\zeta e = D_{\underline{t}a_0k}^{H\delta\bar{y}i*}(t + \Delta_i)$  and  $\zeta e' = D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i)$ . By Claim 8 and part 1 of Claim 9,  $\min\{w, \zeta, w', \zeta'\} > \bar{y}\Delta_i$ . Hence, by Claim 7,  $\Delta_1^{H\bar{y}i}(\zeta', w', \tau_t^i) \leq \Delta_1^{H\bar{y}i}(\zeta, w, \tau_t^i)$  and  $\Delta_2^{H\bar{y}i}(\zeta', \tau_t^i) \geq \Delta_2^{H\bar{y}i}(\zeta, \tau_t^i)$ . Thus,

$$\frac{\Delta_2^{H\bar{y}i}(\zeta e', \tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta e', w', \tau_t^i)} \geq \frac{\Delta_2^{H\bar{y}i}(\zeta e, \tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta e, w, \tau_t^i)}, \quad (71)$$

with equality when  $\zeta e' = \zeta e$ . Accordingly, there are two cases. If  $\frac{1}{1-\delta} \frac{\Delta_2^{H\bar{y}i}(\zeta, \tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta, w, \tau_t^i)} \leq k$ , then  $\zeta e' = \zeta e$ , which implies (70). If  $\frac{1}{1-\delta} \frac{\Delta_2^{H\bar{y}i}(\zeta, \tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta, w, \tau_t^i)} \geq k$ , then  $\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) = -k$  but by (71),  $\frac{1}{1-\delta} \frac{\Delta_2^{H\bar{y}i}(\zeta', \tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta', w', \tau_t^i)} \geq k$  as well, so (70) holds. Q.E.D. Lemma 11

For all  $t' \in [\underline{t}, 1]$ , since  $D_{\underline{t}a_0k}^{H\delta\bar{y}i}(\underline{t}) = a_0$ ,

$$\begin{aligned} D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t') &= a_0 + \Delta_i \sum_{t \in S_{\bar{y}^i, t}^i, t < \tau_{t'}^i} \Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) \\ &= a_0 - \Delta_i \sum_{t \in S_{\bar{y}^i, t}^i, t < \tau_{t'}^i} \min \left\{ \frac{1}{1-\delta} \frac{\Delta_2^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i), t)}{\Delta_1^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), t)}, k \right\} \\ &= a_0 - \int_{t=\underline{t}}^{\tau_{t'}^i} \min \left\{ \frac{1}{1-\delta} \frac{\Delta_2^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i), \tau_t^i)}{\Delta_1^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t + \Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, k \right\} dt, \end{aligned} \quad (72)$$

By (54) and (24), for all  $t' \in [\underline{t}, 1]$ ,

$$D_{\underline{t}a_1k}^{H\delta\bar{y}}(t') = a_1 - \int_{t=\underline{t}}^{t'} \min \left\{ \frac{1}{1-\delta} \frac{v_t(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t)}{v_D(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t)}, k \right\} dt. \quad (73)$$

Define the analogous quantities to  $\Delta_1^{H\bar{y}i}$  and  $\Delta_2^{H\bar{y}i}$  with  $v^{H\bar{y}i}$  replaced by  $v^{H\bar{y}}$ : for any  $D', D'' \in [0, \bar{y}]$ , let

$$\begin{aligned} \Delta_1^{H\bar{y}}(D', D'', t) &= \frac{v^{H\bar{y}}(D', t) - v^{H\bar{y}}(D'', t)}{D' - D''} \\ &\in \left( k_0 \left( 1 - \frac{\max\{D', D''\}}{\bar{y}} \right), k_1 \left( 1 - \frac{\min\{D', D''\}}{\bar{y}} \right) \right) \text{ and} \end{aligned} \quad (74)$$

$$\Delta_2^{H\bar{y}}(D', t, \Delta_i) = \frac{v^{H\bar{y}}(D', t + \Delta_i) - v^{H\bar{y}}(D', t)}{\Delta_i} \in \left( \frac{k_0 (D')^2 [3\bar{y} - 2D']}{6\bar{y}^2}, k_1 D' \right), \quad (75)$$

where the bounds follow from (44) and (45) and imply that

$$\frac{\Delta_2^{H\bar{y}}(D', t, \Delta_i)}{\Delta_1^{H\bar{y}}(D', D'', t)} \in \left( \frac{k_0(D')^2 [3\bar{y} - 2D']}{6\bar{y}k_1(\bar{y} - \min\{D', D''\})}, \frac{\bar{y}k_1 D'}{k_0(\bar{y} - \max\{D', D''\})} \right). \quad (76)$$

If  $a < \bar{y}$  then by (59), (76), (48), and (55), for all  $D, D' \in (\bar{y}\Delta'_i, a]$  and all  $t \in [0, 1]$ , the ratios  $\frac{\Delta_2^{H\bar{y}i}(D', \tau_t^i)}{\Delta_1^{H\bar{y}i}(D', D'', \tau_t^i)}$ ,  $\frac{\Delta_2^{H\bar{y}}(D', t, \Delta_i)}{\Delta_1^{H\bar{y}}(D', t, \tau_t^i)}$ ,  $\frac{v_t^{H\bar{y}}(D', \tau_t^i)}{v_D^{H\bar{y}}(D', \tau_t^i)}$ , and  $\frac{v_t^{H\bar{y}}(D', t)}{v_D^{H\bar{y}}(D', t)}$  are all at most  $(1 - \delta)k_a$ . Let

$$\kappa = (1 - \delta) \min\{k, k_a\} < \infty. \quad (77)$$

It follows that by (72), (73) and the triangle inequality that for all  $t' \in [\underline{t}, 1]$ ,

$$\begin{aligned} (1 - \delta) \left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t') - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t') \right| &= \left| \int_{t=\underline{t}}^{\tau_t^i} \min \left\{ \frac{\Delta_2^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), \tau_t^i)}{\Delta_1^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} dt \right. \\ &\quad \left. - \int_{t=\underline{t}}^{t'} \min \left\{ \frac{v_t^{H\bar{y}}(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t)}{v_D^{H\bar{y}}(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), t)}, \kappa \right\} dt + a_1 - a_0 \right| \\ &\leq |a_1 - a_0| + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \end{aligned}$$

where

$$\begin{aligned} A_1 &= \left| \int_{t=\tau_{t'}^i}^{t'} \min \left\{ \frac{\Delta_2^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), \tau_t^i)}{\Delta_1^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} dt \right|, \\ A_2 &= \int_{t=\underline{t}}^{t'} \left| \min \left\{ \frac{\Delta_2^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), \tau_t^i)}{\Delta_1^{H\bar{y}i}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} \right. \\ &\quad \left. - \min \left\{ \frac{\Delta_2^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), \tau_t^i, \Delta_i)}{\Delta_1^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} \right| dt, \\ A_3 &= \int_{t=\underline{t}}^{t'} \left| \min \left\{ \frac{\Delta_2^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), \tau_t^i, \Delta_i)}{\Delta_1^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} \right. \\ &\quad \left. - \min \left\{ \frac{v_t^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), \tau_t^i)}{v_D^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} \right| dt, \\ A_4 &= \int_{t=\underline{t}}^{t'} \left| \min \left\{ \frac{v_t^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i), \tau_t^i)}{v_D^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} - \min \left\{ \frac{v_t^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}{v_D^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} \right| dt, \\ A_5 &= \int_{t=\underline{t}}^{t'} \left| \min \left\{ \frac{v_t^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}{v_D^{H\bar{y}}(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t), \tau_t^i)}, \kappa \right\} - \min \left\{ \frac{v_t^{H\bar{y}}(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), \tau_t^i)}{v_D^{H\bar{y}}(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t), \tau_t^i)}, \kappa \right\} \right| dt, \end{aligned}$$



and

$$A_6 = \int_{t=\underline{t}}^{t'} \left| \min \left\{ \frac{v_t^{H\bar{y}} \left( D_{ta_1k}^{H\delta\bar{y}}(t), \tau_t^i \right)}{v_D^{H\bar{y}} \left( D_{ta_1k}^{H\delta\bar{y}}(t), \tau_t^i \right)}, \kappa \right\} - \min \left\{ \frac{v_t^{H\bar{y}} \left( D_{ta_1k}^{H\delta\bar{y}}(t), t \right)}{v_D^{H\bar{y}} \left( D_{ta_1k}^{H\delta\bar{y}}(t), t \right)}, \kappa \right\} \right| dt.$$

Clearly,  $A_1 \leq \kappa \Delta_i$ . For  $A_2, A_3, A_4$ , and  $A_5$ , we require the following claim.

LEMMA 12 *For any  $a, b, c, d \geq 0$ ,  $\max\{|a-b|, |c-d|\}$  is an upper bound on both*

$$|\min\{a, c\} - \min\{b, d\}|$$

and  $|\max\{a, c\} - \max\{b, d\}|$ .

PROOF OF LEMMA 12. We prove the result for  $|\min\{a, c\} - \min\{b, d\}|$ ; the proof for  $|\max\{a, c\} - \max\{b, d\}|$  is identical with the keyword min replaced max throughout. First, assume  $a \geq b$  and  $c \geq d$ . Then  $|\min\{a, c\} - \min\{b, d\}| = \min\{a, c\} - \min\{b, d\}$ . And  $\max\{|a-b|, |c-d|\} = \max\{a-b, c-d\}$ . But

$$\min\{a, c\} - \min\{b, d\} \leq \max\{a-b, c-d\}$$

since

$$\begin{aligned} \min\{a, c\} &\leq \min\{\max\{a, b+c-d\}, \max\{c, d+a-b\}\} \\ &= \min\{b + \max\{a-b, c-d\}, d + \max\{a-b, c-d\}\} \\ &= \min\{b, d\} + \max\{a-b, c-d\}. \end{aligned}$$

The other cases (in which  $b > a$  or  $c > d$  or both) are analogous. Q.E.D. Lemma 12

This Lemma leads to the following useful bound.

LEMMA 13 *For any  $a, b, \kappa \in (0, \infty)$  and  $c, d \in [0, \infty)$ ,*

$$\left| \min\left\{\frac{a}{c}, \kappa\right\} - \min\left\{\frac{b}{d}, \kappa\right\} \right| \leq \left| \frac{a-b}{\max\{c, a/\kappa\}} \right| + \frac{b}{\max\{d, b/\kappa\}} \frac{\max\{|c-d|, |a-b|/\kappa\}}{\max\{c, a/\kappa\}}. \quad (78)$$

PROOF OF LEMMA 13. For any  $a, b \geq 0$  and  $c, d > 0$ ,

$$\left| \frac{a}{c} - \frac{b}{d} \right| = \left| \frac{ad - bc}{cd} \right| \leq \left| \frac{ad - bd}{cd} \right| + \left| \frac{bd - bc}{cd} \right| = \left| \frac{a - b}{c} \right| + \frac{b}{d} \left| \frac{d - c}{c} \right|. \quad (79)$$

Moreover, for any  $a > 0$  and  $c, \kappa \geq 0$ ,

$$\min \left\{ \frac{a}{c}, \kappa \right\} = \kappa \min \left\{ \frac{a}{c\kappa}, 1 \right\} = \kappa \min \left\{ \frac{a}{c\kappa}, \frac{a}{a} \right\} = \frac{a\kappa}{\max \{c\kappa, a\}} = \frac{a}{\max \{c, a/\kappa\}}. \quad (80)$$

By (79) and (80) and using Lemma (in that order),

$$\begin{aligned} \left| \min \left\{ \frac{a}{c}, \kappa \right\} - \min \left\{ \frac{b}{d}, \kappa \right\} \right| &= \left| \frac{a}{\max \{c, a/\kappa\}} - \frac{b}{\max \{d, b/\kappa\}} \right| \\ &\leq \left| \frac{a - b}{\max \{c, a/\kappa\}} \right| + \frac{b}{\max \{d, b/\kappa\}} \left| \frac{\max \{d, b/\kappa\} - \max \{c, a/\kappa\}}{\max \{c, a/\kappa\}} \right| \\ &\leq \left| \frac{a - b}{\max \{c, a/\kappa\}} \right| + \frac{b}{\max \{d, b/\kappa\}} \frac{\max \{|c - d|, |a - b|/\kappa\}}{\max \{c, a/\kappa\}}. \end{aligned}$$

Q.E.D.-Lemma 13

Let  $D'_t = D_{ta_0k}^{H\delta\bar{y}i}(t + \Delta_i) \in (\bar{y}\Delta'_i, a_0)$ ,  $D''_t = D_{ta_0k}^{H\delta\bar{y}i}(t) \in (\bar{y}\Delta'_i, a_0]$ ,  $a_t = \Delta_2^{H\bar{y}i}(D'_t, \tau_t^i)$ ,  $b_t = \Delta_2^{H\bar{y}}(D'_t, \tau_t^i, \Delta_i)$ ,  $c_t = \Delta_1^{H\bar{y}i}(D'_t, D''_t, \tau_t^i)$ , and  $d_t = \Delta_1^{H\bar{y}}(D'_t, D''_t, \tau_t^i)$ . By (78),

$$A_2 \leq \int_{t=\underline{t}}^{t'_{\underline{t}}} \left| \frac{a_t - b_t}{\max \{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'_{\underline{t}}} \left( \frac{b_t}{\max \{d_t, b_t/\kappa\}} \frac{\max \{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max \{c_t, a_t/\kappa\}} \right) dt,$$

By (35), (36), (74), (75), and parts 3 and 4 of Claim 3, for any  $\varepsilon > 0$  there is an  $i^* < \infty$  such that if  $i > i^*$ ,  $\max \{|a_t - b_t|, |c_t - d_t|\} < \varepsilon$ . By (75),  $b_t \in \left[ \frac{k_0(D'_t)^2}{6\bar{y}}, k_1\bar{y} \right]$ . By part 1 of Claim 3,  $a_t > k_0 \frac{(D')^2(3\bar{y} - 2D')}{6\bar{y}^2} > k_0 \frac{(D')^2}{6\bar{y}}$  where  $D' = D'_t - 2\bar{y}\Delta'_i < \bar{y}$ . Hence,  $\min \{a_t, b_t\} > \frac{k_0(D'_t - 2\bar{y}\Delta'_i)^2}{6\bar{y}}$ . By (35), (74), and part 2 of Claim 3,  $\min \{c_t, d_t\} \geq k_0 \left( 1 - \frac{D''_t}{\bar{y}} \right)$  so since  $D'_t \geq D''_t - k\Delta_i$ , both  $\max \{c_t, a_t/\kappa\}$  and  $\max \{d_t, b_t/\kappa\}$  are at least

$$k_0 \max \left\{ 1 - \frac{D''_t}{\bar{y}}, \frac{(D'_t - 2\bar{y}\Delta'_i)^2}{6\bar{y}\kappa} \right\} \geq k_0 \max \left\{ 1 - \frac{D''_t}{\bar{y}}, \frac{(D''_t - k\Delta_i - 2\bar{y}\Delta'_i)^2}{6\bar{y}\kappa} \right\},$$

which is bounded below by a strictly positive constant  $\kappa'$  for large enough  $i$  as  $\bar{y} > 0$ .

Collecting these bounds,  $A_2 \leq \varepsilon \kappa_2 [t' - \underline{t}]$  where  $\kappa_2 = \frac{1}{\kappa'} \left[ 1 + \frac{k_1\bar{y}}{\kappa'} \max \{1, 1/\kappa\} \right] \in (0, \infty)$ .

By (78), redefining  $a_t = \Delta_2^{H\bar{y}}(D'_t, \tau_t^i, \Delta_i)$ ,  $b_t = v_t^{H\bar{y}}(D'_t, \tau_t^i)$ ,  $c_t = \Delta_1^{H\bar{y}}(D'_t, D''_t, \tau_t^i)$ , and  $d_t = v_D^{H\bar{y}}(D''_t, \tau_t^i)$ ,

$$A_3 \leq \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left( \frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}} \right) dt.$$

By (44),  $b_t \leq k_1 \bar{y}$ . By the Mean Value Theorem, there is a  $t \in [\tau_t^i, \tau_t^i + \Delta_i]$  such that  $v_t^{H\bar{y}}(D'_t, t) = a_t$ . Thus, by (42),

$$|a_t - b_t| = \left| v_t^{H\bar{y}}(D'_t, t) - b_t \right| = \left| v_t^{H\bar{y}}(D'_t, t) - v_t^{H\bar{y}}(D'_t, \tau_t^i) \right| \leq k_2 \bar{y} \Delta_i.$$

Also by the Mean Value Theorem, there is a  $D \in [D'_t, D''_t]$  such that  $v_D^{H\bar{y}}(D, \tau_t^i) = c_t$ . By part 3 of Claim 9 and (77),

$$|D''_t - D'_t| \leq \Delta_i \min\{k, k_a\} = \frac{\kappa}{(1-\delta)} \Delta_i. \quad (81)$$

Hence,  $|c_t - d_t| = \left| v_D^{H\bar{y}}(D, \tau_t^i) - v_D^{H\bar{y}}(D''_t, \tau_t^i) \right| \leq \frac{k_1}{\bar{y}} |D''_t - D'_t| \leq \frac{k_1 \kappa}{\bar{y}(1-\delta)} \Delta_i$ . By (44), (45), (74), and (75),  $\min\{a_t, b_t\} \geq \frac{k_0(D'_t)^2}{6\bar{y}}$ , and  $\min\{c_t, d_t\} \geq k_0 \left(1 - \frac{D''_t}{\bar{y}}\right)$  so as shown in the prior paragraph, both  $\max\{c_t, a_t/\kappa\}$  and  $\max\{d_t, b_t/\kappa\}$  are at least  $\kappa' > 0$ . Collecting these bounds,  $A_3 \leq \Delta_i \kappa_3 [t' - \underline{t}]$  where  $\kappa_3 = \frac{\bar{y}}{\kappa'} \left[ k_2 + \frac{k_1}{\kappa'} \max\left\{ \frac{k_1 \kappa}{\bar{y}(1-\delta)}, \frac{k_2 \bar{y}}{\kappa} \right\} \right] \in (0, \infty)$ .

By (78), redefining  $a_t = v_t^{H\bar{y}}(D'_t, \tau_t^i)$ ,  $b_t = v_t^{H\bar{y}}(D''_t, \tau_t^i)$ , and  $c_t = d_t = v_D^{H\bar{y}}(D''_t, \tau_t^i)$ ,

$$A_4 \leq \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left( \frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{|a_t - b_t|/\kappa}{\max\{c_t, a_t/\kappa\}} \right) dt,$$

By (44),  $b_t \leq k_1 \bar{y}$ . By (47) and (81),

$$|a_t - b_t| = \left| v_t^{H\bar{y}}(D'_t, \tau_t^i) - v_t^{H\bar{y}}(D''_t, \tau_t^i) \right| \leq \frac{k_1 \kappa}{(1-\delta) \bar{y}} \Delta_i.$$

By (44) and (45),  $\min\{a_t, b_t\} \geq \frac{k_0(D'_t)^2}{6\bar{y}}$ , and  $\min\{c_t, d_t\} \geq k_0 \left(1 - \frac{D''_t}{\bar{y}}\right)$  so as shown in the prior paragraph, both  $\max\{c_t, a_t/\kappa\}$  and  $\max\{d_t, b_t/\kappa\}$  are at least  $\kappa' > 0$ . Collecting these bounds,  $A_4 \leq \Delta_i \kappa_4 [t' - \underline{t}]$ , where  $\kappa_4 = \frac{k_1}{\kappa'(1-\delta)} \left[ \frac{\kappa}{\bar{y}} + \frac{k_1}{\kappa'} \right] \in (0, \infty)$ .

By (78), redefining  $a_t = v_t^{H\bar{y}}(D'_t, \tau_t^i)$ ,  $b_t = v_t^{H\bar{y}}(D_{\underline{t}a_1k}^{H\delta\bar{y}}(\tau_t^i), \tau_t^i)$ ,  $c_t = v_D^{H\bar{y}}(D''_t, \tau_t^i)$ , and  $d_t = v_D^{H\bar{y}}(D_{\underline{t}a_1k}^{H\delta\bar{y}}(\tau_t^i), \tau_t^i)$ ,

$$A_5 \leq \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left( \frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}} \right) dt.$$

By (44),  $b_t \leq k_1 \bar{y}$ . By (42),  $|a_t - b_t| \leq k_2 \bar{y} \left| D_t'' - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(\tau_t^i) \right|$ . By (47),

$$|c_t - d_t| \leq \frac{k_1}{\bar{y}} \left| D_t'' - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(\tau_t^i) \right|.$$

By (44), (45),  $\min\{a_t, b_t\} \geq \frac{k_0(D_t'')^2}{6\bar{y}}$  and  $\min\{c_t, d_t\} \geq k_0 \left(1 - \frac{D_t''}{\bar{y}}\right)$  so as shown above, both  $\max\{c_t, a_t/\kappa\}$  and  $\max\{d_t, b_t/\kappa\}$  are at least  $\kappa' > 0$ . Collecting these bounds and using  $D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(t) = D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(\tau_t^i)$ ,

$$\begin{aligned} A_5 &\leq \kappa_5 \int_{t=\underline{t}}^{t'} \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(\tau_t^i) - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(\tau_t^i) \right| dt \leq \kappa_5 [t' - \underline{t}] \max_{t \in [\underline{t}, t']} \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(\tau_t^i) - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(\tau_t^i) \right| \\ &\leq \kappa_5 [t' - \underline{t}] \max_{t \in [\underline{t}, t']} \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(t) - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t) \right|, \end{aligned}$$

where  $\kappa_5 = \frac{\bar{y}}{\kappa'} \left[ k_2 + \frac{k_1}{\kappa'} \max\left\{ \frac{k_1}{\bar{y}}, \frac{k_2 \bar{y}}{\kappa} \right\} \right] \in (0, \infty)$ .

Now redefine  $a_t = v_t^{H\bar{y}} \left( D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t), \tau_t^i \right)$ ,  $b_t = v_t^{H\bar{y}} \left( D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t), t \right)$ ,  $c_t = v_D^{H\bar{y}} \left( D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t), \tau_t^i \right)$ , and  $d_t = v_D^{H\bar{y}} \left( D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t), t \right)$ . By (78),

$$A_6 \leq \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left( \frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}} \right) dt.$$

By (44),  $b_t \leq k_1 \bar{y}$ . By (42),  $|a_t - b_t| \leq k_2 \bar{y} \Delta_i$ . By (46),  $|c_t - d_t| \leq k_1 \Delta_i$ . By (44) and (45),  $\min\{a_t, b_t\} \geq \frac{k_0(D_t'')^2}{6\bar{y}}$  and  $\min\{c_t, d_t\} \geq k_0 \left(1 - \frac{D_t''}{\bar{y}}\right)$  so as shown above, both  $\max\{c_t, a_t/\kappa\}$  and  $\max\{d_t, b_t/\kappa\}$  are at least  $\kappa' > 0$ . Collecting these bounds,  $A_6 \leq \Delta_i \kappa_6 [t' - \underline{t}]$  where  $\kappa_6 = \frac{\bar{y}}{\kappa'} \left[ k_2 + \frac{k_1}{\kappa'} \max\left\{ k_1, \frac{k_2 \bar{y}}{\kappa} \right\} \right] \in (0, \infty)$ .

Summarizing our findings and since  $\lim_{i \rightarrow \infty} \Delta_i = 0$  and  $t' \in [\underline{t}, 1]$ , for all  $\varepsilon > 0$  there is an  $i^* < \infty$  such that if  $i > i^*$ , then

$$(1 - \delta) \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(t') - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t') \right| \leq |a_1 - a_0| + \kappa'' \varepsilon + \kappa_5 [t' - \underline{t}] \max_{t \in [\underline{t}, t']} \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(t) - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t) \right|$$

where  $\kappa'' = \kappa + (\kappa_2 + \kappa_3 + \kappa_4 + \kappa_6) [1 - \underline{t}]$ . So for any  $t'' \in [\underline{t}, t']$ ,

$$\begin{aligned} &(1 - \delta) \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(t'') - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t'') \right| \\ &\leq |a_1 - a_0| + \kappa'' \varepsilon + \kappa_5 [t'' - \underline{t}] \max_{t \in [\underline{t}, t'']} \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(t) - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t) \right| \\ &\leq |a_1 - a_0| + \kappa'' \varepsilon + \kappa_5 [t' - \underline{t}] \max_{t \in [\underline{t}, t']} \left| D_{\underline{t}a_0 k}^{H\delta \bar{y}i}(t) - D_{\underline{t}a_1 k}^{H\delta \bar{y}}(t) \right| \end{aligned}$$

and therefore,

$$\begin{aligned} & (1 - \delta) \max_{t \in [\underline{t}, t']} \left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t) \right| \\ & \leq |a_1 - a_0| + \kappa'' \varepsilon + \kappa_5 [t' - \underline{t}] \max_{t \in [\underline{t}, t']} \left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t) \right|. \end{aligned}$$

Now for  $t' \in [\underline{t}, \underline{t} + b]$  where  $b = \frac{1-\delta}{2\kappa_5} > 0$ ,  $(1 - \delta) - \kappa_5 [t' - \underline{t}] \geq \frac{1-\delta}{2}$ , so

$$\max_{t \in [\underline{t}, t']} \left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t) \right| \leq \frac{2}{1-\delta} (|a_1 - a_0| + \kappa'' \varepsilon),$$

whence  $\max_{t \in [\underline{t}, \underline{t} + b]} \left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t) \right| \leq \frac{2}{1-\delta} (|a_1 - a_0| + \kappa'' \varepsilon)$ . In particular,

$$\left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(\underline{t} + b) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(\underline{t} + b) \right| \leq \frac{2}{1-\delta} (|a_1 - a_0| + \kappa'' \varepsilon).$$

Let  $a_2 = D_{\underline{t}a_0k}^{H\delta\bar{y}i}(\underline{t} + b)$  and  $a_3 = D_{\underline{t}a_1k}^{H\delta\bar{y}}(\underline{t} + b)$ . Since  $D_{\underline{t}a_0k}^{H\delta\bar{y}i}$  and  $D_{\underline{t}a_1k}^{H\delta\bar{y}}$  are decreasing functions,  $\max\{a_2, a_3\} < a$  so in the above reasoning we can use the same constant  $a$  and thus the same constant  $k_a$  and thus the same  $\kappa''$ . Accordingly,

$$\begin{aligned} \max_{t \in [\underline{t} + b, \underline{t} + 2b]} \left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t) \right| & \leq \frac{2}{1-\delta} (|a_3 - a_2| + \kappa'' \varepsilon) \\ & \leq \frac{2}{1-\delta} \left( \frac{2}{1-\delta} (|a_1 - a_0| + \kappa'' \varepsilon) + \kappa'' \varepsilon \right) \\ & = \left( \frac{2}{1-\delta} \right)^2 |a_1 - a_0| + \left[ \frac{2}{1-\delta} + \left( \frac{2}{1-\delta} \right)^2 \right] \kappa'' \varepsilon. \end{aligned}$$

Iterating this reasoning  $n = \left\lceil \frac{1-\underline{t}}{b} \right\rceil$  (which does not depend on  $i$ ) times, we obtain

$$\max_{t \in [\underline{t}, 1]} \left| D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) - D_{\underline{t}a_1k}^{H\delta\bar{y}}(t) \right| \leq \left( \frac{2}{1-\delta} \right)^n |a_1 - a_0| + \kappa'' \varepsilon \sum_{i=1}^n \left[ \left( \frac{2}{1-\delta} \right)^i \right].$$

Since the constants multiplying  $|a_1 - a_0|$  and  $\varepsilon$  are independent of  $i$ ,  $t \in [\underline{t}, 1]$ ,  $\bar{y} \in (0, \mathbf{y})$ , and  $H \in \mathcal{H}$ , the result follows. Q.E.D. Claim 10

CLAIM 14 For all  $\underline{t} \in [0, 1)$ , there exist unique solutions  $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$  and  $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$  to  $CP_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$  and  $DP_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ , respectively. They are decreasing in  $t$  and, in the case of  $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ , continuous in  $t \in [\underline{t}, 1]$ .

PROOF OF CLAIM 14. Define, for all  $t \in [\underline{t}, 1]$ ,

$$D_{\underline{t}}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\bar{t}y k_a}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}}(t), \quad (82)$$

and for all  $t \in S_i^t$ ,  $D_{\underline{t}}^{H\delta\bar{y}i}(t) = \lim_{a \uparrow \bar{y}} D_{\bar{t}y k_a}^{H\delta\bar{y}i}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t)$ . The four limits exist by part 2 of Claims 5 and 9 and the Monotone Convergence Theorem. The pair of limits that appears in each equation are equal by part 4 of Claims 5 and 9. Hence,  $D_{\underline{t}}^{H\delta\bar{y}}$  and  $D_{\underline{t}}^{H\delta\bar{y}i}$  exist and are unique. Moreover, by part 2 of Claims 5 and 9, any solutions  $D_{\bar{t}y\infty}^{H\delta\bar{y}}$  and  $D_{\bar{t}y\infty}^{H\delta\bar{y}i}$  to  $CP_{\bar{t}y\infty}^{H\delta\bar{y}}$  and  $DP_{\bar{t}y\infty}^{H\delta\bar{y}i}$  must satisfy  $D_{\bar{t}y k_a}^{H\delta\bar{y}} \geq D_{\bar{t}y\infty}^{H\delta\bar{y}} \geq D_{\underline{t}a\infty}^{H\delta\bar{y}}$  and  $D_{\bar{t}y k_a}^{H\delta\bar{y}i} \geq D_{\bar{t}y\infty}^{H\delta\bar{y}i} \geq D_{\underline{t}a\infty}^{H\delta\bar{y}i}$  for any  $a \in (0, \bar{y}]$ , and thus must coincide with  $D_{\underline{t}}^{H\delta\bar{y}}$  and  $D_{\underline{t}}^{H\delta\bar{y}i}$ , respectively. It remains to show that  $D_{\underline{t}}^{H\delta\bar{y}}$  and  $D_{\underline{t}}^{H\delta\bar{y}i}$  solve  $CP_{\bar{t}y\infty}^{H\delta\bar{y}}$  and  $DP_{\bar{t}y\infty}^{H\delta\bar{y}i}$ : that

$$D_{\underline{t}}^{H\delta\bar{y}}(\underline{t}) = \bar{y} \text{ and, for } t \in (\underline{t}, 1), \quad \frac{dD_{\underline{t}}^{H\delta\bar{y}}(t)}{dt} = f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t), t\right) \quad (83)$$

and

$$D_{\underline{t}}^{H\delta\bar{y}i}(\underline{t}) = \bar{y} \text{ and, for } t \in S_i^t \setminus \{1\}, \quad D_{\underline{t}}^{H\delta\bar{y}i}(t + \Delta_i) = D_{\underline{t}}^{H\delta\bar{y}i*}(t + \Delta_i) \quad (84)$$

where  $D_{\underline{t}}^{H\delta\bar{y}i*}(t + \Delta_i)$  is the (by Claim 8) unique solution  $D^* \in (\bar{y}\Delta'_i, D_{\underline{t}}^{H\delta\bar{y}i}(t))$  to

$$v^{H\bar{y}i}(D^*, t + \Delta_i) - \delta v^{H\bar{y}i}(D^*, t) = (1 - \delta) v^{H\bar{y}i}\left(D_{\underline{t}}^{H\delta\bar{y}i}(t), t\right). \quad (85)$$

First,  $D_{\underline{t}}^{H\delta\bar{y}}(\underline{t}) = \lim_{a \uparrow \bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}}(\underline{t}) = \lim_{a \uparrow \bar{y}} a = \bar{y}$ . Second, for  $t \in (\underline{t}, 1)$ , we must show that  $dD_{\underline{t}}^{H\delta\bar{y}}(t)/dt = f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t), t\right)$ . By (50) and (24),  $f^{H\delta\bar{y}}(\zeta, t)$  is continuous in  $t$  so, by (54),  $f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t), t\right)$  is the limit of the derivatives  $dD_{\underline{t}a\infty}^{H\delta\bar{y}}(t)/dt$  as  $a$  goes to  $\bar{y}$ . We now invoke the following well known result.<sup>4</sup>

**THEOREM 15** *Let  $(f_n)_{n=1}^\infty$  be a sequence of real-valued functions on  $t \in [\underline{t}, 1]$ . Suppose that they converge uniformly to some function  $f$ . Assume also that the sequence of derivatives  $(f'_n)_{n=1}^\infty$  converges, uniformly in  $t \in [\underline{t}, 1]$ , to some continuous function. Then  $f$  is differentiable and  $\lim_{n \rightarrow \infty} f'_n(t) = f'(t)$  for all  $t \in [\underline{t}, 1]$ .*

<sup>4</sup>See, e.g., Theorem 9.13 in Apostol (1981, p. 229).

Let  $f_n = D_{\underline{t}a_n^\infty}^{H\delta\bar{y}}$  where  $(a_n)_{n=1}^\infty$  is an increasing sequence that converges to  $\bar{y}$ . By part 4 of Claim 5,  $(f_n)_{n=1}^\infty$  converges to  $f = D_{\underline{t}}^{H\delta\bar{y}}$ , uniformly in  $t \in [\underline{t}, 1]$ . Fix  $w \in (t, 1)$ . We will show that the sequence of derivatives  $(f'_n)_{n=1}^\infty$  converges, uniformly in  $t \in [w, 1]$ , to  $f'$ . The result then follows by taking  $w \rightarrow \underline{t}$ . By (48) and (24), letting  $b = \frac{1}{1-\delta} \frac{k_0}{6k_1}$  and  $x = f_n(w)$ ,

$$\begin{aligned} x &= f_n(\underline{t}) + \int_{\underline{t}}^w f'_n(t) dt = a_n + \int_{\underline{t}}^w f^{H\delta\bar{y}}(f_n(t), t) dt < a_n - \frac{b}{\bar{y}} \int_{\underline{t}}^w [f_n(t)]^2 dt \\ &< a_n - \frac{b}{\bar{y}} \int_{\underline{t}}^w x^2 dt = a_n - \frac{b}{\bar{y}} x^2 (w - \underline{t}) \end{aligned}$$

so  $x$  lies below the higher root of  $bx^2(w - \underline{t}) + \bar{y}x - \bar{y}a_n = 0$ , which (since  $a_n \leq \bar{y}$ ) is at most  $\frac{-\bar{y} + \sqrt{\bar{y}^2 + 4b(w - \underline{t})\bar{y}^2}}{2b(w - \underline{t})}$ . If we set this equal to  $\bar{y}(1 - c)$  and solve for  $c$ , we obtain  $\frac{c}{(1-c)^2} = b(w - \underline{t})$ . Hence, for all  $t \in [w, 1]$ ,

$$\frac{f_n(t)}{\bar{y}} \leq \frac{f_n(w)}{\bar{y}} < 1 - c \quad (86)$$

where  $c$  is positive and independent of  $n$ . Thus, for all  $t \in [w, 1]$  and all  $n$ ,

$$|f'_n(t)| = -f^{H\delta\bar{y}}(f_n(t), t) < \frac{1}{1-\delta} \frac{k_1\bar{y}}{k_0} \frac{1-c}{c}$$

which is a finite constant that is independent of  $n$ . Thus, the functions  $(f_n)_{n=1}^\infty$ , as well as their limit (call it  $f_\infty$ ), are Lipschitz on  $t \in [w, 1]$  with the same Lipschitz constant. As  $f^{H\delta\bar{y}}$  is continuous and  $f_\infty$  is Lipschitz continuous, the function

$$\lim_{n \rightarrow \infty} f'_n(t) = \lim_{n \rightarrow \infty} f^{H\delta\bar{y}}(f_n(t), t) = f^{H\delta\bar{y}}\left(\lim_{n \rightarrow \infty} f_n(t), t\right) = f^{H\delta\bar{y}}(f_\infty(t), t)$$

is continuous in  $t \in [w, 1]$ . Moreover, convergence of  $f'_n(t)$  is uniform since, for all  $n, n' \geq 1$ , by (50), (24), and (86), letting  $k_5 = \frac{k_1}{k_0 c (1-\delta)} \left[1 + \frac{k_1(1-c)}{k_0 c}\right] \in (0, \infty)$ ,

$$\begin{aligned} |f'_n(t) - f'_{n'}(t)| &= \left| f^{H\delta\bar{y}}(f_n(t), t) - f^{H\delta\bar{y}}(f_{n'}(t), t) \right| \leq k_5 |f_n(t) - f_{n'}(t)| \\ &= k_5 \left| D_{\underline{t}a_n^\infty}^{H\delta\bar{y}}(\underline{t}) - D_{\underline{t}a_{n'}^\infty}^{H\delta\bar{y}}(\underline{t}) \right| \leq \bar{y} - \min\{a_n, a_{n'}\}, \end{aligned}$$

where the last inequality is from part 4 of Claim 5. Hence,  $(f'_n)_{n=1}^\infty$  is a uniform (in  $t \in [w, 1]$ ) Cauchy sequence and thus converges uniformly. This proves existence and uniqueness.

By part 1 of claim 9,  $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$  is decreasing in  $t$ . It remains to show that  $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$  is decreasing and continuous in  $t \in [\underline{t}, 1]$ . By L-H and (24),  $f^{H\delta\bar{y}}(D, t)$  is finite and negative for all  $D < \bar{y}$ . Thus, by (83),  $D_{\underline{t}}^{H\delta\bar{y}}$  is decreasing and continuous in  $t \in (\underline{t}, 1]$ . By (83), for continuity at  $t = \underline{t}$  we must show that  $\lim_{t \rightarrow \underline{t}} D_{\underline{t}}^{H\delta\bar{y}}(t) = \bar{y}$ . By (82),  $D_{\underline{t}}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) = \lim_{a \uparrow \bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)$ . By  $\text{CP}_{\underline{t}ak}^{H\delta\bar{y}}$ ,  $D_{\underline{t}ak}^{H\delta\bar{y}}(\underline{t}) = a$ . By the triangle inequality, for any  $a \in (0, \bar{y})$ ,

$$\left| D_{\underline{t}}^{H\delta\bar{y}}(t) - \bar{y} \right| = \left| D_{\underline{t}}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| + \left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - a \right| + |a - \bar{y}|.$$

By parts 2 and 4 of Claim 5,  $\left| D_{\underline{t}}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \leq \left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \leq |\bar{y} - a|$ . Finally,  $\left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - a \right| = \left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(\underline{t}) \right| \leq \frac{k_1 \bar{y} a}{(1-\delta)k_0(\bar{y}-a)} |t - \underline{t}|$  by part 3 of Claim 5. Collecting terms and letting  $a = \bar{y} - \sqrt{t - \underline{t}} \leq \bar{y}$ ,  $\left| D_{\underline{t}}^{H\delta\bar{y}}(t) - \bar{y} \right| \leq \left[ 2 + \frac{k_1 \bar{y}^2}{(1-\delta)k_0} \right] \sqrt{t - \underline{t}}$  which goes to zero as  $t \downarrow \underline{t}$ . Q.E.D.Claim 14

In light of Claim 14, part 5 of Theorem 1 is implied by part 2 of Claim 5, setting  $\underline{t} = 0$ ,  $a = a' = \bar{y}$ , and  $k = k' = \infty$ .

We now turn to the convergence of the discrete solutions to the continuous one. By part 2 of Claims 5 and 9, for all  $a \in (0, \bar{y})$ ,

$$D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) \leq D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t) \leq D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)$$

and thus, by the triangle inequality,

$$\left| D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t) \right| \leq \max \left\{ \begin{array}{l} \left| D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| + \left| D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) \right|, \\ \left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) \right| + \left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \end{array} \right\}. \quad (87)$$

Fix  $\varepsilon > 0$ . By Claim 10, there is an  $i^*$ , independent of  $\bar{y}$ ,  $H$ , and  $t$ , such that for  $i > i^*$ ,  $\left| D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right|$  and  $\left| D_{\underline{t}\bar{y}k}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k}^{H\delta\bar{y}}(t) \right|$  are each less than  $\varepsilon/2$ . By part 4 of Claim 5, for all  $a \in [\bar{y} - \varepsilon/2, \bar{y})$ ,  $\left| D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \right| \leq \varepsilon/2$ . But (87) holds for all  $a \in (0, \bar{y})$ , so it holds in particular for  $a \in [\bar{y} - \varepsilon/2, \bar{y})$ . Thus, for all  $\varepsilon > 0$  there is an  $i^*$ , independent of  $\bar{y}$ ,  $H$ , and  $t$ , such that for  $i > i^*$ ,  $\left| D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t) \right| \leq \varepsilon$ . We now set  $\underline{t} = 0$  to obtain the desired result. Together with Claim 14, this proves that the unique solution  $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$  to  $\text{DP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$  converges to the unique solution  $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$  to  $\text{CP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$  as  $i \rightarrow \infty$ , uniformly in  $H \in \mathcal{H}$ ,  $\bar{y} \in [0, \mathbf{y}]$ , and  $t \in [0, 1]$ .



We next prove that  $p^{H\delta\bar{y}}$  and thus  $u^{H\delta\bar{y}}$  is both continuous and decreasing in the type  $t$ . First,  $p^{H\delta\bar{y}}(t) = v^{H\bar{y}}\left(D_{t/\bar{y}\infty}^{H\delta\bar{y}}(t), t\right)$ . By (22) and L-H,  $v^{H\bar{y}}$  is continuous in both arguments. Since  $D_{t/\bar{y}\infty}^{H\delta\bar{y}}$  is also continuous in  $t$ , so is  $p^{H\delta\bar{y}}$ . By (25),

$$\begin{aligned}\frac{d}{dt}\left[v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right] &= v_1^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\frac{dD^{H\delta\bar{y}}}{dt} + v_2^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right) \\ &= \delta v_1^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\frac{dD^{H\delta\bar{y}}}{dt}\end{aligned}$$

which is negative by (25), (44), and (45). Hence,  $p^{H\delta\bar{y}}$  is decreasing in  $t$ .

We now turn to convergence of  $p^{H\delta\bar{y}i}$  to  $p^{H\delta\bar{y}}$  and thus of  $u^{H\delta\bar{y}i}$  to  $u^{H\delta\bar{y}}$ . For any  $t \in [0, 1]$ , recall that  $\tau_t^i = \Delta_i \lfloor t/\Delta_i \rfloor$ . As  $D^{H\delta\bar{y}i}(t) = D^{H\delta\bar{y}i}(\tau_t^i)$ ,

$$\begin{aligned}\left|p^{H\delta\bar{y}i}(t) - p^{H\delta\bar{y}}(t)\right| &= \left|p^{H\delta\bar{y}i}(\tau_t^i) - p^{H\delta\bar{y}}(t)\right| \\ &= \left|v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right| \leq A'_1 + A'_2 + A'_3\end{aligned}$$

where

$$\begin{aligned}A'_1 &= \left|v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right)\right|, \\ A'_2 &= \left|v^{H\bar{y}}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), \tau_t^i\right)\right|, \text{ and} \\ A'_3 &= \left|v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right|.\end{aligned}$$

By (22), (26), and since  $H(0|\tau_t^i)$  is zero,

$$\frac{A'_1}{\bar{y}} \leq \int_{z=0}^{D^{H\delta\bar{y}i}(t)/\bar{y}} \left|H(z|\tau_t^i) - H\left(\Delta_i \left\lfloor \frac{z}{\Delta_i} \right\rfloor | \tau_t^i\right)\right| dz \leq k_1 \Delta'_i,$$

where the second inequality is by L-H. By (22),  $A'_2 \leq 2\left|D^{H\delta\bar{y}i}(t) - D^{H\delta\bar{y}}(t)\right|$ . By (22) and L-H,

$$A'_3 \leq \bar{y} \int_{z=0}^{D^{H\delta\bar{y}}(t)/\bar{y}} |H(z|t) - H(z|\tau_t^i)| dz \leq \bar{y}k_1 |t - \tau_t^i| \leq \bar{y}k_1 \Delta_i.$$

Hence, by the prior result, for all  $\varepsilon > 0$  there is an  $i^* < \infty$  such that if  $i > i^*$ ,

$$\left|p^{H\delta\bar{y}i}(t) - p(t)\right| \leq \bar{y}k_1 \Delta'_i + 2\left|D^{H\delta\bar{y}i}(t) - D^{H\delta\bar{y}}(t)\right| + \bar{y}k_1 \Delta_i < \varepsilon,$$

for all  $t \in [0, 1]$ ,  $\bar{y} \in (0, \bar{y}]$ , and  $H \in \mathcal{H}$ , as claimed. Hence,  $p^{H\delta\bar{y}i}$  converges uniformly to  $p^{H\delta\bar{y}}$ .

We now show uniform convergence of  $\Pi^{H\delta\bar{y}i}$  to  $\Pi^{H\delta\bar{y}}$ :

$$\left| \Pi^{H\delta\bar{y}i}(t) - \Pi^{H\delta\bar{y}}(t) \right| \leq \left| E^i[\bar{y}Z|t] - E^\infty[\bar{y}Z|t] \right| + \left| u^{H\delta\bar{y}i}(t) - u^{H\delta\bar{y}}(t) \right|$$

and for  $z \in [(c-1)\Delta'_i, c\Delta'_i]$ ,  $c = \left\lfloor \frac{z}{\Delta'_i} \right\rfloor + 1$ , so

$$\begin{aligned} \left| E^i[\bar{y}Z|t] - E^\infty[\bar{y}Z|t] \right| &= \left| \sum_{c=1}^{1/\Delta'_i} \bar{y}c\Delta'_i [H(c\Delta'_i|t) - H((c-1)\Delta'_i|t)] - \int_{z=0}^1 \bar{y}z dH(z|t) \right| \\ &= \bar{y} \left| \sum_{c=1}^{1/\Delta'_i} \int_{z=(c-1)\Delta'_i}^{c\Delta'_i} \left( \left\lfloor \frac{z}{\Delta'_i} \right\rfloor + 1 \right) \Delta'_i dH(z|t) - \int_{z=0}^1 z dH(z|t) \right| \\ &\leq \bar{y}\Delta'_i \int_{z=0}^1 \left| \left\lfloor \frac{z}{\Delta'_i} \right\rfloor + 1 - \frac{z}{\Delta'_i} \right| dH(z|t) \leq \bar{y}\Delta'_i. \end{aligned} \quad (88)$$

Thus,  $E^i[\bar{y}Z|t]$  converges uniformly to  $E^\infty[\bar{y}Z|t]$  and hence  $\Pi^{H\delta\bar{y}i}$  converges uniformly to  $\Pi^{H\delta\bar{y}}$ .

As there is no mention of  $G$  in the statement of the problems  $DP^{H\delta\bar{y}i}$  and  $CP^{H\delta\bar{y}}$ , any solutions  $D^{H\delta\bar{y}i}$  and  $D^{H\delta\bar{y}}$  to these problems for one distribution  $G$  are also solutions for any other distribution  $\widehat{G}$  that satisfies our assumptions.<sup>5</sup> Since, moreover, the solutions  $D^{H\delta\bar{y}i}$  and  $D^{H\delta\bar{y}}$  are unique by parts 1 and 2 of the theorem, they must be independent of  $G$ . Hence, convergence of  $D^{H\delta\bar{y}i}$ ,  $p^{H\delta\bar{y}i}$ ,  $u^{H\delta\bar{y}i}$ , and  $\Pi^{H\delta\bar{y}i}$  is uniform in  $G$  as well.

We now show that  $Eu^{GH\delta\bar{y}i}$  converges uniformly to  $Eu^{GH\delta\bar{y}}$ . Since  $G$  has no atoms,  $G(0) = 0$ ; hence,

$$Eu^{GH\delta\bar{y}i} = \sum_{c=1}^{1/\Delta_i} u^{H\delta\bar{y}i}(c\Delta_i) [G(c\Delta_i) - G((c-1)\Delta_i)] = \int_{t=0}^1 u^{H\delta\bar{y}i}(\tau_t^i + \Delta_i) dG(t)$$

and thus,  $Eu^{GH\delta\bar{y}i} - Eu^{GH\delta\bar{y}} = A_1'' + A_2'' - A_3''$  where

$$A_1'' = \int_{t=0}^1 \left[ u^{H\delta\bar{y}i}(\tau_t^i + \Delta_i) - u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) \right] dG(t),$$

$A_2'' = \int_{t=0}^1 u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) dG(t)$ , and  $A_3'' = \int_{t=0}^1 u^{H\delta\bar{y}}(t) dG(t)$ . As shown above, for all  $\epsilon > 0$  there is an  $i^*$  such that for all models  $i > i^*$ , parameters  $\bar{y}$  in  $(0, \mathbf{y}]$ , and conditional

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<sup>5</sup>In particular,  $\widehat{G}$  must have support  $[0, 1]$ .

distribution functions  $H$  in  $\mathcal{H}$ ,

$$|A_1''| \leq \int_{t=0}^1 \left| u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) - u^{H\delta\bar{y}}(\tau_t^i) \right| dG(t) < \int_{t=0}^1 \frac{\varepsilon}{2} dG(t) = \frac{\varepsilon}{2}.$$

Moreover, since  $u^{H\delta\bar{y}}$  is a nonincreasing function of  $t$ , it follows that

$$u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) \leq u^{H\delta\bar{y}}(t) \leq u^{H\delta\bar{y}}(\tau_t^i),$$

so  $A_2'' \leq A_3'' \leq A_4''$  where

$$A_4'' = \int_{t=0}^1 u^{H\delta\bar{y}}(\tau_t^i) dG(t) = \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}(\tau_t^i) dG(t) + \int_{t=\Delta_i}^1 u^{H\delta\bar{y}}(\tau_t^i) dG(t).$$

But  $\tau_{t+\Delta_i}^i = \tau_t^i + \Delta_i$ . Hence, letting  $t' = t - \Delta_i$  and renaming  $t'$  to  $t$ ,

$$\int_{t=\Delta_i}^1 u^{H\delta\bar{y}}(\tau_t^i) dG(t) = \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) dG(t + \Delta_i).$$

Now let  $i^*$  also be large enough that  $\Delta_{i^*} < \frac{\varepsilon}{2(1-\delta)\bar{y}[2k_3+k_4]}$ . Then by Lipschitz- $G$  and since, by (21),  $u^{H\delta\bar{y}} \in [0, (1-\delta)\bar{y}]$ , for all  $i > i^*$ ,

$$\begin{aligned} |A_2'' - A_3''| &\leq |A_2'' - A_4''| \\ &= \left| \int_{t=0}^1 u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) dG(t) - \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}(\tau_t^i) dG(t) - \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) dG(t + \Delta_i) \right| \\ &\leq \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}(\tau_t^i) dG(t) + \int_{t=1-\Delta_i}^1 u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) dG(t) \\ &\quad + \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}(\tau_t^i + \Delta_i) |\psi(t) - \psi(t + \Delta_i)| dt \\ &\leq (1-\delta)\bar{y}k_3\Delta_i + (1-\delta)\bar{y}k_3\Delta_i + (1-\delta)\bar{y}k_4\Delta_i < \frac{\varepsilon}{2}. \end{aligned}$$

For all  $i > i^*$ ,  $G$  in  $\mathcal{G}$ ,  $H$  in  $\mathcal{H}$ , and  $\bar{y} \in (0, \mathbf{y}]$ ,  $|Eu^{GH\delta\bar{y}i} - Eu^{GH\delta\bar{y}}| < \varepsilon$  as claimed.

We now show that  $E^i[\bar{y}Z]$  converges uniformly to  $E^\infty[\bar{y}Z]$ . Combined with the preceding result, this will show that  $E\Pi^{GH\delta\bar{y}i}$  converges uniformly to  $E\Pi^{GH\delta\bar{y}}$ . First,

$$E^i[\bar{y}Z] = \sum_{c=1}^{1/\Delta_i} E^i[\bar{y}Z|t = c\Delta_i][G(c\Delta_i) - G((c-1)\Delta_i)] = \int_{t=0}^1 E^i[\bar{y}Z|\tau_t^i + \Delta_i] dG(t)$$

so by the triangle inequality,  $|E^i[\bar{y}Z] - E^\infty[\bar{y}Z]| \leq A_1''' + A_2'''$  where

$$A_1''' = \int_{t=0}^1 |E^i[\bar{y}Z|\tau_t^i + \Delta_i] - E^\infty[\bar{y}Z|\tau_t^i + \Delta_i]| dG(t)$$

and  $A_2''' = \int_{t=0}^1 |E^\infty [\bar{y}Z|\tau_t^i + \Delta_i] - E^\infty [\bar{y}Z|t]| dG(t)$ . By (88),  $A_1''' \leq \bar{y}\Delta_i'$ . Integrating by parts,  $E^\infty [Z|t] = 1 - \int_{z=0}^1 H(z|t) dz$ . Thus, by L-H,

$$\begin{aligned} |E^\infty [\bar{y}Z|t = \tau_t^i + \Delta_i] - E^\infty [\bar{y}Z|t]| &\leq \bar{y} \int_{z=0}^1 |H(z|\tau_t^i + \Delta_i) - H(z|t)| dz \\ &\leq \bar{y}k_1 \int_{z=0}^1 |\tau_t^i + \Delta_i - t| dz \leq \bar{y}k_1\Delta_i, \end{aligned}$$

so  $A_2''' \leq \bar{y}k_1\Delta_i$ . Hence,  $E^i [\bar{y}Z]$  converges uniformly to  $E^\infty [\bar{y}Z]$  as claimed.

As for homogeneity in  $\bar{y}$ , using (24), equation (25) for  $\bar{y} = 1$  can be rewritten as

$$\frac{dD^{H\delta 1}}{dt} = \frac{1}{1 - \delta} \frac{\int_{z=0}^{D^{H\delta 1}} \frac{\partial H(z|t)}{\partial t} dz}{1 - H(D^{H\delta 1}|t)}.$$

Hence, for any solution  $D^{H\delta\bar{y}}$  to  $CP^{H\delta\bar{y}}$ ,  $D^{H\delta 1} = \bar{y}^{-1}D^{H\delta\bar{y}}$  is a solution to  $CP^{H\delta 1}$ . But both solutions  $D^{H\delta\bar{y}}$  and  $D^{H\delta 1}$  are unique as shown above. Hence,  $D^{H\delta\bar{y}}$  must equal  $\bar{y}D^{H\delta 1}$ . Thus, the expected payout  $p^{H\delta\bar{y}}(t) = v^{H\bar{y}}(D^{H\delta\bar{y}}(t), t)$  must equal  $v^{H\bar{y}}(\bar{y}D^{H\delta 1}(t), t)$ . But by (22), for any  $D$ ,  $v^{H\bar{y}}(D, t)$  equals  $\bar{y}v^{H1}(\frac{D}{\bar{y}}, t)$ , so  $p^{H\delta\bar{y}}(t) = \bar{y}p^{H\delta 1}(t)$ ,  $u^{H\delta\bar{y}}(t) = \bar{y}u^{H\delta 1}(t)$ , and  $E u^{GH\delta\bar{y}} = \bar{y}E u^{GH\delta 1}$  as claimed. These properties hold for  $\Pi^{H\delta\bar{y}}$  and  $E\Pi^{GH\delta\bar{y}}$  as well since  $E^\infty [\bar{y}Z|t]$  is homogeneous of degree 1 in  $\bar{y}$ .

As for model  $i$ , by (20),

$$v^{H\bar{y}i}(D, t) = \bar{y}v^{H1i}\left(\frac{D}{\bar{y}}, t\right), \quad (89)$$

so if  $D^{H\delta 1i}$  solves  $DP^{H\delta 1i}$ , then  $\bar{y}^{-1}D^{H\delta 1i}$  solves  $DP^{H\delta\bar{y}i}$ . But both solutions  $D^{H\delta\bar{y}i}$  and  $D^{H\delta 1i}$  are unique as shown above. Hence,  $D^{H\delta\bar{y}i}$  must equal  $\bar{y}D^{H\delta 1i}$ . Thus, by (89), the expected payout  $p^{H\delta\bar{y}i}(t) = v^{H\bar{y}i}(D^{H\delta\bar{y}i}(t), t)$  must equal  $\bar{y}v^{H1i}(D^{H\delta 1i}(t), t) = \bar{y}p^{H\delta 1i}(t)$  and so  $u^{H\delta\bar{y}i}(t)$  equals  $\bar{y}u^{H\delta 1i}(t)$  as claimed. These properties hold for  $\Pi^{H\delta\bar{y}i}$  and  $E\Pi^{GH\delta\bar{y}i}$  as well since  $E^i [\bar{y}Z|t]$  is homogeneous of degree 1 in  $\bar{y}$ . This completes the proof of Theorem 1. Q.E.D.

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